Math 376: Differential Forms

The goal of this homework is to rephrase much of what we have talked about (concerning Stoke's theorem, etc.) in a way that's suitable for even higher dimensional generalizations. What follows should be considered a very small taste of what are known as *differential forms*.

In what follows, fix an open set $U \subseteq \mathbb{R}^n$. Whether or not it is explicitly stated, all functions below are assumed to be smooth–i.e., infinitely differentiable (all partial derivatives of all orders exist and are continuous). For $0 \leq k \leq n$, we define an infinite dimensional vector space $\Omega^k(U)$, called the differential k-forms on U, as follows.[1]

- $\Omega^0(U)$ is the set of all smooth functions $f : U \to \mathbb{R}$.
- $\Omega^1(U)$ is defined to be all formal sums,

  $$f_1(x)\,dx_1 + f_2(x)\,dx_2 + \cdots + f_n(x)\,dx_n,$$

  where $f_1, \ldots, f_n : U \to \mathbb{R}$ are any smooth functions, and $dx_1, \ldots, dx_n$ are just formal vectors in our vector space. Note that elements of $\Omega^1(U)$ can be thought of as functions $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$. We define addition in the usual way:

  $$\begin{aligned}&(f_1(x)\,dx_1 + f_2(x)\,dx_2 + \cdots + f_n(x)\,dx_n) + (g_1(x)\,dx_1 + g_2(x)\,dx_2 + \cdots + g_n(x)\,dx_n)\\ &= (f_1 + g_1)\,dx_1 + \cdots + (f_n + g_n)\,dx_n.\end{aligned}$$

  Scalar multiplication is defined in a similar way.
- $\Omega^2(U)$ is defined to be all formal sums,

  $$\sum_{1 \leq i < j \leq n} f_{i,j}(x)\,dx_i \wedge dx_j, \tag{1}$$

  where, again, the $f_{i,j} : U \to \mathbb{R}$ are any smooth functions, and $dx_i \wedge dx_j$ is just a formal element of our vector space. We identify $dx_i \wedge dx_j$ with $-dx_j \wedge dx_i$. Thus $dx_i \wedge dx_i = 0$ and every sum of the form

  $$\sum_{1 \leq i, j \leq n} f_{i,j}(x)\,dx_i \wedge dx_j,$$

  is of the form (1).
- $\Omega^k(U)$ is defined to be all sums of the form,

  $$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1,\ldots,i_k}\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2}$$

  As before we identify $dx_i \wedge dx_j$ with $-dx_j \wedge dx_i$ so that, for instance, $dx_1 \wedge dx_2 \wedge dx_3 = -dx_2 \wedge dx_1 \wedge dx_3 = dx_2 \wedge dx_3 \wedge dx_1$. This is why we restrict attention to $i_1 < i_2 < \cdots < i_k$ in (2).

Problem 1. Explain why, with the above definition, $\Omega^k(U)$ is zero dimensional if $k > n$.

[1] When $k > n$, one takes $\Omega^k(U)$ to be the zero dimensional vector space.

There is a linear map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ given by the following,

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{l=1}^{n} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{3}$$

and extending by linearity (every element of $\Omega^k(U)$ can be written as a finite sum of the terms discussed in (3), so (3) uniquely determines d). Note that, in the sum on the right hand side of (3), there are terms where $l \geq i_1$. In these cases, we use repeated application of the identity $dx_l \wedge dx_i = -dx_i \wedge dx_l$.

Example 1. In $\mathbb{R}^3$, we often write dx, dy, dz instead of dx_1, dx_2, dx_3. Consider, in $\mathbb{R}^3$,

$$d(f dy) = \frac{\partial f}{\partial x} dx \wedge dy + \frac{\partial f}{\partial y} dy \wedge dy + \frac{\partial f}{\partial z} dz \wedge dy = \frac{\partial f}{\partial x} dx \wedge dy - \frac{\partial f}{\partial z} dy \wedge dz,$$

where we have used $dy \wedge dy = 0$.

Problem 2. Compute, in $\mathbb{R}^3$,

$$d(f dx + g dy + h dz).$$

Problem 3. Relate divergence with d. Hint: take $n = 3$ and think of a function $F = (F_1, F_2, F_3) : U \to \mathbb{R}^3$ as a 2-form $F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$.

Problem 4. Relate curl with d. Hint: take $n = 3$, and proceed as in the previous problem, but now thinking of d acting on 1-forms.

Problem 5. Relate the gradient with d (in any dimension). Hint: think of d acting on 0-forms (functions).

Problem 6. Show that $d^2 = 0$. That is, if one applies d twice, one always gets the 0 vector. Conclude that the divergence of the curl of a function is always 0.

We say a k-form, ω, is closed if $d\omega = 0$. We say a k-form, ω, is exact if $\omega = d\gamma$ for some $k-1$ form γ. Problem 6 shows that all exact forms are closed.

Problem 7. By Problem 5, we know that the gradient of a function can be seen as an exact 1-form. Thus a necessary condition that a 1-form be a gradient of a 0 form is that it be closed. Consider the case $n = 2$. For what open sets $U \subseteq \mathbb{R}^2$ do we have that all closed 1-forms are exact?

If we define a k dimensional parameterized surface, S, in U as $S = r(T)$, where $r : T \to S$ is a nice function and $T \subset \mathbb{R}^k$ is a nice set. Say $r(u_1, \ldots, u_k) = (X_1(u_1, \ldots, u_k), \ldots, X_n(u_1, \ldots, u_k))$. We can define the integral of k-forms over S as follows. If

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1, \ldots, i_k}(x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we define,

$$\int_S \omega = \int_T \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1, \ldots, i_k}(r(u_1, \ldots, u_k)) \frac{\partial(X_{i_1}, \ldots, X_{i_k})}{\partial(u_1, \ldots, u_k)} du_1 \cdots du_k.$$

With this notation, the theorems we have been discussing in class can be generalized in the following way:

$$\int_S d\omega = \oint_{\partial S} \omega,$$

where ∂U denotes the boundary of U and $\oint$ denotes an integral with a chosen "orientation" which we have not made precise,[2] and ω is a $k-1$ form. In this generality, this theorem is referred to as Stokes' theorem.

Remark 2. In the above, we have used the usual coordinate system on $\mathbb{R}^n$. At the heart of the power of differential forms is the fact that one would get the same objects of study if one used any coordinate system. One can even go further and define all of the above without reference to any coordinate system. For instance, in $\mathbb{R}^3$ all of the above can be defined without reference to the coordinates x, y, z (so without ever writing dx, dy, dz, etc.).

[2] In the case of Stokes' theorem, in class, this was the direction in which we traversed the boundary of S.