EXERCISES

1 What are the coordinates of the unlabeled vertex of the smaller of the two right triangles in the figure at the beginning of this section?

2 What are the coordinates of the unlabeled vertex of the larger of the two right triangles in the figure at the beginning of this section?

3 Find the slope of the line that contains the points $(3,4)$ and $(7,13)$.

4 Find the slope of the line that contains the points $(2,11)$ and $(6,-5)$.

5 Find a number w such that the line containing the points $(1,w)$ and $(3,7)$ has slope 5.

6 Find a number d such that the line containing the points $(d,4)$ and $(-2,9)$ has slope -3.

7 Suppose the tuition per semester at Euphoria State University is \$525 plus \$200 for each unit taken.

(a) What is the tuition for a semester in which a student is taking 10 units ?

(b) Find a linear function t such that $t(u)$ is the tuition in dollars for a semester in which a student is taking u units.

(c) Find the total tuition for a student who takes 12 semesters to accumulate the 120 units needed to graduate.

(d) Find a linear function g such that $g(s)$ is the total tuition for a student who takes s semesters to accumulate the 120 units needed to graduate.

8 Suppose the tuition per semester at Luxim University is \$900 plus \$850 for each unit taken.

(a) What is the tuition for a semester in which a student is taking 15 units ?

(b) Find a linear function t such that $t(u)$ is the tuition in dollars for a semester in which a student is taking u units.

(c) Find the total tuition for a student who takes 8 semesters to accumulate the 120 units needed to graduate.

(d) Find a linear function g such that $g(s)$ is the total tuition for a student who takes s semesters to accumulate the 120 units needed to graduate.

9 Suppose your cell phone company offers two calling plans. The pay-per-call plan charges \$14 per month plus 3 cents for each minute. The unlimited-calling plan charges a flat rate of \$29 per month for unlimited calls.

(a) What is your monthly cost in dollars for making 400 minutes per month of calls on the pay-per-call plan?

(b) Find a linear function c such that $c(m)$ is your monthly cost in dollars for making m minutes of phone calls per month on the pay-per-call plan.

(c) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

10 Suppose your cell phone company offers two calling plans. The pay-per-call plan charges \$11 per month plus 4 cents for each minute. The unlimited-calling plan charges a flat rate of \$25 per month for unlimited calls.

(a) What is your monthly cost in dollars for making 600 minutes per month of calls on the pay-per-call plan?

(b) Find a linear function c such that $c(m)$ is your monthly cost in dollars for making m minutes of phone calls per month on the pay-per-call plan.

(c) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

11 Find the equation of the line in the xy-plane with slope 2 that contains the point $(7,3)$.

12 Find the equation of the line in the xy-plane with slope -4 that contains the point $(-5,-2)$.

13 Find the equation of the line that contains the points $(2,-1)$ and $(4,9)$.

14 Find the equation of the line that contains the points $(-3,2)$ and $(-5,7)$.

15 Find a number t such that the point $(3,t)$ is on the line containing the points $(7,6)$ and $(14,10)$.

16 Find a number t such that the point $(-2,t)$ is on the line containing the points $(5,-2)$ and $(10,-8)$.

17 Find a function s such that $s(d)$ is the number of seconds in d days.

18 Find a function s such that $s(w)$ is the number of seconds in w weeks.

19 Find a function f such that $f(m)$ is the number of inches in m miles.

20 Find a function m such that $m(f)$ is the number of miles in f feet.

The exact conversion between the English measurement system and the metric system is given by the equation 1 inch = 2.54 centimeters.

21 Find a function k such that $k(m)$ is the number of kilometers in m miles.

22 Find a function M such that $M(m)$ is the number of miles in m meters.

23 Find a function f such that $f(c)$ is the number of inches in c centimeters.

24 Find a function m such that $m(f)$ is the number of meters in f feet.

25 Find a number c such that the point $(c, 13)$ is on the line containing the points $(-4, -17)$ and $(6, 33)$.

26 Find a number c such that the point $(c, -19)$ is on the line containing the points $(2, 1)$ and $(4, 9)$.

27 Find a number t such that the point $(t, 2t)$ is on the line containing the points $(3, -7)$ and $(5, -15)$.

28 Find a number t such that the point $(t, \frac{t}{2})$ is on the line containing the points $(2, -4)$ and $(-3, -11)$.

29 Find the equation of the line in the xy-plane that contains the point $(3, 2)$ and that is parallel to the line $y = 4x - 1$.

30 Find the equation of the line in the xy-plane that contains the point $(-4, -5)$ and that is parallel to the line $y = -2x + 3$.

31 Find the equation of the line that contains the point $(2, 3)$ and that is parallel to the line containing the points $(7, 1)$ and $(5, 6)$.

32 Find the equation of the line that contains the point $(-4, 3)$ and that is parallel to the line containing the points $(3, -7)$ and $(6, -9)$.

33 Find a number t such that the line containing the points $(t, 2)$ and $(3, 5)$ is parallel to the line containing the points $(-1, 4)$ and $(-3, -2)$.

34 Find a number t such that the line containing the points $(-3, t)$ and $(2, -4)$ is parallel to the line containing the points $(5, 6)$ and $(-2, 4)$.

35 Find the intersection in the xy-plane of the lines $y = 5x + 3$ and $y = -2x + 1$.

36 Find the intersection in the xy-plane of the lines $y = -4x + 5$ and $y = 5x - 2$.

37 Find a number b such that the three lines in the xy-plane given by the equations $y = 2x + b$, $y = 3x - 5$, and $y = -4x + 6$ have a common intersection point.

38 Find a number m such that the three lines in the xy-plane given by the equations $y = mx + 3$, $y = 4x + 1$, and $y = 5x + 7$ have a common intersection point.

39 Find the equation of the line in the xy-plane that contains the point $(4, 1)$ and that is perpendicular to the line $y = 3x + 5$.

40 Find the equation of the line in the xy-plane that contains the point $(-3, 2)$ and that is perpendicular to the line $y = -5x + 1$.

41 Find a number t such that the line in the xy-plane containing the points $(t, 4)$ and $(2, -1)$ is perpendicular to the line $y = 6x - 7$.

42 Find a number t such that the line in the xy-plane containing the points $(-3, t)$ and $(4, 3)$ is perpendicular to the line $y = -5x + 999$.

43 Find a number t such that the line containing the points $(4, t)$ and $(-1, 6)$ is perpendicular to the line that contains the points $(3, 5)$ and $(1, -2)$.

44 Find a number t such that the line containing the points $(t, -2)$ and $(-3, 5)$ is perpendicular to the line that contains the points $(4, 7)$ and $(1, 11)$.

PROBLEMS

Some problems require considerably more thought than the exercises.

45 Find the equation of the line in the xy-plane that has slope m and intersects the x-axis at $(c, 0)$.

46 Show that the composition of two linear functions is a linear function.

47 Show that if f and g are linear functions, then the graphs of $f \circ g$ and $g \circ f$ have the same slope.

48 Show that a linear function is increasing if and only if the slope of its graph is positive.

49 Show that a linear function is decreasing if and only if the slope of its graph is negative.

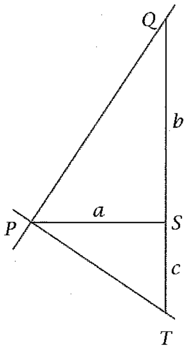
50 Show that every nonconstant linear function is a one-to-one function.

51 Show that if f is the linear function defined by $f(x) = mx + b$, where $m \neq 0$, then the inverse function f^{-1} is defined by the formula $f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}$.

52 Show that the linear function f defined by $f(x) = mx + b$ is an odd function if and only if $b = 0$.

53 Show that the linear function f defined by $f(x) = mx + b$ is an even function if and only if $m = 0$.

54 We used similar triangles to show that the product of the slopes of two perpendicular lines equals -1. The steps below outline an alternative proof that avoids the use of similar triangles but uses more algebra instead. Use the figure below, which is the same as the figure used earlier except that there is now no need to label the angles.



QP is perpendicular to PT.

(a) Apply the Pythagorean Theorem to triangle PSQ to find the length of the line segment PQ in terms of a and b.

(b) Apply the Pythagorean Theorem to triangle PST to find the length of the line segment PT in terms of a and c.

(c) Apply the Pythagorean Theorem to triangle QPT to find the length of the line segment QT in terms of the lengths of the line segments of PQ and PT calculated in the first two parts of this problem.

(d) As can be seen from the figure, the length of the line segment QT equals $b + c$. Thus set the formula for length of the line segment QT, as calculated in the previous part of this problem, equal to $b + c$, and solve the resulting equation for c in terms of a and b.

(e) Use the result in the previous part of this problem to show that the slope of the line containing P and Q times the slope of the line containing P and T equals -1.

55 Suppose a and b are nonzero numbers. Where does the line in the xy-plane given by the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

intersect the coordinate axes?

56 Show that the points $(-84, -14)$, $(21, 1)$, and $(98, 12)$ lie on a line.

57 Show that the points $(-8, -65)$, $(1, 52)$, and $(3, 77)$ do not lie on a line.

58 Change just one of the six numbers in the problem above so that the resulting three points do lie on a line.

59 Show that for every number t, the point $(5-3t, 7-4t)$ is on the line containing the points $(2, 3)$ and $(5, 7)$.

60 Suppose (x_1, y_1) and (x_2, y_2) are the endpoints of a line segment.

(a) Show that the line containing the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_1, y_1) has slope $\frac{y_2-y_1}{x_2-x_1}$.

(b) Show that the line containing the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_2, y_2) has slope $\frac{y_2-y_1}{x_2-x_1}$.

(c) Explain why parts (a) and (b) of this problem imply that the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ lies on the line containing the endpoints (x_1, y_1) and (x_2, y_2).

61 The Kelvin temperature scale is defined by $K = C + 273.15$, where K is the temperature on the Kelvin scale and C is the temperature on the Celsius scale. (Thus -273.15 degrees Celsius, which is the temperature at which all atomic movement ceases and thus is the lowest possible temperature, corresponds to 0 on the Kelvin scale.)

(a) Find a function F such that $F(x)$ equals the temperature on the Fahrenheit scale corresponding to temperature x on the Kelvin scale.

(b) Explain why the graph of the function F from part (a) is parallel to the graph of the function f obtained in Example 5.

Isaac Newton showed that a comet's orbit around a star lies either on an ellipse or on a parabola (rare) or on a hyperbola with the star at one of the foci. For example, if units are chosen so that the orbit of a comet is the upper branch of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$, then the star must be located at $(0, 5)$.

The next result generalizes the example above. The verification of this result is outlined in Problems 105–107. To do this verification, simply use the ideas from the example above.

Formula for the foci of a hyperbola

If a and b are nonzero numbers, then the foci of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are the points

$$(0, -\sqrt{a^2 + b^2}) \quad \text{and} \quad (0, \sqrt{a^2 + b^2}).$$

Sometimes when a new comet is discovered, there are not enough observations to determine whether the comet is in an elliptical orbit or in a hyperbolic orbit. The distinction is important, because a comet in a hyperbolic orbit will disappear and never again be visible from earth.

EXERCISES

For Exercises 1–12, use the following information: If an object is thrown straight up into the air from height H feet at time 0 with initial velocity V feet per second, then at time t seconds the height of the object is $h(t)$ feet, where

$$h(t) = -16.1t^2 + Vt + H.$$

This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

1 Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.

(a) How long before the ball hits the ground?

(b) How long before the ball reaches its maximum height?

(c) What is the ball's maximum height?

2 Suppose a ball is tossed straight up into the air from height 4 feet with initial velocity 40 feet per second.

(a) How long before the ball hits the ground?

(b) How long before the ball reaches its maximum height?

(c) What is the ball's maximum height?

3 Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?

4 Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball stay in the air for 3 seconds?

5 Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach its maximum height after 1 second?

6 Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach its maximum height after 2 seconds?

7 Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach a height of 50 feet?

8 Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach a height of 70 feet?

9 Suppose a notebook computer is accidentally knocked off a shelf that is six feet high. How long before the computer hits the ground?

10 Suppose a notebook computer is accidentally knocked off a desk that is three feet high. How long before the computer hits the ground?

Some notebook computers have a sensor that detects sudden changes in motion and stops the notebook's hard drive, protecting it from damage.

11 Suppose the motion detection/protection mechanism of a notebook computer takes 0.3 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

12 Suppose the motion detection/protection mechanism of a notebook computer takes 0.4 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

13 Find all numbers x such that

$$\frac{x-1}{x+3} = \frac{2x-1}{x+2}.$$

14 Find all numbers x such that

$$\frac{3x+2}{x-2} = \frac{2x-1}{x-1}.$$

15 Find two numbers w such that the points $(3,1)$, $(w,4)$, and $(5,w)$ all lie on a straight line.

16 Find two numbers r such that the points $(-1,4)$, $(r,2r)$, and $(1,r)$ all lie on a straight line.

For Exercises 17–22, find the vertex of the graph of the given function f.

17 $f(x) = 7x^2 - 12$

18 $f(x) = -9x^2 - 5$

19 $f(x) = (x-2)^2 - 3$

20 $f(x) = (x+3)^2 + 4$

21 $f(x) = (2x-5)^2 + 6$

22 $f(x) = (7x+3)^2 + 5$

23 Find the only numbers x and y such that

$$x^2 - 6x + y^2 + 8y = -25.$$

24 Find the only numbers x and y such that

$$x^2 + 5x + y^2 - 3y = -\tfrac{17}{2}.$$

25 Find the point on the line $y = 3x + 1$ in the xy-plane that is closest to the point $(2,4)$.

26 Find the point on the line $y = 2x - 3$ in the xy-plane that is closest to the point $(5,1)$.

27 Find a number t such that the distance between $(2,3)$ and $(t,2t)$ is as small as possible.

28 Find a number t such that the distance between $(-2,1)$ and $(3t,2t)$ is as small as possible.

For Exercises 29–32, for the given function f:

(a) ***Write $f(x)$ in the form $a(x-h)^2 + k$.***

(b) ***Find the value of x where $f(x)$ attains its minimum value or its maximum value.***

(c) ***Sketch the graph of f on an interval of length 2 centered at the number where f attains its minimum or maximum value.***

(d) ***Find the vertex of the graph of f.***

29 $f(x) = x^2 + 7x + 12$

30 $f(x) = 5x^2 + 2x + 1$

31 $f(x) = -2x^2 + 5x - 2$

32 $f(x) = -3x^2 + 5x - 1$

33 Find a number c such that the graph of $y = x^2+6x+c$ has its vertex on the x-axis.

34 Find a number c such that the graph of $y = x^2+5x+c$ in the xy-plane has its vertex on the line $y = x$.

35 Find two numbers whose sum equals 10 and whose product equals 7.

36 Find two numbers whose sum equals 6 and whose product equals 4.

37 Find two positive numbers whose difference equals 3 and whose product equals 20.

38 Find two positive numbers whose difference equals 4 and whose product equals 15.

39 Find the minimum value of $x^2 - 6x + 2$.

40 Find the minimum value of $3x^2 + 5x + 1$.

41 Find the maximum value of $7 - 2x - x^2$.

42 Find the maximum value of $9 + 5x - 4x^2$.

43 Suppose the graph of f is a parabola with vertex at $(3,2)$. Suppose $g(x) = 4x + 5$. What are the coordinates of the vertex of the graph of $f \circ g$?

44 Suppose the graph of f is a parabola with vertex at $(-5,4)$. Suppose $g(x) = 3x - 1$. What are the coordinates of the vertex of the graph of $f \circ g$?

45 Suppose the graph of f is a parabola with vertex at $(3,2)$. Suppose $g(x) = 4x + 5$. What are the coordinates of the vertex of the graph of $g \circ f$?

46 Suppose the graph of f is a parabola with vertex at $(-5,4)$. Suppose $g(x) = 3x - 1$. What are the coordinates of the vertex of the graph of $g \circ f$?

47 Suppose the graph of f is a parabola with vertex at (t,s). Suppose $g(x) = ax + b$, where a and b are numbers with $a \neq 0$. What are the coordinates of the vertex of the graph of $f \circ g$?

48 Suppose the graph of f is a parabola with vertex at (t,s). Suppose $g(x) = ax + b$, where a and b are numbers with $a \neq 0$. What are the coordinates of the vertex of the graph of $g \circ f$?

49 Suppose $h(x) = x^2 + 3x + 4$, with the domain of h being the set of positive numbers. Evaluate $h^{-1}(7)$.

50 Suppose $h(x) = x^2 + 2x - 5$, with the domain of h being the set of positive numbers. Evaluate $h^{-1}(4)$.

51 Suppose f is the function whose domain is the interval $[1, \infty)$ with

$$f(x) = x^2 + 3x + 5.$$

(a) What is the range of f?

(b) Find a formula for f^{-1}.

(c) What is the domain of f^{-1}?

(d) What is the range of f^{-1}?

52 Suppose g is the function whose domain is the interval $[1, \infty)$ with

$$g(x) = x^2 + 4x + 7.$$

(a) What is the range of g?

(b) Find a formula for g^{-1}.

(c) What is the domain of g^{-1}?

(d) What is the range of g^{-1}?

53 Suppose

$$f(x) = x^2 - 6x + 11.$$

Find the smallest number b such that f is increasing on the interval $[b, \infty)$.

54 Suppose

$$f(x) = x^2 + 8x + 5.$$

Find the smallest number b such that f is increasing on the interval $[b, \infty)$.

55 Find the distance between the points $(3, -2)$ and $(-1, 4)$.

56 Find the distance between the points $(-4, -7)$ and $(-8, -5)$.

57 Find two choices for t such that the distance between $(2, -1)$ and $(t, 3)$ equals 7.

58 Find two choices for t such that the distance between $(3, -2)$ and $(1, t)$ equals 5.

59 Find two choices for b such that $(4, b)$ has distance 5 from $(3, 6)$.

60 Find two choices for b such that $(b, -1)$ has distance 4 from $(3, 2)$.

61 Find two points on the horizontal axis whose distance from $(3, 2)$ equals 7.

62 Find two points on the horizontal axis whose distance from $(1, 4)$ equals 6.

63 Find two points on the vertical axis whose distance from $(5, -1)$ equals 8.

64 Find two points on the vertical axis whose distance from $(2, -4)$ equals 5.

65 A ship sails north for 2 miles and then west for 5 miles. How far is the ship from its starting point?

66 A ship sails east for 7 miles and then south for 3 miles. How far is the ship from its starting point?

67 Find the equation of the circle in the xy-plane centered at $(3, -2)$ with radius 7.

68 Find the equation of the circle in the xy-plane centered at $(-4, 5)$ with radius 6.

69 Find two choices for b such that $(5, b)$ is on the circle with radius 4 centered at $(3, 6)$.

70 Find two choices for b such that $(b, 4)$ is on the circle with radius 3 centered at $(-1, 6)$.

71 Find the center and radius of the circle

$$x^2 - 8x + y^2 + 2y = -14.$$

72 Find the center and radius of the circle

$$x^2 + 5x + y^2 - 6y = 3.$$

73 Find the two points where the circle of radius 2 centered at the origin intersects the circle of radius 3 centered at $(3, 0)$.

74 Find the two points where the circle of radius 3 centered at the origin intersects the circle of radius 4 centered at $(5, 0)$.

75 Suppose units are chosen so that the orbit of a planet around a star is the ellipse

$$4x^2 + 5y^2 = 1.$$

What is the location of the star? (Assume that the first coordinate of the star is positive.)

76 Suppose units are chosen so that the orbit of a planet around a star is the ellipse

$$3x^2 + 7y^2 = 1.$$

What is the location of the star? (Assume that the first coordinate of the star is positive.)

77 Suppose units are chosen so that the orbit of a comet around a star is the upper branch of the hyperbola

$$3y^2 - 2x^2 = 5.$$

What is the location of the star?

78 Suppose units are chosen so that the orbit of a comet around a star is the upper branch of the hyperbola

$$4y^2 - 5x^2 = 7.$$

What is the location of the star?

PROBLEMS

79 Show that
$$(a+b)^2 = a^2 + b^2$$
if and only if $a = 0$ or $b = 0$.

80 Explain why
$$x^2 + 4x + y^2 - 10y \ge -29$$
for all real numbers x and y.

81 Show that a quadratic function f defined by $f(x) = ax^2 + bx + c$ is an even function if and only if $b = 0$.

82 Show that if f is a nonconstant linear function and g is a quadratic function, then $f \circ g$ and $g \circ f$ are both quadratic functions.

83 Suppose
$$2x^2 + 3x + c > 0$$
for every real number x. Show that $c > \frac{9}{8}$.

84 Suppose
$$3x^2 + bx + 7 > 0$$
for every real number x. Show that $|b| < 2\sqrt{21}$.

85 Suppose
$$at^2 + 5t + 4 > 0$$
for every real number t. Show that $a > \frac{25}{16}$.

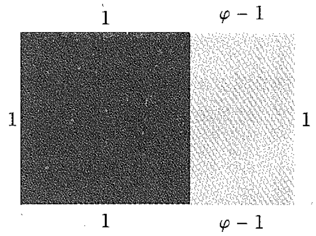
86 Suppose $a \ne 0$ and $b^2 \ge 4ac$. Verify by direct substitution that if
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
then $ax^2 + bx + c = 0$.

87 Suppose $a \ne 0$ and $b^2 \ge 4ac$. Verify by direct calculation that
$$ax^2 + bx + c =$$
$$a\Big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\Big)\Big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\Big).$$

88 Suppose $f(x) = ax^2 + bx + c$, where $a \ne 0$. Show that the vertex of the graph of f is the point $\big(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\big)$.

89 Suppose b and c are numbers such that the equation
$$x^2 + bx + c = 0$$
has no real solutions. Explain why the equation
$$x^2 + bx - c = 0$$
has two real solutions.

90 Find a number φ such that in the figure below, the yellow rectangle is similar to the large rectangle formed by the union of the blue square and the yellow rectangle.



[*The number φ that solves this problem is called the* **golden ratio** (*the symbol φ is the Greek letter phi*). *Rectangles whose ratio between the length of the long side and the length of the short side equals φ are supposedly the most aesthetically pleasing rectangles. The large rectangle formed by the union of the blue square and the yellow rectangle has the golden ratio, as does the yellow rectangle. Many works of art feature rectangles with the golden ratio.*]

91 Show that there do not exist two real numbers whose sum is 7 and whose product is 13.

92 Suppose f is a quadratic function such that the equation $f(x) = 0$ has exactly one solution. Show that this solution is the first coordinate of the vertex of the graph of f and that the second coordinate of the vertex equals 0.

93 Suppose f is a quadratic function such that the equation $f(x) = 0$ has two real solutions. Show that the average of these two solutions is the first coordinate of the vertex of the graph of f.

94 Find two points, one on the horizontal axis and one on the vertical axis, such that the distance between these two points equals 15.

95 Explain why there does not exist a point on the horizontal axis whose distance from $(5, 4)$ equals 3.

96 Find the distance between the points $(-21, -15)$ and $(17, 28)$.
[*In WolframAlpha, you can do this by typing* `distance from (-21, -15) to (17, 28)` *in an entry box. Note that in addition to the distance in both exact and approximate form, you get a figure showing the two points. Experiment with finding the distance between other pairs of points.*]

97 Find six distinct points whose distance from the origin equals 3.

98 Find six distinct points whose distance from $(3, 1)$ equals 4.

99 Suppose $a > b > 0$. Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(\sqrt{a^2 - b^2}, 0)$.

100 Suppose $a > b > 0$. Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(-\sqrt{a^2 - b^2}, 0)$.

101 Suppose $a > b > 0$. Use the results of the two previous problems to show that $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ are foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

102 Suppose $b > a > 0$. Find a formula in terms of y for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, \sqrt{b^2 - a^2})$.

103 Suppose $b > a > 0$. Find a formula in terms of y for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, -\sqrt{b^2 - a^2})$.

104 Suppose $b > a > 0$. Use the results of the two previous problems to show that $(0, \sqrt{b^2 - a^2})$ and $(0, -\sqrt{b^2 - a^2})$ are foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

105 Suppose a and b are nonzero numbers. Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ to the point $(0, -\sqrt{a^2 + b^2})$.

106 Suppose a and b are nonzero numbers. Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ to the point $(0, \sqrt{a^2 + b^2})$.

107 Suppose a and b are nonzero numbers. Use the results of the two previous problems to show that $(0, -\sqrt{a^2 + b^2})$ and $(0, \sqrt{a^2 + b^2})$ are foci of the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

108 Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to the point $(-\sqrt{2}, -\sqrt{2})$ is $x + \frac{1}{x} + \sqrt{2}$.
[*See Example 6 in Section 2.3 for a graph of* $y = \frac{1}{x}$.]

109 Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to the point $(\sqrt{2}, \sqrt{2})$ is $x + \frac{1}{x} - \sqrt{2}$.

110 Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to $(-\sqrt{2}, -\sqrt{2})$ minus the distance from $(x, \frac{1}{x})$ to $(\sqrt{2}, \sqrt{2})$ equals $2\sqrt{2}$.

111 Explain why the result of the previous problem justifies calling the curve $y = \frac{1}{x}$ a hyperbola with foci at $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–12, use the following information: If an object is thrown straight up into the air from height H feet at time 0 with initial velocity V feet per second, then at time t seconds the height of the object is h(t) feet, where

$$h(t) = -16.1t^2 + Vt + H.$$

This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

1 Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.

(a) How long before the ball hits the ground?

(b) How long before the ball reaches its maximum height?

(c) What is the ball's maximum height?

SOLUTION

(a) Here we have

$$h(t) = -16.1t^2 + 20t + 5.$$

The ball hits the ground when $h(t) = 0$; the quadratic formula shows that this happens when $t \approx 1.46$ seconds (the other solution produced by the quadratic formula has been discarded because it is negative).

(b) Completing the square, we have

$$\begin{aligned} h(t) &= -16.1t^2 + 20t + 5 \\ &= -16.1\Big[t^2 - \frac{20}{16.1}t\Big] + 5 \\ &= -16.1\Big[\Big(t - \frac{10}{16.1}\Big)^2 - \Big(\frac{10}{16.1}\Big)^2\Big] + 5 \\ &= -16.1\Big(t - \frac{10}{16.1}\Big)^2 + \frac{100}{16.1} + 5. \end{aligned}$$

Thus the ball reaches its maximum height when $t = \frac{10}{16.1} \approx 0.62$ seconds.

(c) The solution to part (b) shows that the maximum height of the ball is $\frac{100}{16.1} + 5 \approx 11.2$ feet.

3 Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?

EXERCISES

For Exercises 1–6, evaluate the given expression. Do not use a calculator.

1 $2^5 - 5^2$

2 $4^3 - 3^4$

3 $\dfrac{3^{-2}}{2^{-3}}$

4 $\dfrac{2^{-6}}{6^{-2}}$

5 $(\frac{2}{3})^{-4}$

6 $(\frac{5}{4})^{-3}$

The numbers in Exercises 7–14 are too large to be handled by a calculator. These exercises require an understanding of the concepts.

7 Write 9^{3000} as a power of 3.

8 Write 27^{4000} as a power of 3.

9 Write 5^{4000} as a power of 25.

10 Write 2^{3000} as a power of 8.

11 Write $2^5 \cdot 8^{1000}$ as a power of 2.

12 Write $5^3 \cdot 25^{2000}$ as a power of 5.

13 Write $2^{100} \cdot 4^{200} \cdot 8^{300}$ as a power of 2.

14 Write $3^{500} \cdot 9^{200} \cdot 27^{100}$ as a power of 3.

For Exercises 15–20, simplify the given expression by writing it as a power of a single variable.

15 $x^5(x^2)^3$

16 $y^4(y^3)^5$

17 $y^4(y^2(y^5)^2)^{3/5}$

18 $x(x^4(x^3)^2)^{5/3}$

19 $t^4(t^3(t^{-2})^5)^4$

20 $w^3(w^4(w^{-3})^6)^2$

21 Write $\dfrac{8^{1000}}{2^5}$ as a power of 2.

22 Write $\dfrac{25^{2000}}{5^3}$ as a power of 5.

23 Find integers m and n such that $2^m \cdot 5^n = 16000$.

24 Find integers m and n such that $2^m \cdot 5^n = 0.0032$.

For Exercises 25–32, simplify the given expression.

25 $\dfrac{(x^2)^3 y^8}{x^5(y^4)^3}$

26 $\dfrac{x^{11}(y^3)^2}{(x^3)^5(y^2)^4}$

27 $\dfrac{(x^{-2})^3 y^8}{x^{-5}(y^4)^{-3}}$

28 $\dfrac{x^{-11}(y^3)^{-2}}{(x^{-3})^5(y^2)^4}$

29 $\dfrac{(x^2y^{4/5})^3}{(x^5y^2)^{-4}}$

30 $\dfrac{(x^4y^{3/4})^{-3}}{(x^5y^{-2})^4}$

31 $\left(\dfrac{(x^2y^{-5})^{-4}}{(x^5y^{-2})^{-3}}\right)^2$

32 $\left(\dfrac{(x^{-3}y^5)^{-4}}{(x^{-5}y^{-2})^{-3}}\right)^{-2}$

For Exercises 33–44, find a formula for $f \circ g$ given the indicated functions f and g.

33 $f(x) = x^2$, $g(x) = x^3$

34 $f(x) = x^5$, $g(x) = x^4$

35 $f(x) = 4x^2$, $g(x) = 5x^3$

36 $f(x) = 3x^5$, $g(x) = 2x^4$

37 $f(x) = 4x^{-2}$, $g(x) = 5x^3$

38 $f(x) = 3x^{-5}$, $g(x) = 2x^4$

39 $f(x) = 4x^{-2}$, $g(x) = -5x^{-3}$

40 $f(x) = 3x^{-5}$, $g(x) = -2x^{-4}$

41 $f(x) = x^{1/2}$, $g(x) = x^{3/7}$

42 $f(x) = x^{5/3}$, $g(x) = x^{4/9}$

43 $f(x) = 3 + x^{5/4}$, $g(x) = x^{2/7}$

44 $f(x) = x^{2/3} - 7$, $g(x) = x^{9/16}$

For Exercises 45–56, expand the expression.

45 $(2+\sqrt{3})^2$

46 $(3+\sqrt{2})^2$

47 $(2-3\sqrt{5})^2$

48 $(3-5\sqrt{2})^2$

49 $(2+\sqrt{3})^4$

50 $(3+\sqrt{2})^4$

51 $(3+\sqrt{x})^2$

52 $(5+\sqrt{x})^2$

53 $(3-\sqrt{2x})^2$

54 $(5-\sqrt{3x})^2$

55 $(1+2\sqrt{3x})^2$

56 $(3+2\sqrt{5x})^2$

For Exercises 57–64, find all real numbers x that satisfy the indicated equation.

57 $x - 5\sqrt{x} + 6 = 0$

58 $x - 7\sqrt{x} + 12 = 0$

59 $x - \sqrt{x} = 6$

60 $x - \sqrt{x} = 12$

61 $x^{2/3} - 6x^{1/3} = -8$

62 $x^{2/3} + 3x^{1/3} = 10$

63 $x^4 - 3x^2 = 10$

64 $x^4 - 8x^2 = -15$

65 Evaluate 3^{-2x} if x is a number such that $3^x = 4$.

66 Evaluate 2^{-4x} if x is a number such that $2^x = \frac{1}{3}$.

67 Evaluate 8^x if x is a number such that $2^x = 5$.

68 Evaluate $(\frac{1}{9})^x$ if x is a number such that $3^x = 5$.

For Exercises 69–78, sketch the graph of the given function f on the interval $[-1.3, 1.3]$.

69 $f(x) = x^3 + 1$

70 $f(x) = x^4 + 2$

71 $f(x) = x^4 - 1.5$

72 $f(x) = x^3 - 0.5$

73 $f(x) = 2x^3$

74 $f(x) = 3x^4$

75 $f(x) = -2x^4$

76 $f(x) = -3x^3$

77 $f(x) = -2x^4 + 3$

78 $f(x) = -3x^3 + 4$

For Exercises 79–86, evaluate the indicated quantities. Do not use a calculator because otherwise you will not gain the understanding that these exercises should help you attain.

79 $25^{3/2}$

80 $8^{5/3}$

81 $32^{3/5}$

82 $81^{3/4}$

83 $32^{-4/5}$

84 $8^{-5/3}$

85 $(-8)^{7/3}$

86 $(-27)^{4/3}$

For Exercises 87–98, find a formula for the inverse function f^{-1} of the indicated function f.

87 $f(x) = x^9$

88 $f(x) = x^{12}$

89 $f(x) = x^{1/7}$

90 $f(x) = x^{1/11}$

91 $f(x) = x^{-2/5}$

92 $f(x) = x^{-17/7}$

93 $f(x) = \frac{x^4}{81}$

94 $f(x) = 32x^5$

95 $f(x) = 6 + x^3$

96 $f(x) = x^6 - 5$

97 $f(x) = 4x^{3/7} - 1$

98 $f(x) = 7 + 8x^{5/9}$

For Exercises 99–108, sketch the graph of the given function f on the domain $[-3, -\frac{1}{3}] \cup [\frac{1}{3}, 3]$.

99 $f(x) = \frac{1}{x} + 1$

100 $f(x) = \frac{1}{x^2} + 2$

101 $f(x) = \frac{1}{x^2} - 2$

102 $f(x) = \frac{1}{x} - 3$

103 $f(x) = \frac{2}{x}$

104 $f(x) = \frac{3}{x^2}$

105 $f(x) = -\frac{2}{x^2}$

106 $f(x) = -\frac{3}{x}$

107 $f(x) = -\frac{2}{x^2} + 3$

108 $f(x) = -\frac{3}{x} + 4$

109 Find an integer m such that

$$\big((3 + 2\sqrt{5})^2 - m\big)^2$$

is an integer.

110 Find an integer m such that

$$\big((5 - 2\sqrt{3})^2 - m\big)^2$$

is an integer.

PROBLEMS

111 Suppose m is a positive integer. Explain why 10^m, when written out in the usual decimal notation, is the digit 1 followed by m 0's.

112 Suppose m is an odd integer. Show that the function f defined by $f(x) = x^m$ is an odd function.

113 Suppose m is an even integer. Show that the function f defined by $f(x) = x^m$ is an even function.

114 (a) Verify that $(2^2)^2 = 2^{(2^2)}$.

(b) Show that if m is an integer greater than 2, then

$$(m^m)^m \neq m^{(m^m)}.$$

115 What is the domain of the function $(3 + x)^{1/4}$?

116 What is the domain of the function $(1 + x^2)^{1/8}$?

117 Suppose p and q are rational numbers. Define functions f and g by $f(x) = x^p$ and $g(x) = x^q$. Explain why

$$(f \circ g)(x) = x^{pq}.$$

118 Suppose x is a real number and m, n, and p are positive integers. Explain why

$$x^{m+n+p} = x^m x^n x^p.$$

119 Suppose x is a real number and m, n, and p are positive integers. Explain why

$$((x^m)^n)^p = x^{mnp}.$$

120 Suppose x, y, and z are real numbers and m is a positive integer. Explain why

$$x^m y^m z^m = (xyz)^m.$$

121 Show that if $x \neq 0$, then

$$|x^n| = |x|^n$$

for all integers n.

122 Sketch the graph of the functions $\sqrt{x} + 1$ and $\sqrt{x + 1}$ on the interval $[0, 4]$.

123 Explain why the spoken phrase "the square root of x plus one" could be interpreted in two different ways that would not give the same result.

124 Sketch the graph of the functions $2x^{1/3}$ and $(2x)^{1/3}$ on the interval $[0, 8]$.

125 Sketch the graphs of the functions $x^{1/4}$ and $x^{1/5}$ on the interval $[0, 81]$.

126 Show that $\sqrt{5} \cdot 5^{3/2} = 25$.

127 Show that $\sqrt{2}^3 \sqrt{8}^3 = 64$.

128 Show that $3^{3/2} 12^{3/2} = 216$.

129 Show that $\sqrt{2 + \sqrt{3}} = \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}$.

130 Show that $\sqrt{2 - \sqrt{3}} = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}$.

131 Show that $\sqrt{9 - 4\sqrt{5}} = \sqrt{5} - 2$.

132 Show that $(23 - 8\sqrt{7})^{1/2} = 4 - \sqrt{7}$.

133 Make up a problem similar in form to the problem above, without duplicating anything in this book.

134 Show that $(99 + 70\sqrt{2})^{1/3} = 3 + 2\sqrt{2}$.

135 Show that $(-37 + 30\sqrt{3})^{1/3} = -1 + 2\sqrt{3}$.

136 Show that if x and y are positive numbers with $x \neq y$, then

$$\frac{x - y}{\sqrt{x} - \sqrt{y}} = \sqrt{x} + \sqrt{y}.$$

137 Explain why

$$10^{100}(\sqrt{10^{200} + 1} - 10^{100})$$

is approximately $\frac{1}{2}$.

138 Explain why the equation $\sqrt{x^2} = x$ is not valid for all real numbers x and should be replaced by the equation $\sqrt{x^2} = |x|$.

139 Explain why the equation $\sqrt{x^8} = x^4$ is valid for all real numbers x, with no necessity for using absolute value.

140 Show that if x and y are positive numbers, then

$$\sqrt{x + y} < \sqrt{x} + \sqrt{y}.$$

[*In particular, if x and y are positive numbers, then* $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$.]

141 Show that if $0 < x < y$, then

$$\sqrt{y} - \sqrt{x} < \sqrt{y - x}.$$

142 Explain why

$$\sqrt{x} < \sqrt[3]{x} \quad \text{if} \quad 0 < x < 1$$

and

$$\sqrt{x} > \sqrt[3]{x} \quad \text{if} \quad x > 1.$$

Sketch the graphs of the functions $\sqrt{x}$ and $\sqrt[3]{x}$ on the interval $[0, 4]$.

143 Using the result that $\sqrt{2}$ is irrational (proved in Section 0.1), show that $2^{5/2}$ is irrational.

144 Using the result that $\sqrt{2}$ is irrational, explain why $2^{1/6}$ is irrational.

145 Give an example of three irrational numbers x, y, and z such that xyz is a rational number.

146 Suppose you have a calculator that can only compute square roots. Explain how you could use this calculator to compute $7^{1/8}$.

147 Suppose you have a calculator that can only compute square roots and can multiply. Explain how you could use this calculator to compute $7^{3/4}$.

Fermat's Last Theorem states that if n is an integer greater than 2, then there do not exist positive integers x, y, and z such that

$$x^n + y^n = z^n.$$

Fermat's Last Theorem was not proved until 1994, although mathematicians had been trying to find a proof for centuries.

148 Use Fermat's Last Theorem to show that if n is an integer greater than 2, then there do not exist positive rational numbers x and y such that

$$x^n + y^n = 1.$$

[*Hint:* Use proof by contradiction: Assume there exist rational numbers $x = \frac{m}{p}$ and $y = \frac{q}{r}$ such that $x^n + y^n = 1$; then show that this assumption leads to a contradiction of Fermat's Last Theorem.]

149 Use Fermat's Last Theorem to show that if n is an integer greater than 2, then there do not exist positive rational numbers x, y, and z such that

$$x^n + y^n = z^n.$$

[*The equation $3^2 + 4^2 = 5^2$ shows the necessity of the hypothesis that $n > 2$.*]

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1-6, evaluate the given expression. Do not use a calculator.

1 $2^5 - 5^2$

SOLUTION $2^5 - 5^2 = 32 - 25 = 7$

3 $\dfrac{3^{-2}}{2^{-3}}$

SOLUTION $\dfrac{3^{-2}}{2^{-3}} = \dfrac{2^3}{3^2} = \dfrac{8}{9}$

5 $\left(\frac{2}{3}\right)^{-4}$

SOLUTION $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16}$

The numbers in Exercises 7-14 are too large to be handled by a calculator. These exercises require an understanding of the concepts.

7 Write 9^{3000} as a power of 3.

Graphs of Polynomials

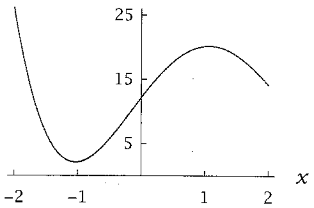
Computers can draw graphs of polynomials better than humans. However, some human thought is usually needed to select an appropriate interval on which to graph a polynomial.

EXAMPLE 8

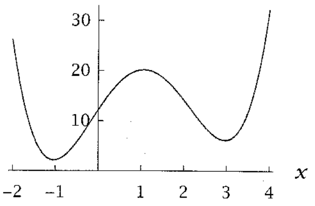
Let p be the polynomial defined by

$$p(x) = x^4 - 4x^3 - 2x^2 + 13x + 12.$$

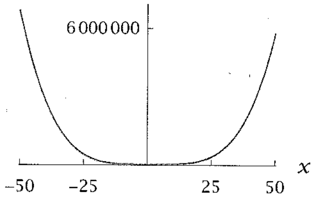
Find an interval that does a good job of illustrating the key features of the graph of p.



The graph of p on the interval $[-2, 2]$.



The graph of p on the interval $[-2, 4]$.



The graph of p on the interval $[-50, 50]$.

SOLUTION If we ask a computer to graph this polynomial on the interval $[-2, 2]$, we obtain the the first graph shown here. Because $p(x)$ behaves like x^4 for very large values of x, this first graph does not depict enough features of p.

Often a bit of experimentation is needed to find an appropriate interval to illustrate the key features of the graph. For this polynomial p, the interval $[-2, 4]$ works well, as shown in the second graph here.

The second graph here shows the graph of p beginning to look like the graph of x^4 when $|x|$ is large. Thus the interval $[-2, 4]$ provides a more complete representation of the behavior of p than does $[-2, 2]$, which was the first interval we used.

We also now see that the graph of p above contains three points that might be thought of as either the top of a peak (at $x \approx 1$) or the bottom of a valley (at $x \approx -1$ and $x \approx 3$).

To search for additional behavior of p, we might try graphing p on a much larger interval, as shown in the third graph here.

The third graph shows no peaks or valleys, even though we know that it contains a total of at least three peaks and valleys. The scale needed to display the graph on the interval $[-50, 50]$ made the peaks and valleys so small that we cannot see them. Thus using this large interval hid some key features that were visible when we used the interval $[-2, 4]$.

Conclusion: A good choice for graphing this function is the interval $[-2, 4]$.

EXERCISES

Suppose

$$p(x) = x^2 + 5x + 2,$$
$$q(x) = 2x^3 - 3x + 1, \quad s(x) = 4x^3 - 2.$$

In Exercises 1–18, write the indicated expression as a polynomial.

1 $(p + q)(x)$

2 $(p - q)(x)$

3 $(3p - 2q)(x)$

4 $(4p + 5q)(x)$

5 $(pq)(x)$

6 $(ps)(x)$

7 $(p(x))^2$

8 $(q(x))^2$

9 $(p(x))^2 s(x)$

10 $(q(x))^2 s(x)$

11 $(p \circ q)(x)$

12 $(q \circ p)(x)$

13 $(p \circ s)(x)$

14 $(s \circ p)(x)$

15 $(q \circ (p + s))(x)$

16 $((q + p) \circ s)(x)$

17 $\dfrac{q(2 + x) - q(2)}{x}$

18 $\dfrac{s(1 + x) - s(1)}{x}$

19 Factor $x^8 - y^8$ as nicely as possible.

20 Factor $x^{16} - y^8$ as nicely as possible.

21 Find all real numbers x such that

$$x^6 - 8x^3 + 15 = 0.$$

22 Find all real numbers x such that

$$x^6 - 3x^3 - 10 = 0.$$

23 Find all real numbers x such that

$$x^4 - 2x^2 - 15 = 0.$$

24 Find all real numbers x such that

$$x^4 + 5x^2 - 14 = 0.$$

25 Find a number b such that 3 is a zero of the polynomial p defined by

$$p(x) = 1 - 4x + bx^2 + 2x^3.$$

26 Find a number c such that -2 is a zero of the polynomial p defined by

$$p(x) = 5 - 3x + 4x^2 + cx^3.$$

27 Find a polynomial p of degree 3 such that -1, 2, and 3 are zeros of p and $p(0) = 1$.

28 Find a polynomial p of degree 3 such that -2, -1, and 4 are zeros of p and $p(1) = 2$.

29 Find all choices of b, c, and d such that 1 and 4 are the only zeros of the polynomial p defined by

$$p(x) = x^3 + bx^2 + cx + d.$$

30 Find all choices of b, c, and d such that -3 and 2 are the only zeros of the polynomial p defined by

$$p(x) = x^3 + bx^2 + cx + d.$$

PROBLEMS

31 Give an example of two polynomials of degree 4 whose sum has degree 3.

32 Find a polynomial p of degree 2 with integer coefficients such that 2.1 and 4.1 are zeros of p.

33 Find a polynomial p with integer coefficients such that $2^{3/5}$ is a zero of p.

34 Show that if p and q are nonzero polynomials with $\deg p < \deg q$, then $\deg(p + q) = \deg q$.

35 Give an example of polynomials p and q such that $\deg(pq) = 8$ and $\deg(p + q) = 5$.

36 Give an example of polynomials p and q such that $\deg(pq) = 8$ and $\deg(p + q) = 2$.

37 Suppose $q(x) = 2x^3 - 3x + 1$.

(a) Show that the point $(2, 11)$ is on the graph of q.

(b) Show that the slope of a line containing $(2, 11)$ and a point on the graph of q very close to $(2, 11)$ is approximately 21.

[*Hint:* Use the result of Exercise 17.]

38 Suppose $s(x) = 4x^3 - 2$.

(a) Show that the point $(1, 2)$ is on the graph of s.

(b) Give an estimate for the slope of a line containing $(1, 2)$ and a point on the graph of s very close to $(1, 2)$.

[*Hint:* Use the result of Exercise 18.]

39 Give an example of polynomials p and q of degree 3 such that $p(1) = q(1)$, $p(2) = q(2)$, and $p(3) = q(3)$, but $p(4) \neq q(4)$.

40 Suppose p and q are polynomials of degree 3 such that $p(1) = q(1)$, $p(2) = q(2)$, $p(3) = q(3)$, and $p(4) = q(4)$. Explain why $p = q$.

41 Explain why the polynomial p defined by

$$p(x) = x^6 + 7x^5 - 2x - 3$$

has a zero in the interval $(0, 1)$.

For Problems 42–43, let p be the polynomial defined by

$$p(x) = x^6 - 87x^4 - 92x + 2.$$

42 (a) Use a computer or calculator to sketch a graph of p on the interval $[-5, 5]$.

(b) Is $p(x)$ positive or negative for x near ∞?

(c) Is $p(x)$ positive or negative for x near $-\infty$?

(d) Explain why the graph from part (a) does not accurately show the behavior of $p(x)$ for large values of x.

43 (a) Evaluate $p(-2)$, $p(-1)$, $p(0)$, and $p(1)$.

(b) Explain why the results from part (a) imply that p has a zero in the interval $(-2, -1)$ and p has a zero in the interval $(0, 1)$.

(c) Show that p has at least four zeros in the interval $[-10, 10]$.
[*Hint:* We already know from part (*b*) that p has at least two zeros is the interval $[-10, 10]$. You can show the existence of other zeros by finding integers n such that one of the numbers $p(n)$, $p(n + 1)$ is positive and the other is negative.]

44 A new snack shop on campus finds that the number of students following it on *Twitter* at the end of each of its first five weeks in business is 23, 89, 223, 419, and 647. A clever employee discovers that the number of students following the new snack shop on *Twitter* after w weeks is $p(w)$, where p is defined by

$$p(w) = 7 + 3w + 5w^2 + 9w^3 - w^4.$$

Indeed, with p defined as above, we have $p(1) = 23$, $p(2) = 89$, $p(3) = 223$, $p(4) = 419$, and $p(5) = 647$. Explain why the polynomial p defined above cannot give accurate predictions for the number of followers on *Twitter* for weeks far into the future.

45 A textbook states that the rabbit population on a small island is observed to be

$$1000 + 120t - 0.4t^4,$$

where t is the time in months since observations of the island began. Explain why the formula above cannot correctly give the number of rabbits on the island for large values of t.

46 Verify that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

47 Verify that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

48 Verify that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

49 Verify that $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

50 Write the polynomial $x^4 + 16$ as the product of two polynomials of degree 2.
[*Hint:* Use the result from the previous problem with x replaced by $\frac{x}{2}$.]

51 Show that

$$(a + b)^3 = a^3 + b^3$$

if and only if $a = 0$ or $b = 0$ or $a = -b$.

52 Suppose d is a real number. Show that

$$(d + 1)^4 = d^4 + 1$$

if and only if $d = 0$.

53 Without doing any calculations or using a calculator, explain why

$$x^2 + 87559743x - 787727821$$

has no integer zeros.
[*Hint:* If x is an odd integer, is the expression above even or odd? If x is an even integer, is the expression above even or odd?]

54 Suppose M and N are odd integers. Explain why

$$x^2 + Mx + N$$

has no integer zeros.

55 Suppose M and N are odd integers. Explain why

$$x^2 + Mx + N$$

has no rational zeros.

56 Suppose $p(x) = 3x^7 - 5x^3 + 7x - 2$.

(a) Show that if m is a zero of p, then

$$\frac{2}{m} = 3m^6 - 5m^2 + 7.$$

(b) Show that the only possible integer zeros of p are -2, -1, 1, and 2.

(c) Show that no integer is a zero of p.

57 Suppose a, b, and c are integers and that

$$p(x) = ax^3 + bx^2 + cx + 9.$$

Explain why every zero of p that is an integer is contained in the set $\{-9, -3, -1, 1, 3, 9\}$.

58 Suppose $p(x) = 2x^5 + 5x^4 + 2x^3 - 1$. Show that -1 is the only integer zero of p.

59 Suppose $p(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n$ are integers. Suppose m is a nonzero integer that is a zero of p. Show that a_0/m is an integer.
[*This result shows that to find integer zeros of a polynomial with integer coefficients, we need only look at divisors of its constant term.*]

60 Suppose $p(x) = 2x^6 + 3x^5 + 5$.

(a) Show that if $\frac{M}{N}$ is a zero of p, then

$$2M^6 + 3M^5N + 5N^6 = 0.$$

(b) Show that if M and N are integers with no common factors and $\frac{M}{N}$ is a zero of p, then $5/M$ and $2/N$ are integers.

(c) Show that the only possible rational zeros of p are -5, -1, $-\frac{1}{2}$, and $-\frac{5}{2}$.

(d) Show that no rational number is a zero of p.

61 Suppose $p(x) = 2x^4 + 9x^3 + 1$.

(a) Show that if $\frac{M}{N}$ is a zero of p, then

$$2M^4 + 9M^3N + N^4 = 0.$$

(b) Show that if M and N are integers with no common factors and $\frac{M}{N}$ is a zero of p, then $M = -1$ or $M = 1$.

(c) Show that if M and N are integers with no common factors and $\frac{M}{N}$ is a zero of p, then $N = -2$ or $N = 2$ or $N = -1$ or $N = 1$.

(d) Show that $-\frac{1}{2}$ is the only rational zero of p.

62 Suppose $p(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n$ are integers. Suppose M and N are nonzero integers with no common factors and $\frac{M}{N}$ is a zero of p. Show that a_0/M and a_n/N are integers. [*Thus to find rational zeros of a polynomial with integer coefficients, we need only look at fractions whose numerator is a divisor of the constant term and whose denominator is a divisor of the coefficient of highest degree. This result is called the* **Rational Zeros Theorem** *or the* **Rational Roots Theorem**.]

63 Explain why the polynomial p defined by

$$p(x) = x^6 + 100x^2 + 5$$

has no real zeros.

64 Give an example of a polynomial of degree 5 that has exactly two zeros.

65 Give an example of a polynomial of degree 8 that has exactly three zeros.

66 Give an example of a polynomial p of degree 4 such that $p(7) = 0$ and $p(x) \ge 0$ for all real numbers x.

67 Give an example of a polynomial p of degree 6 such that $p(0) = 5$ and $p(x) \ge 5$ for all real numbers x.

68 Give an example of a polynomial p of degree 8 such that $p(2) = 3$ and $p(x) \ge 3$ for all real numbers x.

69 Explain why there does not exist a polynomial p of degree 7 such that $p(x) \ge -100$ for every real number x.

70 Explain why the composition of two polynomials is a polynomial.

71 Show that if p and q are nonzero polynomials, then

$$\deg(p \circ q) = (\deg p)(\deg q).$$

72 In the first figure in the solution to Example 6, the graph of the polynomial p clearly lies below the x-axis for x in the interval $[5000, 10000]$. Yet in the second figure in the same solution, the graph of p seems to be on or above the x-axis for all values of p in the interval $[0, 1000000]$. Explain.

73 Suppose t is a zero of the polynomial p defined by

$$p(x) = 3x^5 + 7x^4 + 2x + 6.$$

Show that $\frac{1}{t}$ is a zero of the polynomial q defined by

$$q(x) = 3 + 7x + 2x^4 + 6x^5.$$

74 Generalize the problem above.

75 Suppose q is a polynomial of degree 4 such that $q(0) = -1$. Define p by

$$p(x) = x^5 + q(x).$$

Explain why p has a zero on the interval $(0, \infty)$.

76 Suppose q is a polynomial of degree 5 such that $q(1) = -3$. Define p by

$$p(x) = x^6 + q(x).$$

Explain why p has at least two zeros.

77 Suppose

$$p(x) = x^5 + 2x^3 + 1.$$

(a) Find two distinct points on the graph of p.

(b) Explain why p is an increasing function.

(c) Find two distinct points on the graph of p^{-1}.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Suppose $\quad p(x) = x^2 + 5x + 2,$

$$q(x) = 2x^3 - 3x + 1, \quad s(x) = 4x^3 - 2.$$

In Exercises 1–18, write the indicated expression as a polynomial.

1 $(p + q)(x)$

SOLUTION

$$(p + q)(x) = (x^2 + 5x + 2) + (2x^3 - 3x + 1)$$
$$= 2x^3 + x^2 + 2x + 3$$

3 $(3p - 2q)(x)$

SOLUTION

$$(3p - 2q)(x) = 3(x^2 + 5x + 2) - 2(2x^3 - 3x + 1)$$
$$= 3x^2 + 15x + 6 - 4x^3 + 6x - 2$$
$$= -4x^3 + 3x^2 + 21x + 4$$

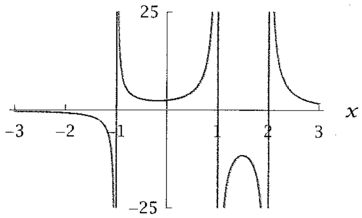
5 $(pq)(x)$

SOLUTION

$$(pq)(x) = (x^2 + 5x + 2)(2x^3 - 3x + 1)$$
$$= x^2(2x^3 - 3x + 1)$$
$$+ 5x(2x^3 - 3x + 1) + 2(2x^3 - 3x + 1)$$
$$= 2x^5 + 10x^4 + x^3 - 14x^2 - x + 2$$

EXAMPLE 9 Discuss the asymptotes of the graph of the rational function r defined by

$$r(x) = \frac{x^2+5}{x^3-2x^2-x+2}.$$



The graph of r on the interval $[-3,3]$, truncated on the vertical axis to the interval $[-25,25]$.

SOLUTION Because the numerator of $r(x)$ has degree less than the denominator, $r(x)$ is close to 0 for x near ∞ and for x near $-\infty$. Thus the x-axis is an asymptote of the graph of r, as can be seen in the graph here.

The strikingly different behavior of the graph here as compared to previous graphs that we have seen occurs near $x = -1$, $x = 1$, and $x = 2$; those three lines are shown here in red. To understand what is happening here, verify that the denominator of $r(x)$ is zero if $x = -1$, $x = 1$, or $x = 2$. Thus the numbers -1, 1, and 2 are not in the domain of r, because division by 0 is not defined.

The red lines above are the vertical asymptotes of the graph of r. The x-axis is also an asymptote of this graph.

For values of x very close to $x = -1$, $x = 1$, or $x = 2$, the denominator of $r(x)$ is very close to 0, but the numerator is always at least 5. Dividing a number larger than 5 by a number very close to 0 produces a number with very large absolute value, which explains the behavior of the graph of r near $x = -1$, $x = 1$, and $x = 2$. In other words, the lines $x = -1$, $x = 1$, and $x = 2$ are asymptotes of the graph of r.

EXERCISES

For Exercises 1–4, write the domain of the given function r as a union of intervals.

1 $r(x) = \dfrac{5x^3-12x^2+13}{x^2-7}$

2 $r(x) = \dfrac{x^5+3x^4-6}{2x^2-5}$

3 $r(x) = \dfrac{4x^7+8x^2-1}{x^2-2x-6}$

4 $r(x) = \dfrac{6x^9+x^5+8}{x^2+4x+1}$

For Exercises 5–8, find the asymptotes of the graph of the given function r.

5 $r(x) = \dfrac{6x^4+4x^3-7}{2x^4+3x^2+5}$

6 $r(x) = \dfrac{6x^6-7x^3+3}{3x^6+5x^4+x^2+1}$

7 $r(x) = \dfrac{3x+1}{x^2+x-2}$

8 $r(x) = \dfrac{9x+5}{x^2-x-6}$

In Exercises 9–26, write the indicated expression as a ratio of polynomials, assuming that

$$r(x) = \frac{3x+4}{x^2+1}, \quad s(x) = \frac{x^2+2}{2x-1}, \quad t(x) = \frac{5}{4x^3+3}.$$

9 $(r+s)(x)$

10 $(r-s)(x)$

11 $(s-t)(x)$

12 $(s+t)(x)$

13 $(3r-2s)(x)$

14 $(4r+5s)(x)$

15 $(rs)(x)$

16 $(rt)(x)$

17 $(r(x))^2$

18 $(s(x))^2$

19 $(r(x))^2 t(x)$

20 $(s(x))^2 t(x)$

21 $(r \circ s)(x)$

22 $(s \circ r)(x)$

23 $(r \circ t)(x)$

24 $(t \circ r)(x)$

25 $\frac{s(1+x)-s(1)}{x}$

26 $\frac{t(x-1)-t(-1)}{x}$

For Exercises 27–32, suppose

$$r(x) = \frac{x+1}{x^2+3} \quad \textit{and} \quad s(x) = \frac{x+2}{x^2+5}.$$

27 What is the domain of r?

28 What is the domain of s?

29 Find two distinct numbers x such that $r(x) = \frac{1}{4}$.

30 Find two distinct numbers x such that $s(x) = \frac{1}{8}$.

31 What is the range of r?

32 What is the range of s?

In Exercises 33–38, write each expression as the sum of a polynomial and a rational function whose numerator has smaller degree than its denominator.

33 $\dfrac{2x+1}{x-3}$

34 $\dfrac{4x-5}{x+7}$

35 $\dfrac{x^2}{3x-1}$

36 $\dfrac{x^2}{4x+3}$

37 $\dfrac{x^6+3x^3+1}{x^2+2x+5}$

38 $\dfrac{x^6-4x^2+5}{x^2-3x+1}$

39 Find a number c such that $r(10^{100}) \approx 6$, where

$$r(x) = \frac{cx^3+20x^2-15x+17}{5x^3+4x^2+18x+7}.$$

40 Find a number c such that $r(2^{1000}) \approx 5$, where

$$r(x) = \frac{3x^4-2x^3+8x+7}{cx^4-9x+2}.$$

41 A bicycle company finds that its average cost per bicycle for producing n thousand bicycles is $a(n)$ dollars, where

$$a(n) = 700\frac{4n^2+3n+50}{16n^2+3n+35}.$$

What will be the approximate cost per bicycle when the company is producing many bicycles?

42 A bicycle company finds that its average cost per bicycle for producing n thousand bicycles is $a(n)$ dollars, where

$$a(n) = 800\frac{3n^2+n+40}{16n^2+2n+45}.$$

What will be the approximate cost per bicycle when the company is producing many bicycles?

43 Suppose you start driving a car on a chilly fall day. As you drive, the heater in the car makes the temperature inside the car $F(t)$ degrees Fahrenheit at time t minutes after you started driving, where

$$F(t) = 40 + \frac{30t^3}{t^3+100}.$$

(a) What was the temperature in the car when you started driving?

(b) What was the approximate temperature in the car ten minutes after you started driving?

(c) What will be the approximate temperature in the car after you have been driving for a long time?

44 Suppose you start driving a car on a hot summer day. As you drive, the air conditioner in the car makes the temperature inside the car $F(t)$ degrees Fahrenheit at time t minutes after you started driving, where

$$F(t) = 90 - \frac{18t^2}{t^2+65}.$$

(a) What was the temperature in the car when you started driving?

(b) What was the approximate temperature in the car 15 minutes after you started driving?

(c) What will be the approximate temperature in the car after you have been driving for a long time?

PROBLEMS

45 Suppose $s(x) = \dfrac{x^2+2}{2x-1}$.

(a) Show that the point $(1, 3)$ is on the graph of s.

(b) Show that the slope of a line containing $(1, 3)$ and a point on the graph of s very close to $(1, 3)$ is approximately -4.

[*Hint:* Use the result of Exercise 25.]

46 Suppose $t(x) = \dfrac{5}{4x^3+3}$.

(a) Show that the point $(-1, -5)$ is on the graph of t.

(b) Give an estimate for the slope of a line containing $(-1, -5)$ and a point on the graph of t very close to $(-1, -5)$.

[*Hint:* Use the result of Exercise 26.]

47 Explain why the composition of a polynomial and a rational function (in either order) is a rational function.

48 Explain why the composition of two rational functions is a rational function.

49 Suppose p is a polynomial and t is a number. Explain why there is a polynomial G such that

$$\frac{p(x)-p(t)}{x-t} = G(x)$$

for every number $x \neq t$.

50 Suppose r is the function with domain $(0, \infty)$ defined by

$$r(x) = \frac{1}{x^4+2x^3+3x^2}$$

for each positive number x.

(a) Find two distinct points on the graph of r.

(b) Explain why r is a decreasing function on $(0, \infty)$.

(c) Find two distinct points on the graph of r^{-1}.

51 Suppose p is a nonzero polynomial with at least one (real) zero. Explain why

- there exist real numbers $t_1, t_2, \dots, t_m$ and a polynomial G such that G has no (real) zeros and

$$p(x) = (x - t_1)(x - t_2) \dots (x - t_m)G(x)$$

for every real number x;

- each of the numbers $t_1, t_2, \dots, t_m$ is a zero of p;
- p has no zeros other than $t_1, t_2, \dots, t_m$.

52 Suppose p and q are polynomials and the horizonal axis is an asymptote of the graph of $\frac{p}{q}$. Explain why

$$\deg p < \deg q.$$

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–4, write the domain of the given function r as a union of intervals.

1 $r(x) = \dfrac{5x^3 - 12x^2 + 13}{x^2 - 7}$

SOLUTION Because we have no other information about the domain of r, we assume the domain of r is the set of numbers where the expression defining r makes sense, which means where the denominator is not 0. The denominator of the expression defining r is 0 if $x = -\sqrt{7}$ or $x = \sqrt{7}$. Thus the domain of r is the set of numbers other than $-\sqrt{7}$ and $\sqrt{7}$. In other words, the domain of r is $(-\infty, -\sqrt{7}) \cup (-\sqrt{7}, \sqrt{7}) \cup (\sqrt{7}, \infty)$.

3 $r(x) = \dfrac{4x^7 + 8x^2 - 1}{x^2 - 2x - 6}$

SOLUTION To find where the expression defining r does not make sense, apply the quadratic formula to the equation $x^2 - 2x - 6 = 0$, getting $x = 1 - \sqrt{7}$ or $x = 1 + \sqrt{7}$. Thus the domain of r is the set of numbers other than $1 - \sqrt{7}$ and $1 + \sqrt{7}$. In other words, the domain of r is $(-\infty, 1 - \sqrt{7}) \cup (1 - \sqrt{7}, 1 + \sqrt{7}) \cup (1 + \sqrt{7}, \infty)$.

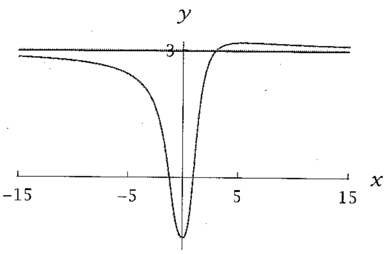
For Exercises 5–8, find the asymptotes of the graph of the given function r.

5 $r(x) = \dfrac{6x^4 + 4x^3 - 7}{2x^4 + 3x^2 + 5}$

SOLUTION The denominator of this rational function is never 0, so we only need to worry about the behavior of r near $\pm\infty$. For $|x|$ very large, we have

$$r(x) = \frac{6x^4 + 4x^3 - 7}{2x^4 + 3x^2 + 5} = \frac{6x^4\left(1 + \frac{2}{3x} - \frac{7}{6x^4}\right)}{2x^4\left(1 + \frac{3}{2x^2} + \frac{5}{2x^4}\right)} \approx 3.$$

Thus the line $y = 3$ is an asymptote of the graph of r, as shown below:



The graph of $\dfrac{6x^4 + 4x^3 - 7}{2x^4 + 3x^2 + 5}$ *on the interval* $[-15, 15]$.

7 $r(x) = \dfrac{3x + 1}{x^2 + x - 2}$

SOLUTION The denominator of this rational function is 0 when

$$x^2 + x - 2 = 0.$$

Solving this equation either by factoring or using the quadratic formula, we get $x = -2$ or $x = 1$. Because the degree of the numerator is less than the degree of the denominator, the value of this function is close to 0 when $|x|$ is large. Thus the asymptotes of the graph of r are the lines $x = -2$, $x = 1$, and $y = 0$, as shown below: