MATH 542 - ASSIGNMENT 1 - INTEGRAL DOMAINS

For this assignment assume that the ring R is an **integral domain**. That is R is commutative with 1, $1 \neq 0$ and for $a, b \in R$, $ab = 0 \Rightarrow \; a = 0$ or $b = 0$. We say p **divides** a (and write $b \mid a$) if $a = pb$ for some $b \in R$.

(1) Suppose $n \geq 2$, for which n is $\mathbb{Z}/n\mathbb{Z}$ is an integral domain? Justify your answer.

(2) Show that $F[x]$, the ring of polynomials with coefficients in the field F, is an integral domain.

(3) Recall that a **unit** in a ring R is any element with a multiplicative inverse. We say that $a, b \in R$ are **associates** if $a = ub$ for some unit $u \in R$. Suppose R is an integral domain. Show that if $a, b \in R$ and $a \mid b$ and $b \mid a$ then a and b are associates.

(4) What are the units in $\mathbb{Z}$? What are the units in $F[x]$?

(5) For $a \in R$, define $Ra = \{ra \mid r \in \mathbb{R}\}$. This is called the **principal ideal** generated by a. It is also denoted by (a). Show that if $(a) \subset (b)$ then $b \mid a$.

(6) Recall that a **principal ideal domain** (or PID for short) is an integral domain where all ideals are principal. We will show that $\mathbb{Z}$ and $F[x]$ are PIDs on the next assignment. Assuming we know this, give a description of all ideals of $\mathbb{Z}$ and $F[x]$.

(7) A **greatest common divisor** (or gcd) of $a, b \in R$ is an element g such that $g \mid a$ and $g \mid b$ and with the property that if f also divides both a and b then $f \mid d$. Show that any two *gcd* of a and b are associates.

(8) A **least common multiple** (or lcm) of $a, b \in R$ is an element e such that any multiple of both a and b is also a multiple of e. Show that in a PID there always exists a least common multiple. Is it unique?