MATH 375 HOMEWORK 1
DUE WEDNESDAY OCT. 7
COMPLEX NUMBERS

Let $A = \{(a,b) \mid a,b \in \mathbb{R}\}$ with the usual notion of addition (component-wise). Define multiplication on A to be

$$(a,b) \cdot (c,d) = (ac - bd, bc + ad).$$

For $z = (a,b) \in A$ define the *conjugate* of $z = (a,b)$ to be

$$\bar{z} = \overline{(a,b)} = (a,-b).$$

For $z = (a,b) \in A$ define the *length* of z to be $|z| = \sqrt{a^2 + b^2}$.

(1) Show that $(0,0) \cdot (a,b) = (0,0)$. (We will write $0 = (0,0)$).
(2) Show that $(1,0) \cdot (a,b) = (1,0) \cdot (a,b) = (a,b)$. (We will write $1 = (1,0)$).
(3) Show that if we let $i^2 = -1$ then $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$. (i.e. A can be identified with $\mathbb{C}$, the set of complex numbers.)
(4) Show that $z \cdot \bar{z} = |z|^2$.
(5) Show that if $z \neq 0 \in A$ then $z^{-1} = \dfrac{\bar{z}}{|z|^2}$. (i.e. Show $z \cdot z^{-1} = 1$.)

We can define vectors spaces with complex scalars instead of real scalars. Almost everything works as before. Since we can view $\mathbb{R} \subset \mathbb{C}$ by viewing $x \in \mathbb{R}$ as $(x,0) \in \mathbb{C}$, any vector space over $\mathbb{C}$ can also be thought of as a vector space over $\mathbb{R}$.

(7) Show that $\mathbb{C}$ is a real vector space of dimension 2 and a complex vector space of dimension 1.

An inner product on a complex vector space V is a way of assigning a complex number $\langle x,y\rangle$ to each pair of vectors $x,y \in V$ where all of the same conditions are satisfied as for real inner products except the symmetry condition is now replaced with

$$\langle x,y\rangle = \overline{\langle y,x\rangle}.$$

A complex vector space V with an inner product is called a *Unitary space*.

(8) Show that $\langle x, cy\rangle = \bar{c}\,\langle x,y\rangle$ for $c \in \mathbb{C}$ and $x,y \in V$.
(9) Show that $\|z\|^2 = \langle z,z\rangle$ is always a real number. (Hint: use the conjugate).
(10) Prove the Cauchy-Schwartz inequality for Unitary spaces:

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\,\langle y,y\rangle$$

(Hint: This is done in your book but try to follow your notes instead and think about where you have to change to proof to make it work for complex inner products).