LIMITS FOR MATH 171

1. Infinite limits

We define what it means for a function $f(x)$ to go to infinity (or minus infinity) as x goes to infinity (or minus infinity).

Definition 1.1. *We say that a function $f(x)$ goes to infinity as x goes to infinity and write*

$$\lim_{x\to\infty} f(x) = \infty$$

if for all $M > 0$ there exists a bound N such that if $x > N$ then $f(x) > M$.

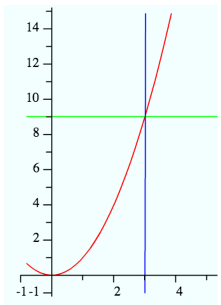

Figure 1. This is the graph of the function $f(x) = x^2$. The horizontal green line is at height $M = 9$. The vertical blue line is the bound $N = 3$. If $x > 3$ then $f(x) > 9$. In this example if we want $f(x) > M$ then I should pick $x > \sqrt{M}$.

Definition 1.2. *We say that a function $f(x)$ goes to negative infinity as x goes to infinity and write*

$$\lim_{x\to\infty} f(x) = -\infty$$

if for all $M < 0$ there exists a bound N such that if $x > N$ then $f(x) < M$.

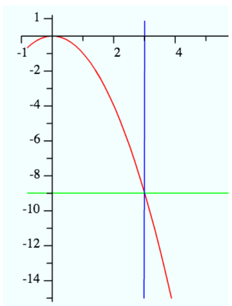

Figure 2. This is the graph of the function $f(x) = -x^2$. The horizontal green line is at height $M = -9$. The vertical blue line is the bound $N = 3$. If $x > 3$ then $f(x) < -9$. In this example if we want $f(x) < M$ then I should pick $x > \sqrt{|M|}$.

Definition 1.3. *We say that a function $f(x)$ goes to infinity as x goes to negative infinity and write*

$$\lim_{x \to -\infty} f(x) = \infty$$

if for all $M > 0$ there exists a bound N such that if $x < N$ then $f(x) > M$.

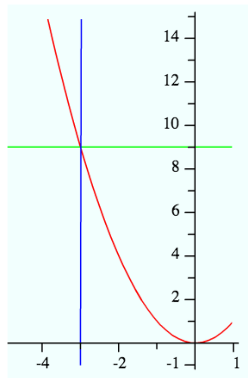

FIGURE 3. This is the graph of the function $f(x) = x^2$. The horizontal green line is at height $M = 9$. The vertical blue line is the bound $N = -3$. If $x < -3$ then $f(x) > 9$. In this example if we want $f(x) > M$ then I should pick $x < -\sqrt{M}$.

Definition 1.4. *We say that a function $f(x)$ goes to negative infinity as x goes to negative infinity and write*

$$\lim_{x \to -\infty} f(x) = -\infty$$

if for all $M < 0$ there exists a bound N such that if $x < N$ then $f(x) < M$.

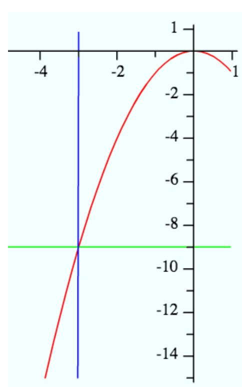

FIGURE 4. This is the graph of the function $f(x) = -x^2$. The horizontal green line is at height $M = -9$. The vertical blue line is the bound $N = -3$. If $x < -3$ then $f(x) < -9$. In this example if we want $f(x) < M$ then I should pick $x < -\sqrt{|M|}$.

Problem. Show that $\lim_{x \to -\infty} -2x + 1 = \infty$.

Scratch work. To solve this problem we need to use the third definition. We want $-2x + 1 > M$ as long as x is a large negative number. How negative should I make x? We rewrite the inequality to express x in terms of the lower bound M:

$$\begin{aligned} -2x + 1 &> M \\ -2x &> M - 1 \\ x &< -1/2(M - 1) \end{aligned}$$

Write up. If $x < -1/2(M - 1)$ then $-2x + 1 > M$ since

$$\begin{aligned} x &< -1/2(M - 1) \\ -2x &> M - 1 \\ -2x + 1 &> M. \end{aligned}$$

Therefore, by definition, we have that $\lim_{x \to -\infty} -2x + 1 = -\infty$.

Remark. Note that the write up basically follows the same steps as the scratch work but in the reverse order.

2. How to do an $\epsilon - \delta$ problem for a linear function

Definition $\lim_{x \to a} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

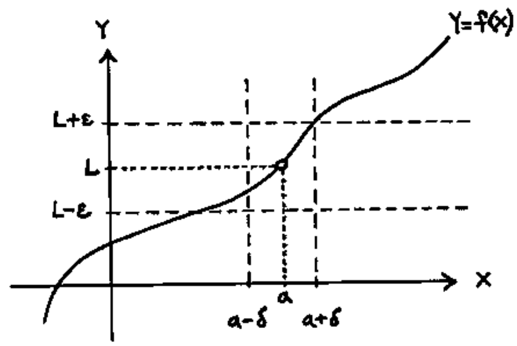

Figure 5. If we plug in x values into $f(x)$ that are with distance δ of a then $f(x)$ is within distance ϵ of L.

Problem. Show that $\lim_{x\to 3} 4x + 1 = 13$.

Scratch work: Your goal is to find a δ in terms of ϵ such that

if $0 < |x - 3| < \delta$ then $|(4x + 1) - 13| < \epsilon$.

You should think of δ as a function $\delta(\epsilon)$.

First you have to do some scratch work on a scrap piece of paper:

$$\begin{aligned} |(4x + 1) - 13| &< \epsilon \\ |4x - 12| &< \epsilon \\ 4|x - 3| &< \epsilon \\ |x - 3| &< \epsilon/4 \end{aligned}$$

What I have done is simplified the expression so that it contains $|x - 3|$.

Now I see that if I pick $\delta = \epsilon/4$ then $0 < |x - 3| < \delta$ implies that $|(4x + 1) - 13| < \epsilon$.

Now you are ready to write up the problem.

Writing up the problem:

Given $\epsilon > 0$ let $\delta = \epsilon/4$. Then if $0 < |x - 3| < \delta$ we have that

$$\begin{aligned} |(4x + 1) - 13| &= |4x - 12| \\ &= 4|x - 3| \\ &< 4 \cdot \epsilon/4. \end{aligned}$$

In other words $|(4x + 1) - 13| < \epsilon$. Therefore by definition $\lim_{x\to 3} 4x + 1 = 13$.

Remark. Note again that the write up basically follows the same steps as the scratch work but in the reverse order.

2.1. **Homework.** The following problems were also listed on the webpage.

(1) Show that the limit of $-2x+1$ as x goes to infinity is negative infinity. i.e. show that

$$\lim_{x\to\infty} -2x+1 = -\infty$$

(2) Show that the limit of $-3x-1$ as x goes to negative infinity is positive infinity. i.e. show that

$$\lim_{x\to-\infty} -3x-1 = \infty$$

(3) Show that the limit of $-2x^2+4x-2$ as x goes to infinity is negative infinity. i.e. show that

$$\lim_{x\to\infty} -2x^2+4x-2 = -\infty$$

(hint: factor it!)

2.2. **Comments.** The point of these definitions is to introduce you to rigorous thinking. Yes it's "easy to see" that x^2 goes off to ∞ as x goes to infinity or that that $2x+1$ gets "arbitrarily close" to 3 as x gets close to 1. But if you ask yourself what do these statements really mean you will find yourself having to write down the above definitions.

2.3. **Bonus exercises.** Write down definitions from the following:

$$\lim_{x\to\infty} f(x) = L$$
$$\lim_{x\to-\infty} f(x) = L$$
$$\lim_{x\to a} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty$$

2.4. **Calculus Notes.** This material is also covered in your calculus notes in Chapter 3 Sections 5.5 and 9