

MATH 376 HOMEWORK 8
DUE MONDAY APRIL 3

Definition. Let f be a bounded function on $[a, b]$. Define

$$M(f) = \sup\{f(x) \mid x \in [a, b]\} \text{ and } m(f) = \inf\{f(x) \mid x \in [a, b]\}$$

Recall the “Extreme value theorem” from one variable calculus:

Theorem. If f is continuous on a closed interval $[a, b]$ then there exist $c, d \in [a, b]$ such that $f(c) = M(f)$ and $f(d) = m(f)$.

The goal of this home work is to prove the “small span theorem”.

Theorem. Let f be a continuous function on the closed interval $[a, b]$. Then for every $\epsilon > 0$ there is a partition of $[a, b]$ into a finite number of subintervals such that on each subinterval of the partition

$$M(f) - m(f) < \epsilon.$$

We prove this theorem by contradiction.

- (1) Suppose that that above “small span theorem” is false for some function f . Write out the negation of the theorem starting with “There exists ϵ_0 such that .. ”
- (2) Suppose c is the midpoint of $[a, b]$. Explain why if the theorem is does not hold with $\epsilon = \epsilon_0$ for f on $[a, b]$ then it must not hold for $\epsilon = \epsilon_0$ on at least one of the subintervals $[a, c]$ or $[c, d]$. Call this a “bad” subinterval.
- (3) Let $[a_1, b_1] = [a, c]$ if $[a, c]$ is a “bad” subinterval (otherwise let $[a_1, b_1] = [c, b]$). Now splitting $[a_1, b_1]$ in half we define $[a_2, b_2]$ to be the first half of $[a_1, b_1]$ if it is “bad” otherwise we set it to be the second half. Repeat this to get $[a_n, b_n]$. What is $M(f) - m(f)$ on $[a_n, b_n]$? What is the size of $[a_n, b_n]$?
- (4) Let $A = \{a, a_1, a_2, \dots\}$ be the collection of left hand endpoints of these intervals. Why does A have a supremum? Let $\alpha = \sup A$. Show that $\alpha \in [a, b]$.
- (5) Suppose $\alpha \in (a, b)$. Argue that for some $\delta > 0$ there is an open ball of radius δ around α such that $M(f) - m(f) < \epsilon_0$ on that ball. (Hint: continuity)
- (6) Show that for n large enough $[a_n, b_n]$ is contained in this ball.
- (7) Explain why the last statement produces a contradiction.
- (8) What if $\alpha = a$ or $\alpha = b$? How does the proof need to be changed?
- (9) Extend this theorem to $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$. (What about arbitrary dimensions?)