EXERCISES

1 Find all numbers t such that $(\frac{1}{3}, t)$ is a point on the unit circle.

2 Find all numbers t such that $(\frac{3}{5}, t)$ is a point on the unit circle.

3 Find all numbers t such that $(t, -\frac{2}{5})$ is a point on the unit circle.

4 Find all numbers t such that $(t, -\frac{3}{7})$ is a point on the unit circle.

5 Find the points where the line through the origin with slope 3 intersects the unit circle.

6 Find the points where the line through the origin with slope 4 intersects the unit circle.

For Exercises 7–14, sketch the unit circle and the radius corresponding to the given angle. Include an arrow to show the direction in which the angle is measured from the positive horizontal axis.

7 $20°$	10 $330°$	13 $-75°$
8 $80°$	11 $460°$	14 $-170°$
9 $160°$	12 $-10°$	

15 What is the angle between the hour hand and the minute hand on a clock at 4 o'clock?

16 What is the angle between the hour hand and the minute hand on a clock at 5 o'clock?

17 What is the angle between the hour hand and the minute hand on a clock at 4:30?

18 What is the angle between the hour hand and the minute hand on a clock at 7:15?

19 What is the angle between the hour hand and the minute hand on a clock at 1:23?

20 What is the angle between the hour hand and the minute hand on a clock at 11:17?

For Exercises 21–24, give the answers to the nearest second.

21 At what time between 1 o'clock and 2 o'clock are the hour hand and the minute hand of a clock pointing in the same direction?

22 At what time between 4 o'clock and 5 o'clock are the hour hand and the minute hand of a clock pointing in the same direction?

23 Find two times between 1 o'clock and 2 o'clock when the hour hand and the minute hand of a clock are perpendicular.

24 Find two times between 4 o'clock and 5 o'clock when the hour hand and the minute hand of a clock are perpendicular.

25 Suppose an ant walks counterclockwise on the unit circle from the point $(1,0)$ to the endpoint of the radius corresponding to $70°$. How far has the ant walked?

26 Suppose an ant walks counterclockwise on the unit circle from the point $(1,0)$ to the endpoint of the radius corresponding to $130°$. How far has the ant walked?

27 What angle corresponds to a circular arc on the unit circle with length $\frac{\pi}{5}$?

28 What angle corresponds to a circular arc on the unit circle with length $\frac{\pi}{6}$?

29 What angle corresponds to a circular arc on the unit circle with length $\frac{5}{2}$?

30 What angle corresponds to a circular arc on the unit circle with length 1?

For Exercises 31–32, assume the surface of the earth is a sphere with diameter 7926 miles.

31 Approximately how far does a ship travel when sailing along the equator in the Atlantic Ocean from longitude $20°$ west to longitude $30°$ west?

32 Approximately how far does a ship travel when sailing along the equator in the Pacific Ocean from longitude $170°$ west to longitude $120°$ west?

33 Find the lengths of both circular arcs on the unit circle connecting the points $(1,0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

34 Find the lengths of both circular arcs on the unit circle connecting the points $(1,0)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

For Exercises 35–40, find the endpoint of the radius of the unit circle corresponding to the given angle.

35 $120°$	37 $-30°$	39 $390°$
36 $240°$	38 $-150°$	40 $510°$

For Exercises 41–46, find the angle corresponding to the radius of the unit circle ending at the given point. Among the infinitely many possible correct solutions, choose the one with the smallest absolute value.

41 $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	44 $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
42 $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$	45 $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
43 $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	46 $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

47 Find the lengths of both circular arcs on the unit circle connecting the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and the endpoint of the radius corresponding to $130°$.

48 Find the lengths of both circular arcs on the unit circle connecting the point $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and the endpoint of the radius corresponding to $50°$.

49 Find the lengths of both circular arcs on the unit circle connecting the point $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and the endpoint of the radius corresponding to $125°$.

50 Find the lengths of both circular arcs on the unit circle connecting the point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and the endpoint of the radius corresponding to $20°$.

51 What is the slope of the radius of the unit circle corresponding to $30°$?

52 What is the slope of the radius of the unit circle corresponding to $60°$?

PROBLEMS

Some problems require considerably more thought than the exercises.

53 Suppose m is a real number. Find the points where the line through the origin with slope m intersects the unit circle.

For Problems 54–56, suppose a spider moves along the edge of a circular web at a distance of 3 cm from the center.

54 If the spider begins on the far right side of the web and creeps counterclockwise until it reaches the top of the web, approximately how far does it travel?

55 If the spider crawls along the edge of the web a distance of 2 cm, approximately what is the angle formed by the line segment from the center of the web to the spider's starting point and the line segment from the center of the web to the spider's finishing point?

56 Place the origin of the coordinate plane at the center of the web. What are the coordinates of the spider when it reaches the point directly southwest of the center?

Use the following information for Problems 57–62: A grad is a unit of measurement for angles that is sometimes used in surveying, especially in some European countries. A complete revolution once around a circle is 400 grads.

[These problems may help you work comfortably with angles in units other than degrees. In the next section we will introduce radians, the most important units used for angles.]

57 How many grads in a right angle?

58 The angles in a triangle add up to how many grads?

59 How many grads in each angle of an equilateral triangle?

60 Convert $37°$ to grads.

61 Convert 37 grads to degrees.

62 Discuss advantages and disadvantages of using grads as compared to degrees.

63 Verify that each of the following points is on the unit circle:

(a) $(\frac{3}{5}, \frac{4}{5})$ (b) $(\frac{5}{13}, \frac{12}{13})$ (c) $(\frac{8}{17}, \frac{15}{17})$

The next problem shows that the unit circle contains infinitely many points for which both coordinates are rational.

64 Show that if m and n are integers, not both zero, then

$$\left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}\right)$$

is a point on the unit circle.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Do not read these worked-out solutions before attempting to do the exercises yourself. Otherwise you may mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises and check your answers here. If you get stuck on an exercise, then look at the worked-out solution here.

1 Find all numbers t such that $(\frac{1}{3}, t)$ is a point on the unit circle.

SOLUTION For $(\frac{1}{3}, t)$ to be a point on the unit circle means that the sum of the squares of the coordinates equals 1. In other words,

$$\left(\tfrac{1}{3}\right)^2 + t^2 = 1.$$

This simplifies to the equation $t^2 = \frac{8}{9}$, which implies that $t = \frac{\sqrt{8}}{3}$ or $t = -\frac{\sqrt{8}}{3}$. Because $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$, we can rewrite this as $t = \frac{2\sqrt{2}}{3}$ or $t = -\frac{2\sqrt{2}}{3}$.

3 Find all numbers t such that $(t, -\frac{2}{5})$ is a point on the unit circle.

EXERCISES

In Exercises 1–8, convert each angle to radians.

1 $15°$

2 $40°$

3 $-45°$

4 $-60°$

5 $270°$

6 $240°$

7 $1080°$

8 $1440°$

In Exercises 9–16, convert each angle to degrees.

9 4π radians

10 6π radians

11 $\frac{\pi}{9}$ radians

12 $\frac{\pi}{10}$ radians

13 3 radians

14 5 radians

15 $-\frac{2\pi}{3}$ radians

16 $-\frac{3\pi}{4}$ radians

For Exercises 17–24, sketch the unit circle and the radius corresponding to the given angle. Include an arrow to show the direction in which the angle is measured from the positive horizontal axis.

17 $\frac{5\pi}{18}$ radians

18 $\frac{1}{2}$ radian

19 2 radians

20 5 radians

21 $\frac{11\pi}{5}$ radians

22 $-\frac{\pi}{12}$ radians

23 -1 radian

24 $-\frac{8\pi}{9}$ radians

25 Suppose an ant walks counterclockwise on the unit circle from the point $(0, 1)$ to the endpoint of the radius corresponding to $\frac{5\pi}{4}$ radians. How far has the ant walked?

26 Suppose an ant walks counterclockwise on the unit circle from the point $(-1, 0)$ to the endpoint of the radius corresponding to 6 radians. How far has the ant walked?

27 Find the lengths of both circular arcs of the unit circle connecting the point $(1, 0)$ and the endpoint of the radius corresponding to 3 radians.

28 Find the lengths of both circular arcs of the unit circle connecting the point $(1, 0)$ and the endpoint of the radius corresponding to 4 radians.

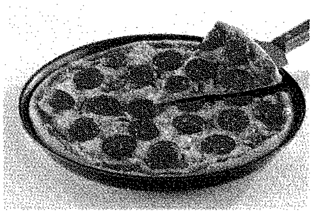
29 Find the lengths of both circular arcs of the unit circle connecting the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and the point whose radius corresponds to 1 radian.

30 Find the lengths of both circular arcs of the unit circle connecting the point $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and the point whose radius corresponds to 2 radians.

31 For a 16-inch pizza, find the area of a slice with angle $\frac{3}{4}$ radians.

32 For a 14-inch pizza, find the area of a slice with angle $\frac{4}{5}$ radians.

33 Suppose a slice of a 12-inch pizza has an area of 20 square inches. What is the angle of this slice?



34 Suppose a slice of a 10-inch pizza has an area of 15 square inches. What is the angle of this slice?

35 Suppose a slice of pizza with an angle of $\frac{5}{6}$ radians has an area of 21 square inches. What is the diameter of this pizza?

36 Suppose a slice of pizza with an angle of 1.1 radians has an area of 25 square inches. What is the diameter of this pizza?

For each of the angles in Exercises 37–42, find the endpoint of the radius of the unit circle that corresponds to the given angle.

37 $\frac{5\pi}{6}$ radians

38 $\frac{7\pi}{6}$ radians

39 $-\frac{\pi}{4}$ radians

40 $-\frac{3\pi}{4}$ radians

41 $\frac{5\pi}{2}$ radians

42 $\frac{11\pi}{2}$ radians

For each of the angles in Exercises 43–46, find the slope of the radius of the unit circle that corresponds to the given angle.

43 $\frac{5\pi}{6}$ radians

44 $\frac{7\pi}{6}$ radians

45 $-\frac{\pi}{4}$ radians

46 $-\frac{3\pi}{4}$ radians

PROBLEMS

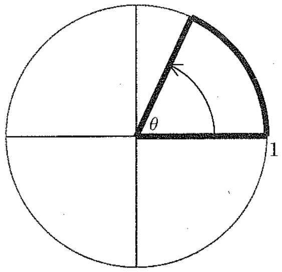
47 Find the formula for the length of a circular arc corresponding to θ radians on a circle of radius r.

48 Most dictionaries define acute angles and obtuse angles in terms of degrees. Restate these definitions in terms of radians.

49 Find a formula for converting from radians to grads. [See the note before Problem 57 in Section 4.1 for the definition of grads.]

50 Find a formula for converting from grads to radians.

51 Suppose the region bounded by the thickened radii and circular arc shown here is removed. Find a formula (in terms of θ) for the perimeter of the remaining region inside the unit circle.



Assume $0 < \theta < 2\pi$.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

In Exercises 1–8, convert each angle to radians.

1 $15°$

SOLUTION Start with the equation

$$360° = 2\pi \text{ radians.}$$

Divide both sides by 360 to obtain

$$1° = \frac{\pi}{180} \text{ radians.}$$

Now multiply both sides by 15, obtaining

$$15° = \frac{15\pi}{180} \text{ radians} = \frac{\pi}{12} \text{ radians.}$$

3 $-45°$

SOLUTION Start with the equation

$$360° = 2\pi \text{ radians.}$$

Divide both sides by 360 to obtain

$$1° = \frac{\pi}{180} \text{ radians.}$$

Now multiply both sides by -45, obtaining

$$-45° = -\frac{45\pi}{180} \text{ radians} = -\frac{\pi}{4} \text{ radians.}$$

5 $270°$

SOLUTION Start with the equation

$$360° = 2\pi \text{ radians.}$$

Divide both sides by 360 to obtain

$$1° = \frac{\pi}{180} \text{ radians.}$$

Now multiply both sides by 270, obtaining

$$270° = \frac{270\pi}{180} \text{ radians} = \frac{3\pi}{2} \text{ radians.}$$

7 $1080°$

SOLUTION Start with the equation

$$360° = 2\pi \text{ radians.}$$

Divide both sides by 360 to obtain

$$1° = \frac{\pi}{180} \text{ radians.}$$

Now multiply both sides by 1080, obtaining

$$1080° = \frac{1080\pi}{180} \text{ radians} = 6\pi \text{ radians.}$$

In Exercises 9–16, convert each angle to degrees.

9 4π radians

SOLUTION Start with the equation

$$2\pi \text{ radians} = 360°.$$

Multiply both sides by 2, obtaining

$$4\pi \text{ radians} = 2 \cdot 360° = 720°.$$

11 $\frac{\pi}{9}$ radians

SOLUTION Start with the equation

$$2\pi \text{ radians} = 360°.$$

Divide both sides by 2 to obtain

$$\pi \text{ radians} = 180°.$$

Now divide both sides by 9, obtaining

$$\frac{\pi}{9} \text{ radians} = \frac{180°}{9} = 20°.$$

13 3 radians

SOLUTION Start with the equation

$$2\pi \text{ radians} = 360°.$$

EXERCISES

Give exact values for the quantities in Exercises 1–10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

1 (a) $\cos(3\pi)$ (b) $\sin(3\pi)$

2 (a) $\cos(-\frac{3\pi}{2})$ (b) $\sin(-\frac{3\pi}{2})$

3 (a) $\cos\frac{11\pi}{4}$ (b) $\sin\frac{11\pi}{4}$

4 (a) $\cos\frac{15\pi}{4}$ (b) $\sin\frac{15\pi}{4}$

5 (a) $\cos\frac{2\pi}{3}$ (b) $\sin\frac{2\pi}{3}$

6 (a) $\cos\frac{4\pi}{3}$ (b) $\sin\frac{4\pi}{3}$

7 (a) $\cos 210°$ (b) $\sin 210°$

8 (a) $\cos 300°$ (b) $\sin 300°$

9 (a) $\cos 360045°$ (b) $\sin 360045°$

10 (a) $\cos(-360030°)$ (b) $\sin(-360030°)$

11 Find the smallest number θ larger than 4π such that $\cos\theta = 0$.

12 Find the smallest number θ larger than 6π such that $\sin\theta = \frac{\sqrt{2}}{2}$.

13 Find the four smallest positive numbers θ such that $\cos\theta = 0$.

14 Find the four smallest positive numbers θ such that $\sin\theta = 0$.

15 Find the four smallest positive numbers θ such that $\sin\theta = 1$.

16 Find the four smallest positive numbers θ such that $\cos\theta = 1$.

17 Find the four smallest positive numbers θ such that $\cos\theta = -1$.

18 Find the four smallest positive numbers θ such that $\sin\theta = -1$.

19 Find the four smallest positive numbers θ such that $\sin\theta = \frac{1}{2}$.

20 Find the four smallest positive numbers θ such that $\cos\theta = \frac{1}{2}$.

21 Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{2}{5}$. Evaluate $\sin\theta$.

22 Suppose $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{3}{7}$. Evaluate $\cos\theta$.

23 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{2}{9}$. Evaluate $\cos\theta$.

24 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{8}$. Evaluate $\cos\theta$.

25 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = 0.1$. Evaluate $\sin\theta$.

26 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = 0.3$. Evaluate $\sin\theta$.

27 Find the smallest number x such that

$$\sin(e^x) = 0.$$

28 Find the smallest number x such that

$$\cos(e^x + 1) = 0.$$

29 Find the smallest positive number x such that

$$\sin(x^2 + x + 4) = 0.$$

30 Find the smallest positive number x such that

$$\cos(x^2 + 2x + 6) = 0.$$

31 Let θ be the acute angle between the positive horizontal axis and the line with slope 3 through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

32 Let θ be the acute angle between the positive horizontal axis and the line with slope 4 through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

PROBLEMS

33 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\cos\theta = \frac{6}{7}$.

(b) Sketch another radius, different from the one in part (a), also illustrating $\cos\theta = \frac{6}{7}$.

34 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\sin\theta = -0.8$.

(b) Sketch another radius, different from the one in part (a), also illustrating $\sin\theta = -0.8$.

35 Find angles u and v such that $\cos u = \cos v$ but $\sin u \neq \sin v$.

36 Find angles u and v such that $\sin u = \sin v$ but $\cos u \neq \cos v$.

37 Show that $\ln(\cos\theta)$ is the average of $\ln(1 - \sin\theta)$ and $\ln(1 + \sin\theta)$ for every θ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

38 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\cos 3.14$. One calculator gives an answer of -0.999999; the other calculator gives an answer of 0.998499. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

39 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\sin 1$. One calculator gives an answer of 0.017452; the other calculator gives an answer of 0.841471. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

40 Suppose m is a real number. Let θ be the acute angle between the positive horizontal axis and the line with slope m through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

41 Explain why there does not exist a real number x such that $2^{\sin x} = \frac{3}{7}$.

42 Explain why $\pi^{\cos x} < 4$ for every real number x.

43 Explain why $\frac{1}{3} < e^{\sin x}$ for every number real number x.

44 Explain why the equation

$$(\sin x)^2 - 4\sin x + 4 = 0$$

has no solutions.

45 Explain why the equation

$$(\cos x)^{99} + 4\cos x - 6 = 0$$

has no solutions.

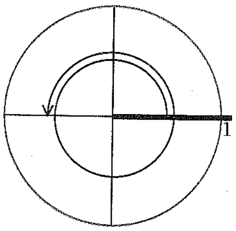
46 Explain why there does not exist a number θ such that $\log\cos\theta = 0.1$.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Give exact values for the quantities in Exercises 1-10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

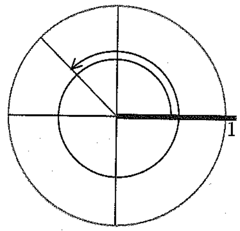
1 (a) $\cos(3\pi)$ (b) $\sin(3\pi)$

SOLUTION Because $3\pi = 2\pi + \pi$, an angle of 3π radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (2π radians) followed by another π radians (180°), as shown below. The endpoint of the corresponding radius is $(-1, 0)$. Thus $\cos(3\pi) = -1$ and $\sin(3\pi) = 0$.



3 (a) $\cos\frac{11\pi}{4}$ (b) $\sin\frac{11\pi}{4}$

SOLUTION Because $\frac{11\pi}{4} = 2\pi + \frac{\pi}{2} + \frac{\pi}{4}$, an angle of $\frac{11\pi}{4}$ radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (2π radians) followed by another $\frac{\pi}{2}$ radians (90°), followed by another $\frac{\pi}{4}$ radians (45°), as shown below. Hence the endpoint of the corresponding radius is $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Thus $\cos\frac{11\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{11\pi}{4} = \frac{\sqrt{2}}{2}$.



5 (a) $\cos\frac{2\pi}{3}$ (b) $\sin\frac{2\pi}{3}$

SOLUTION Because $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$, an angle of $\frac{2\pi}{3}$ radians (as measured counterclockwise from the positive horizontal axis) consists of $\frac{\pi}{2}$ radians (90°) followed by another $\frac{\pi}{6}$ radians (30°), as shown below. The endpoint of the corresponding radius is $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

EXERCISES

1 Find the equation of the line in the xy-plane that goes through the origin and makes an angle of 0.7 radians with the positive x-axis.

2 Find the equation of the line in the xy-plane that goes through the origin and makes an angle of 1.2 radians with the positive x-axis.

3 Find the equation of the line in the xy-plane that contains the point $(3, 2)$ and makes an angle of $41°$ with the positive x-axis.

4 Find the equation of the line in the xy-plane that contains the point $(2, 5)$ and makes an angle of $73°$ with the positive x-axis.

5 Find a number t such that the line through the origin that contains the point $(4, t)$ makes a $22°$ angle with the positive horizontal axis.

6 Find a number w such that the line through the origin that contains the point $(7, w)$ makes a $17°$ angle with the positive horizontal axis.

7 Find the four smallest positive numbers θ such that $\tan\theta = 1$.

8 Find the four smallest positive numbers θ such that $\tan\theta = -1$.

9 Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{1}{5}$. Evaluate:
(a) $\sin\theta$ (b) $\tan\theta$

10 Suppose $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{1}{4}$. Evaluate:
(a) $\cos\theta$ (b) $\tan\theta$

11 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{2}{3}$. Evaluate:
(a) $\cos\theta$ (b) $\tan\theta$

12 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{4}$. Evaluate:
(a) $\cos\theta$ (b) $\tan\theta$

13 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = \frac{4}{5}$. Evaluate:
(a) $\sin\theta$ (b) $\tan\theta$

14 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = \frac{1}{5}$. Evaluate:
(a) $\sin\theta$ (b) $\tan\theta$

15 Suppose $0 < \theta < \frac{\pi}{2}$ and $\tan\theta = \frac{1}{4}$. Evaluate:
(a) $\cos\theta$ (b) $\sin\theta$

16 Suppose $0 < \theta < \frac{\pi}{2}$ and $\tan\theta = \frac{2}{3}$. Evaluate:
(a) $\cos\theta$ (b) $\sin\theta$

17 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\tan\theta = -3$. Evaluate:
(a) $\cos\theta$ (b) $\sin\theta$

18 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\tan\theta = -2$. Evaluate:
(a) $\cos\theta$ (b) $\sin\theta$

Given that

$$\cos 15° = \frac{\sqrt{2+\sqrt{3}}}{2} \quad \textit{and} \quad \sin 22.5° = \frac{\sqrt{2-\sqrt{2}}}{2},$$

in Exercises 19–28 find exact expressions for the indicated quantities.
[These values for $\cos 15°$ and $\sin 22.5°$ will be derived in Examples 3 and 4 in Section 5.5.]

19 $\sin 15°$
20 $\cos 22.5°$
21 $\tan 15°$
22 $\tan 22.5°$
23 $\cot 15°$
24 $\cot 22.5°$
25 $\csc 15°$
26 $\csc 22.5°$
27 $\sec 15°$
28 $\sec 22.5°$

Suppose u and v are in the interval $(0, \frac{\pi}{2})$, with

$$\tan u = 2 \quad \textit{and} \quad \tan v = 3.$$

In Exercises 29–38, find exact expressions for the indicated quantities.

29 $\cot u$
30 $\cot v$
31 $\cos u$
32 $\cos v$
33 $\sin u$
34 $\sin v$
35 $\csc u$
36 $\csc v$
37 $\sec u$
38 $\sec v$

39 Find the smallest number x such that $\tan e^x = 0$.

40 Find the smallest number x such that $\tan e^x$ is undefined.

PROBLEMS

41 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\tan\theta = \frac{1}{7}$.
(b) Sketch another radius, different from the one in part (a), also illustrating $\tan\theta = \frac{1}{7}$.

42 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\tan\theta = 7$.
(b) Sketch another radius, different from the one in part (a), also illustrating $\tan\theta = 7$.

43 Suppose a radius of the unit circle corresponds to an angle whose tangent equals 5, and another radius of the unit circle corresponds to an angle whose tangent equals $-\frac{1}{5}$. Explain why these two radii are perpendicular to each other.

44 Explain why

$$\tan(\theta + \tfrac{\pi}{2}) = -\frac{1}{\tan\theta}$$

for every number θ that is not an integer multiple of $\frac{\pi}{2}$.

45 Explain why the previous problem excluded integer multiples of $\frac{\pi}{2}$ from the allowable values for θ.

46 Find a number θ such that the tangent of θ degrees is larger than 50000.

47 Find a positive number θ such that the tangent of θ degrees is less than -90000.

48 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\tan 89.9$. One calculator replies with an answer of -2.62; the other calculator replies with an answer of 572.96. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

49 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\tan 1$. One calculator replies with an answer of 0.017455; the other calculator replies with an answer of 1.557408. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

50 Explain why

$$|\sin\theta| \le |\tan\theta|$$

for all θ such that $\tan\theta$ is defined.

51 Suppose θ is not an odd multiple of $\frac{\pi}{2}$. Explain why the point $(\tan\theta, 1)$ is on the line containing the point $(\sin\theta, \cos\theta)$ and the origin.

52 Explain why $\log(\cot\theta) = -\log(\tan\theta)$ for every θ in the interval $(0, \frac{\pi}{2})$.

53 In 1768 the Swiss mathematician Johann Lambert proved that if θ is a rational number in the interval $(0, \frac{\pi}{2})$, then $\tan\theta$ is irrational. Use the equation $\tan\frac{\pi}{4} = 1$ to explain why this result implies that π is irrational.
[*This was the first proof that π is irrational.*]

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

1 Find the equation of the line in the xy-plane that goes through the origin and makes an angle of 0.7 radians with the positive x-axis.

SOLUTION A line that makes an angle of 0.7 radians with the positive x-axis has slope $\tan 0.7$. Thus the equation of the line is $y = (\tan 0.7)x$. Because $\tan 0.7 \approx 0.842288$, we could rewrite this as $y \approx 0.842288x$.

3 Find the equation of the line in the xy-plane that contains the point $(3, 2)$ and makes an angle of $41°$ with the positive x-axis.

SOLUTION A line that makes an angle of $41°$ with the positive x-axis has slope $\tan 41°$. Thus the equation of the line is $y - 2 = (\tan 41°)(x - 3)$. Because $\tan 41° \approx 0.869287$, we could rewrite this as $y \approx 0.869287x - 0.60786$.

5 Find a number t such that the line through the origin that contains the point $(4, t)$ makes a $22°$ angle with the positive horizontal axis.

SOLUTION The line through the origin that contains the point $(4, t)$ has slope $\frac{t}{4}$. Thus we want $\tan 22° = \frac{t}{4}$. Hence

$$t = 4\tan 22° \approx 1.6161.$$

7 Find the four smallest positive numbers θ such that $\tan\theta = 1$.

SOLUTION Think of a radius of the unit circle whose endpoint is $(1, 0)$. If this radius moves counterclockwise, forming an angle of θ radians with the positive horizontal axis, then the first and second coordinates of its endpoint first become equal (which is equivalent to having $\tan\theta = 1$) when θ equals $\frac{\pi}{4}$ (which equals $45°$), then again when θ equals $\frac{5\pi}{4}$ (which equals $225°$), then again when θ equals $\frac{9\pi}{4}$ (which equals $360° + 45°$, or $405°$), then again when θ equals $\frac{13\pi}{4}$ (which equals $360° + 225°$, or $585°$), and so on.

Thus the four smallest positive numbers θ such that $\tan\theta = 1$ are $\frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{9\pi}{4}$, and $\frac{13\pi}{4}$.

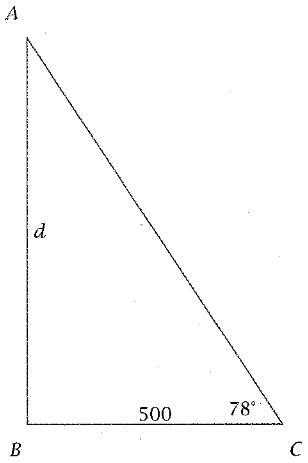
9 Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{1}{5}$. Evaluate:

(a) $\sin\theta$ (b) $\tan\theta$

The next example illustrates another practical application of the ideas in this section.

EXAMPLE 4

A surveyor wishes to measure the distance between points A and B, but a canyon between A and B prevents a direct measurement. Thus the surveyor moves 500 meters perpendicular to the line AB to the point C and measures angle BCA as $78°$ (such angles can be measured with a tool called a *transit level*). What is the distance between the points A and B?



SOLUTION Let d denote the distance from A to B. From the figure (which is not drawn to scale) we have

$$\tan 78° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{d}{500}.$$

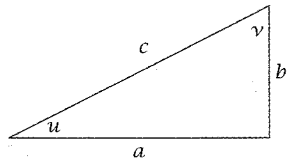
Solving this equation for d, we get

$$d = 500 \tan 78° \approx 2352.$$

Thus the distance between A and B is approximately 2352 meters.

EXERCISES

Use the right triangle below for Exercises 1–20. This triangle is not drawn to scale corresponding to the data in the exercises.



1 Suppose $a = 2$ and $b = 7$. Evaluate

(a) c (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

2 Suppose $a = 3$ and $b = 5$. Evaluate

(a) c (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

3 Suppose $b = 2$ and $c = 7$. Evaluate

(a) a (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

4 Suppose $b = 4$ and $c = 6$. Evaluate

(a) a (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

5 Suppose $a = 5$ and $u = 17°$. Evaluate

(a) b (b) c.

6 Suppose $b = 3$ and $v = 38°$. Evaluate

(a) a (b) c.

7 Suppose $u = 27°$. Evaluate

(a) $\cos v$ (b) $\sin v$ (c) $\tan v$.

8 Suppose $v = 48°$. Evaluate

(a) $\cos u$ (b) $\sin u$ (c) $\tan u$.

9 Suppose $c = 8$ and $u = 1$ radian. Evaluate

(a) a (b) b.

10 Suppose $c = 3$ and $v = 0.2$ radians. Evaluate

(a) a (b) b.

11 Suppose $u = 0.7$ radians. Evaluate

(a) $\cos v$ (b) $\sin v$ (c) $\tan v$.

12 Suppose $v = 0.1$ radians. Evaluate

(a) $\cos u$ (b) $\sin u$ (c) $\tan u$.

13 Suppose $c = 4$ and $\cos u = \frac{3}{5}$. Evaluate

(a) a (b) b.

14 Suppose $c = 5$ and $\cos u = \frac{1}{4}$. Evaluate
(a) a (b) b.

15 Suppose $\cos u = \frac{1}{5}$. Evaluate
(a) $\sin u$ (c) $\cos v$ (e) $\tan v$.
(b) $\tan u$ (d) $\sin v$

16 Suppose $\cos u = \frac{2}{3}$. Evaluate
(a) $\sin u$ (c) $\cos v$ (e) $\tan v$.
(b) $\tan u$ (d) $\sin v$

17 Suppose $b = 4$ and $\sin v = \frac{1}{6}$. Evaluate
(a) a (b) c.

18 Suppose $b = 2$ and $\sin v = \frac{5}{7}$. Evaluate
(a) a (b) c.

19 Suppose $\sin v = \frac{1}{3}$. Evaluate
(a) $\cos u$ (c) $\tan u$ (e) $\tan v$.
(b) $\sin u$ (d) $\cos v$

20 Suppose $\sin v = \frac{3}{7}$. Evaluate
(a) $\cos u$ (c) $\tan u$ (e) $\tan v$.
(b) $\sin u$ (d) $\cos v$

21 Find the perimeter of a right triangle that has hypotenuse of length 6 and a 40° angle.

22 Find the perimeter of a right triangle that has hypotenuse of length 8 and a 35° angle.

23 Suppose a 25-foot ladder is leaning against a wall, making a 63° angle with the ground (measured from a perpendicular line from the base of the ladder to the wall). How high up the wall is the end of the ladder?

24 Suppose a 19-foot ladder is leaning against a wall, making a 71° angle with the ground (measured from a perpendicular line from the base of the ladder to the wall). How high up the wall is the end of the ladder?

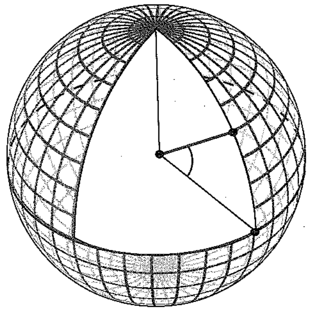
25 Suppose you need to find the height of a tall building. Standing 20 meters from the base of the building, you aim a laser pointer at the closest part of the top of the building. You measure that the laser pointer is 4° tilted from pointing straight up. The laser pointer is held 2 meters above the ground. How tall is the building?

26 Suppose you need to find the height of a tall building. Standing 15 meters from the base of the building, you aim a laser pointer at the closest part of the top of the building. You measure that the laser pointer is 7° tilted from pointing straight up. The laser pointer is held 2 meters above the ground. How tall is the building?

27 A surveyor wishes to measure the distance between points A and B, but buildings between A and B prevent a direct measurement. Thus the surveyor moves 50 meters perpendicular to the line AB to the point C and measures that angle BCA is 87°. What is the distance between the points A and B?

28 A surveyor wishes to measure the distance between points A and B, but a river between A and B prevents a direct measurement. Thus the surveyor moves 200 feet perpendicular to the line AB to the point C and measures that angle BCA is 81°. What is the distance between the points A and B?

For Exercises 29–34, assume the surface of the earth is a sphere with radius 3963 miles. The latitude of a point P on the earth's surface is the angle between the line from the center of the earth to P and the line from the center of the earth to the point on the equator closest to P, as shown below for latitude 40°.



29 Dallas has latitude 32.8° north. Find the radius of the circle formed by the points with the same latitude as Dallas.

30 Cleveland has latitude 41.5° north. Find the radius of the circle formed by the points with the same latitude as Cleveland.

31 Suppose you travel east on the surface of the earth from Dallas (latitude 32.8° north, longitude 96.8° west), always staying at the same latitude as Dallas. You stop when reaching latitude 32.8° north, longitude 84.4° west (directly south of Atlanta). How far have you traveled?

32 Suppose you travel east on the surface of the earth from Cleveland (latitude 41.5° north, longitude 81.7° west), always staying at the same latitude as Cleveland. You stop when reaching latitude 41.5° north, longitude 75.1° west (directly north of Philadelphia). How far have you traveled?

33 How fast is Dallas moving due to the daily rotation of the earth about its axis?

34 How fast is Cleveland moving due to the daily rotation of the earth about its axis?

35 Find the perimeter of an isosceles triangle that has two sides of length 6 and an 80° angle between those two sides.

36 Find the perimeter of an isosceles triangle that has two sides of length 8 and a 130° angle between those two sides.

PROBLEMS

37 In doing several of the exercises in this section, you should have noticed a relationship between $\cos u$ and $\sin v$, along with a relationship between $\sin u$ and $\cos v$. What are these relationships? Explain why they hold.

38 In doing several of the exercises in this section, you should have noticed a relationship between $\tan u$ and $\tan v$. What is this relationship? Explain why it holds.

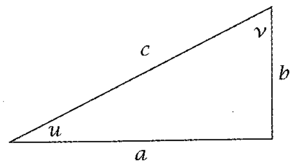
39 Find the lengths of all three sides of a right triangle that has perimeter 29 and has a 42° angle.

40 Find the latitude of your location, then compute how fast you are moving due to the daily rotation of the earth about its axis.

41 Find a formula for the perimeter of an isosceles triangle that has two sides of length c with angle θ between those two sides.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Use the right triangle below for Exercises 1–20. This triangle is not drawn to scale corresponding to the data in the exercises.



1 Suppose $a = 2$ and $b = 7$. Evaluate

(a) c (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

SOLUTION

(a) The Pythagorean Theorem implies that $c^2 = 2^2 + 7^2$. Thus

$$c = \sqrt{2^2 + 7^2} = \sqrt{53}.$$

(b) $\cos u = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{2}{\sqrt{53}} = \dfrac{2\sqrt{53}}{53}$

(c) $\sin u = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{7}{\sqrt{53}} = \dfrac{7\sqrt{53}}{53}$

(d) $\tan u = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{b}{a} = \dfrac{7}{2}$

(e) $\cos v = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{7}{\sqrt{53}} = \dfrac{7\sqrt{53}}{53}$

(f) $\sin v = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{2}{\sqrt{53}} = \dfrac{2\sqrt{53}}{53}$

(g) $\tan v = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b} = \dfrac{2}{7}$

3 Suppose $b = 2$ and $c = 7$. Evaluate

(a) a (b) $\cos u$ (c) $\sin u$ (d) $\tan u$ (e) $\cos v$ (f) $\sin v$ (g) $\tan v$.

SOLUTION

(a) The Pythagorean Theorem implies that $a^2 + 2^2 = 7^2$. Thus

$$a = \sqrt{7^2 - 2^2} = \sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}.$$

(b) $\cos u = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{3\sqrt{5}}{7}$

(c) $\sin u = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{2}{7}$

(d) $\tan u = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{b}{a} = \dfrac{2}{3\sqrt{5}} = \dfrac{2\sqrt{5}}{15}$

(e) $\cos v = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c} = \dfrac{2}{7}$

(f) $\sin v = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c} = \dfrac{3\sqrt{5}}{7}$

(g) $\tan v = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b} = \dfrac{3\sqrt{5}}{2}$

5 Suppose $a = 5$ and $u = 17°$. Evaluate

(a) b (b) c.

SOLUTION

(a) We have

$$\tan 17° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{b}{5}.$$

Solving for b, we get $b = 5 \tan 17° \approx 1.53$.

(b) We have

$$\cos 17° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{5}{c}.$$

Solving for c, we get

$$c = \frac{5}{\cos 17°} \approx 5.23.$$

Similarly, in our trigonometric formulas for $\theta + \pi$, we could replace π by any odd multiple of π. For example, the radius corresponding to $\theta + 5\pi$ is obtained by starting with the radius corresponding to θ and then making two-and-one-half rotations around the circle, ending up with the opposite radius. Thus $\cos(\theta + 5\pi) = -\cos\theta$, $\sin(\theta + 5\pi) = -\sin\theta$, and $\tan(\theta + 5\pi) = \tan\theta$.

The trigonometric identities involving an integer multiple of π can be summarized as follows:

Trigonometric identities with $\theta + n\pi$

$$\cos(\theta + n\pi) = \begin{cases} \cos\theta & \text{if } n \text{ is an even integer} \\ -\cos\theta & \text{if } n \text{ is an odd integer} \end{cases}$$

$$\sin(\theta + n\pi) = \begin{cases} \sin\theta & \text{if } n \text{ is an even integer} \\ -\sin\theta & \text{if } n \text{ is an odd integer} \end{cases}$$

$$\tan(\theta + n\pi) = \tan\theta \quad \text{if } n \text{ is an integer}$$

The first two identities hold for all values of θ. The third identity holds for all values of θ except odd multiples of $\frac{\pi}{2}$; these values are excluded because $\tan(\theta + n\pi)$ and $\tan\theta$ are undefined for such angles.

EXERCISES

The next two exercises emphasize that $\cos^2\theta$ does not equal $\cos(\theta^2)$.

1 For $\theta = 7°$, evaluate each of the following:

(a) $\cos^2\theta$ (b) $\cos(\theta^2)$

2 For $\theta = 5$ radians, evaluate each of the following:

(a) $\cos^2\theta$ (b) $\cos(\theta^2)$

The next two exercises emphasize that $\sin^2\theta$ does not equal $\sin(\theta^2)$.

3 For $\theta = 4$ radians, evaluate each of the following:

(a) $\sin^2\theta$ (b) $\sin(\theta^2)$

4 For $\theta = -8°$, evaluate each of the following:

(a) $\sin^2\theta$ (b) $\sin(\theta^2)$

In Exercises 5–38, find exact expressions for the indicated quantities, given that

$$\cos\tfrac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} \quad \textit{and} \quad \sin\tfrac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

[These values for $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{8}$ will be derived in Examples 3 and 4 in Section 5.5.]

5 $\cos(-\frac{\pi}{12})$
6 $\sin(-\frac{\pi}{8})$
7 $\sin\frac{\pi}{12}$
8 $\cos\frac{\pi}{8}$
9 $\sin(-\frac{\pi}{12})$
10 $\cos(-\frac{\pi}{8})$
11 $\tan\frac{\pi}{12}$
12 $\tan\frac{\pi}{8}$
13 $\tan(-\frac{\pi}{12})$
14 $\tan(-\frac{\pi}{8})$
15 $\cos\frac{25\pi}{12}$
16 $\cos\frac{17\pi}{8}$
17 $\sin\frac{25\pi}{12}$
18 $\sin\frac{17\pi}{8}$
19 $\tan\frac{25\pi}{12}$
20 $\tan\frac{17\pi}{8}$
21 $\cos\frac{13\pi}{12}$
22 $\cos\frac{9\pi}{8}$
23 $\sin\frac{13\pi}{12}$
24 $\sin\frac{9\pi}{8}$
25 $\tan\frac{13\pi}{12}$
26 $\tan\frac{9\pi}{8}$
27 $\cos\frac{5\pi}{12}$
28 $\cos\frac{3\pi}{8}$
29 $\cos(-\frac{5\pi}{12})$
30 $\cos(-\frac{3\pi}{8})$
31 $\sin\frac{5\pi}{12}$
32 $\sin\frac{3\pi}{8}$
33 $\sin(-\frac{5\pi}{12})$
34 $\sin(-\frac{3\pi}{8})$
35 $\tan\frac{5\pi}{12}$
36 $\tan\frac{3\pi}{8}$
37 $\tan(-\frac{5\pi}{12})$
38 $\tan(-\frac{3\pi}{8})$

39 Find the smallest positive number x such that

$$(\cos(x+\pi))(\cos x) + \frac{1}{2} = 0.$$

40 Find the smallest positive number x such that

$$\sin(x+\pi) - \sin x = 1.$$

41 Find the smallest positive number x such that

$$\tan x = 3\tan(\tfrac{\pi}{2} - x).$$

42 Find the smallest positive number x such that

$$(\tan x)(1 + 2\tan(\tfrac{\pi}{2} - x)) = 2 - \sqrt{3}.$$

Suppose u and v are in the interval $(\frac{\pi}{2}, \pi)$, with

$$\tan u = -2 \quad \textit{and} \quad \tan v = -3.$$

In Exercises 43–70, find exact expressions for the indicated quantities.

43 $\tan(-u)$

44 $\tan(-v)$

45 $\cos u$

46 $\cos v$

47 $\cos(-u)$

48 $\cos(-v)$

49 $\sin u$

50 $\sin v$

51 $\sin(-u)$

52 $\sin(-v)$

53 $\cos(u + 4\pi)$

54 $\cos(v - 6\pi)$

55 $\sin(u - 6\pi)$

56 $\sin(v + 10\pi)$

57 $\tan(u + 8\pi)$

58 $\tan(v - 4\pi)$

59 $\cos(u - 3\pi)$

60 $\cos(v + 5\pi)$

61 $\sin(u + 5\pi)$

62 $\sin(v - 7\pi)$

63 $\tan(u - 9\pi)$

64 $\tan(v + 3\pi)$

65 $\cos(\frac{\pi}{2} - u)$

66 $\cos(\frac{\pi}{2} - v)$

67 $\sin(\frac{\pi}{2} - u)$

68 $\sin(\frac{\pi}{2} - v)$

69 $\tan(\frac{\pi}{2} - u)$

70 $\tan(\frac{\pi}{2} - v)$

PROBLEMS

71 Show that

$$(\cos\theta + \sin\theta)^2 = 1 + 2\cos\theta\sin\theta$$

for every number θ.
[*Expressions such as* $\cos\theta\,\sin\theta$ *mean* $(\cos\theta)(\sin\theta)$, *not* $\cos(\theta\sin\theta)$.]

72 Show that

$$\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$$

for every number x that is not an integer multiple of π.

73 (a) Show that

$$x^3 + x^2y + xy^2 + y^3 = (x^2 + y^2)(x + y)$$

for all numbers x and y.

(b) Show that

$$\cos^3\theta + \cos^2\theta\sin\theta + \cos\theta\sin^2\theta + \sin^3\theta = \cos\theta + \sin\theta$$

for every number θ.

74 Show that

$$\cos^4 u + 2\cos^2 u\sin^2 u + \sin^4 u = 1$$

for every number u.

75 Simplify the expression

$$(\tan\theta)\Big(\frac{1}{1 - \cos\theta} - \frac{1}{1 + \cos\theta}\Big).$$

76 Show that

$$\sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta}$$

for all θ except odd multiples of $\frac{\pi}{2}$.

77 Find a formula for $\cos^2\theta$ solely in terms of $\tan^2\theta$.

78 Find a formula for $\tan^2\theta$ solely in terms of $\sin^2\theta$.

79 Explain why $\sin 3° + \sin 357° = 0$.

80 Explain why $\cos 85° + \cos 95° = 0$.

81 Pretend that you are living in the time before calculators and computers existed, and that you have a table showing the cosines and sines of $1°$, $2°$, $3°$, and so on, up to the cosine and sine of $45°$. Explain how you would find the cosine and sine of $71°$, which are beyond the range of your table.

82 Suppose n is an integer. Find formulas for $\sec(\theta + n\pi)$, $\csc(\theta + n\pi)$, and $\cot(\theta + n\pi)$ in terms of $\sec\theta$, $\csc\theta$, and $\cot\theta$.

83 Restate all the results in boxes in the subsection on *Trigonometric Identities Involving a Multiple of* π in terms of degrees instead of in terms of radians.

84 Show that

$$\cos(\pi - \theta) = -\cos\theta$$

for every angle θ.

85 Show that

$$\tan(\theta + \tfrac{\pi}{2}) = -\frac{1}{\tan\theta}$$

for every angle θ that is not an integer multiple of $\frac{\pi}{2}$. Interpret this result in terms of the characterization of the slopes of perpendicular lines.

86 Show that

$$\sin(\pi - \theta) = \sin\theta$$

for every angle θ.

87 Show that

$$\cos(x + \tfrac{\pi}{2}) = -\sin x$$

for every number x.

88 Show that

$$\sin(t + \tfrac{\pi}{2}) = \cos t$$

for every number t.

89 Explain why

$$|\cos(x + n\pi)| = |\cos x|$$

for every number x and every integer n.