MATH 375 HOMEWORK 11
DUE WEDNESDAY NOV. 17
SYMMETRIC AND HERMITIAN LINEAR TRANSFORMATIONS

Let V be a vector space over the complex numbers $\mathbb{C}$ of dimension n.

1) Look up the *Fundamental theorem of algebra* and explain why it implies that any linear transformation $T : V \to V$ has at least one eigenvector.

Suppose V also has a complex inner product $\langle v, w \rangle$. We say that $T : V \to V$ is *Hermitian* if

$$\langle T(v), w \rangle = \langle v, T(w) \rangle$$

2) If λ is an eigenvalue of T show that λ is real. (Hint: Recall $\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$)
3) Show that if v, w are eigenvectors of T corresponding to distinct eigenvalues λ, μ then v and w are orthogonal.
4) Suppose v is an eigenvector of T and $U = L(\{v\})^{\perp}$ is the orthogonal complement of v. Show that U is invariant under T. (i.e $T(U) \subseteq U$.)
5) Prove by induction (on n) that V has an orthonormal basis of eigenvectors. (Hint: #1 and # 4 give the induction step.)

If V is instead a vector space over $\mathbb{R}$ with real inner product $\langle v, w \rangle$ we say that $T : V \to V$ is *symmetric* if $\langle T(v), w \rangle = \langle v, T(w) \rangle$

6) If we were to try to use the proof in #5 replacing $\mathbb{C}$ with $\mathbb{R}$ why would it not work?
7) Nevertheless why does the above proof imply that a symmetric T has a basis of eigenvectors? (Hint: # 2)
8) Show that if A is a symmetric matrix (i.e. $A = A^t$) then $T_A : V \to V$ is a symmetric linear transformation.

   (The complex analogue is that A equals its *adjoint* which is obtained by taking the conjugate of each entry of A^t).

If you are stuck on any of these you can consult Chapter 5 in your textbook or search for the *Spectral Theorem*.