MATH 376 HOMEWORK 8
DUE THURSDAY MARCH 31

Defn. A number B is called a *least upper bound* of a nonempty set $S \subseteq \mathbb{R}$ if B has the following two properties:

- B is an upper bound for S (i.e. for all $s \in S$, $s \leq B$)
- No number less than B is an upper bound for S.

You can prove that if B exists then it is unique and so we call B the *supremum* of S and denote it $\sup S$.

Completeness Axiom. Every non empty set S of real numbers which is bounded above has a supremum; that is, there is a real number B such that $B = \sup S$

(1) Similarity define *infimum* of a set (greatest lower bound) and then prove that the completeness axiom implies that every set S that is bounded below has an infimum.

(2) Prove that if $\sup S$ exists then for any $h > 0$ there is $x \in S$ such that $x > \sup S - h$.

(3) Show that if S and T are two non empty sets such that for all $s \in S$ and $t \in T$ we have $s \leq t$ then $\sup S \leq \inf T$.

(4) In this exercise we will prove that a continuous function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$.
   (a) Show that for any $\alpha \in [a, b]$ there exist an open interval I_α containing α such that f is bounded on I_α.
   (**Hint:** $\epsilon - \delta$ definition of continuity and a triangle inequality)
   (b) Suppose that f is unbounded on $[a, b]$ then note that f is unbounded on one of $[a, c]$ or $[c, b]$ where c is the midpoint of $[a, b]$. Call this new interval on which f is unbounded $[a_1, b_1]$. Repeat this procedure on $[a_1, b_1]$ to get $[a_2, b_2]$ and keep repeating to get $[a_n, b_n]$. What is the length of $[a_n, b_n]$?
   (c) Let $A = \{a, a_1, a_2, \ldots\}$ show that $\sup A \in [a, b]$.
   (d) Let $\alpha = \sup A$. Show that for n large enough $[a_n, b_n]$ must lie inside the open interval I_α from (a).
   (e) Write out a proof that a continuous function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ carefully using the above steps.

(5) Prove that if f is continuous on $[a, b]$ then for every $\epsilon > 0$ there is a partition of $[a, b]$ into a finite number of subintervals $[x_i, x_{i+1}]$ such that the difference between the max and min of f on $[x_i, x_{i+1}]$ is at most ϵ.
   (**Hint:** Suppose this is false (i.e. there is some $\epsilon_0 > 0$ such that $[a, b]$ cannot be partitioned into such intervals. Let c be the midpoint of $[a, b]$. Then for the same ϵ_0 the theorem is false for at least one of $[a, c]$ or $[c, b]$. Adapt the ideas of the above theorem to finish the proof.))

(6) **Bonus.** State and prove versions of the last two problems for continuous functions $f : [a, b] \times [c, d] \to \mathbb{R}$.