When describing a function by a table, there should be no repetitions in the left column, which shows the numbers in the domain of the function. However, repetitions can occur in the right column, which shows the numbers in the range of the function, as in the following example.

EXAMPLE 13

Suppose f is the function completely determined by the table shown here.

(a) What is the domain of f?

(b) What is the range of f?

x	$f(x)$
1	6
2	6
3	−7
5	6

For this function we have $f(1) = f(2) = f(5) = 6$ and $f(3) = -7$.

SOLUTION

(a) The left column of the table contains the numbers 1, 2, 3, and 5. Thus the domain of f is the set $\{1, 2, 3, 5\}$.

(b) The right column of the table contains only two distinct numbers, −7 and 6. Thus the range of f is the set $\{-7, 6\}$.

EXERCISES

For Exercises 1–12, assume

$$f(x) = \frac{x+2}{x^2+1}$$

for every real number x. Evaluate and simplify each of the following expressions.

1 $f(0)$

2 $f(1)$

3 $f(-1)$

4 $f(-2)$

5 $f(2a)$

6 $f(\frac{b}{3})$

7 $f(2a+1)$

8 $f(3a-1)$

9 $f(x^2+1)$

10 $f(2x^2+3)$

11 $f(\frac{a}{b}-1)$

12 $f(\frac{2a}{b}+3)$

For Exercises 13–18, assume

$$g(x) = \frac{x-1}{x+2}.$$

13 Find a number b such that $g(b) = 4$.

14 Find a number b such that $g(b) = 3$.

15 Simplify the expression $\dfrac{g(x)-g(2)}{x-2}$.

16 Simplify the expression $\dfrac{g(x)-g(3)}{x-3}$.

17 Simplify the expression $\dfrac{g(a+t)-g(a)}{t}$.

18 Simplify the expression $\dfrac{g(x+b)-g(x-b)}{2b}$.

For Exercises 19–26, assume f is the function defined by

$$f(t) = \begin{cases} 2t+9 & \text{if } t < 0 \\ 3t-10 & \text{if } t \geq 0. \end{cases}$$

19 Evaluate $f(1)$.

20 Evaluate $f(2)$.

21 Evaluate $f(-3)$.

22 Evaluate $f(-4)$.

23 Evaluate $f(|x|+1)$.

24 Evaluate $f(|x-5|+2)$.

25 Find two different values of t such that $f(t) = 0$.

26 Find two different values of t such that $f(t) = 4$.

27 Using the tax function given in Example 2, find the 2011 federal income tax for a single person whose taxable income that year was $45,000.

28 Using the tax function given in Example 2, find the 2011 federal income tax for a single person whose taxable income that year was $90,000.

For Exercises 29–32, find a number b such that the function f equals the function g.

29 The function f has domain the set of positive numbers and is defined by $f(x) = 5x^2 - 7$; the function g has domain (b, ∞) and is defined by $g(x) = 5x^2 - 7$.

30 The function f has domain the set of numbers with absolute value less than 4 and is defined by $f(x) = \dfrac{3}{x+5}$; the function g has domain the interval $(-b, b)$ and is defined by $g(x) = \dfrac{3}{x+5}$.

31 Both f and g have domain $\{3,5\}$, with f defined on this domain by the formula $f(x) = x^2 - 3$ and g defined on this domain by the formula $g(x) = \frac{18}{x} + b(x-3)$.

32 Both f and g have domain $\{-3,4\}$, with f defined on this domain by the formula $f(x) = 3x + 5$ and g defined on this domain by the formula $g(x) = 15 + \frac{8}{x} + b(x-4)$.

For Exercises 33–38, a formula has been given defining a function f but no domain has been specified. Find the domain of each function f, assuming that the domain is the set of real numbers for which the formula makes sense and produces a real number.

33 $f(x) = \frac{2x+1}{3x-4}$

34 $f(x) = \frac{4x-9}{7x+5}$

35 $f(x) = \frac{\sqrt{x-5}}{x-7}$

36 $f(x) = \frac{\sqrt{2x+3}}{x-6}$

37 $f(x) = \sqrt{|x-6|-1}$

38 $f(x) = \sqrt{|x+5|-3}$

For Exercises 39–44, find the range of h if h is defined by

$$h(t) = |t| + 1$$

and the domain of h is the indicated set.

39 $(1,4]$

40 $[-8,-3)$

41 $[-3,5]$

42 $[-8,2]$

43 $(0,\infty)$

44 $(-\infty,0)$

For Exercises 45–52, assume f and g are functions completely defined by the following tables:

x	$f(x)$
3	13
4	-5
6	$\frac{3}{5}$
7.3	-5

x	$g(x)$
3	3
8	$\sqrt{7}$
8.4	$\sqrt{7}$
12.1	$-\frac{2}{7}$

45 Evaluate $f(6)$.

46 Evaluate $g(8)$.

47 What is the domain of f?

48 What is the domain of g?

49 What is the range of f?

50 What is the range of g?

51 Find two different values of x such that $f(x) = -5$.

52 Find two different values of x such that $g(x) = \sqrt{7}$.

53 Find all functions (displayed as tables) whose domain is the set $\{2,9\}$ and whose range is the set $\{4,6\}$.

54 Find all functions (displayed as tables) whose domain is the set $\{5,8\}$ and whose range is the set $\{1,3\}$.

55 Find all functions (displayed as tables) whose domain is $\{1,2,4\}$ and whose range is $\{-2,1,\sqrt{3}\}$.

56 Find all functions (displayed as tables) whose domain is $\{-1,0,\pi\}$ and whose range is $\{-3,\sqrt{2},5\}$.

57 Find all functions (displayed as tables) whose domain is $\{3,5,9\}$ and whose range is $\{2,4\}$.

58 Find all functions (displayed as tables) whose domain is $\{0,2,8\}$ and whose range is $\{6,9\}$.

PROBLEMS

Some problems require considerably more thought than the exercises.

59 Suppose the only information you know about a function f is that the domain of f is the set of real numbers and

$$f(1) = 1,\ f(2) = 4,\ f(3) = 9,\ \text{and } f(4) = 16.$$

What can you say about the value of $f(5)$? [*Hint:* The answer to this problem is not "25". The shortest correct answer is just one word.]

60 Suppose g and h are functions whose domain is the set of real numbers, with g and h defined on this domain by the formulas

$$g(y) = \frac{4y}{y^2+5} \quad \text{and} \quad h(r) = \frac{4r}{r^2+5}.$$

Are g and h equal functions?

61 Give an example of a function whose domain is $\{2,5,7\}$ and whose range is $\{-2,3,4\}$.

62 Give an example of a function whose domain is $\{3,4,7,9\}$ and whose range is $\{-1,0,3\}$.

63 Find two different functions whose domain is $\{3,8\}$ and whose range is $\{-4,1\}$.

64 Explain why there does not exist a function whose domain is $\{-1,0,3\}$ and whose range is $\{3,4,7,9\}$.

65 Give an example of a function f whose domain is the set of real numbers and such that the values of $f(-1)$, $f(0)$, and $f(2)$ are given by the following table:

x	$f(x)$
-1	$\sqrt{2}$
0	$\frac{17}{3}$
2	-5

66 Give an example of two different functions f and g, both of which have the set of real numbers as their domain, such that $f(x) = g(x)$ for every rational number x.

67 Give an example of a function whose domain equals the set of real numbers and whose range equals the set $\{-1, 0, 1\}$.

68 Give an example of a function whose domain equals the set of real numbers and whose range equals the set of integers.

69 Give an example of a function whose domain is the interval $[0, 1]$ and whose range is the interval $(0, 1)$.

70 Give an example of a function whose domain is the interval $(0, 1)$ and whose range is the interval $[0, 1]$.

71 Give an example of a function whose domain is the set of positive integers and whose range is the set of positive even integers.

72 Give an example of a function whose domain is the set of positive even integers and whose range is the set of positive odd integers.

73 Give an example of a function whose domain is the set of integers and whose range is the set of positive integers.

74 Give an example of a function whose domain is the set of positive integers and whose range is the set of integers.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Do not read these worked-out solutions before attempting to do the exercises yourself. Otherwise you may mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises and check your answers here. If you get stuck on an exercise, then look at the worked-out solution here.

For Exercises 1–12, assume

$$f(x) = \frac{x+2}{x^2+1}$$

for every real number x. Evaluate and simplify each of the following expressions.

1 $f(0)$

SOLUTION $f(0) = \frac{0+2}{0^2+1} = \frac{2}{1} = 2$

3 $f(-1)$

SOLUTION $f(-1) = \dfrac{-1+2}{(-1)^2+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

5 $f(2a)$

SOLUTION $f(2a) = \dfrac{2a+2}{(2a)^2+1} = \dfrac{2a+2}{4a^2+1}$

7 $f(2a+1)$

SOLUTION

$$f(2a+1) = \frac{(2a+1)+2}{(2a+1)^2+1} = \frac{2a+3}{4a^2+4a+2}$$

9 $f(x^2+1)$

SOLUTION

$$f(x^2+1) = \frac{(x^2+1)+2}{(x^2+1)^2+1} = \frac{x^2+3}{x^4+2x^2+2}$$

11 $f(\frac{a}{b} - 1)$

SOLUTION We have

$$f(\tfrac{a}{b} - 1) = \frac{(\frac{a}{b} - 1) + 2}{(\frac{a}{b} - 1)^2 + 1} = \frac{\frac{a}{b} + 1}{\frac{a^2}{b^2} - 2\frac{a}{b} + 2}$$
$$= \frac{ab + b^2}{a^2 - 2ab + 2b^2},$$

where the last expression was obtained by multiplying the numerator and denominator of the previous expression by b^2.

For Exercises 13–18, assume

$$g(x) = \frac{x-1}{x+2}.$$

13 Find a number b such that $g(b) = 4$.

SOLUTION We want to find a number b such that

$$\frac{b-1}{b+2} = 4.$$

Multiply both sides of the equation above by $b + 2$, getting

$$b - 1 = 4b + 8.$$

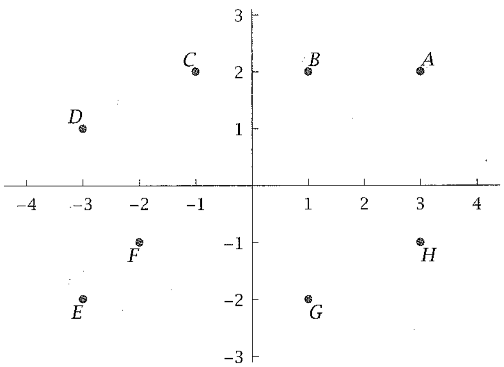
Now solve this equation for b, getting $b = -3$.

15 Simplify the expression $\dfrac{g(x) - g(2)}{x - 2}$.

SOLUTION We begin by evaluating the numerator:

EXERCISES

For Exercises 1-8, give the coordinates of the specified point using the figure below:



1 A **3** C **5** E **7** G

2 B **4** D **6** F **8** H

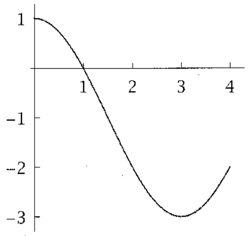
9 Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(1, 2)$, $(-2, 2)$, $(-3, -1)$, $(2, -3)$.

10 Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(2.5, 1)$, $(-1, 3)$, $(-1.5, -1.5)$, $(1, -3)$.

11 Shown below is the graph of a function f.

(a) What is the domain of f?

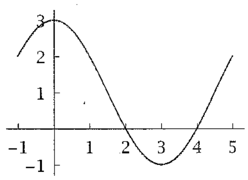
(b) What is the range of f?



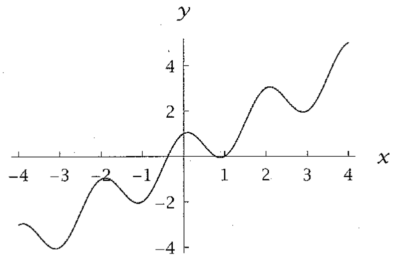
12 Shown below is the graph of a function f.

(a) What is the domain of f?

(b) What is the range of f?



For Exercises 13-24, assume f is the function with domain $[-4, 4]$ whose graph is shown below:



The graph of f.

13 Estimate the value of $f(-4)$.

14 Estimate the value of $f(-3)$.

15 Estimate the value of $f(-2)$.

16 Estimate the value of $f(-1)$.

17 Estimate the value of $f(2)$.

18 Estimate the value of $f(0)$.

19 Estimate the value of $f(4)$.

20 Estimate the value of $f(3)$.

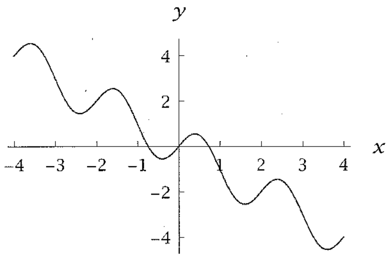
21 Estimate a number b such that $f(b) = 4$.

22 Estimate a negative number b such that $f(b) = 0.5$.

23 How many values of x satisfy the equation $f(x) = \frac{1}{2}$?

24 How many values of x satisfy the equation $f(x) = -3.5$?

For Exercises 25-36, assume g is the function with domain $[-4, 4]$ whose graph is shown below:



The graph of g.

25 Estimate the value of $g(-4)$.

26 Estimate the value of $g(-3)$.

27 Estimate the value of $g(-2)$.

28 Estimate the value of $g(-1)$.

29 Estimate the value of $g(2)$.

30 Estimate the value of $g(1)$.

31 Estimate the value of $g(2.5)$.

32 Estimate the value of $g(1.5)$.

33 Estimate a number b such that $g(b) = 3.5$.

34 Estimate a number b such that $g(b) = -3.5$.

35 How many values of x satisfy the equation $g(x) = -2$?

36 How many values of x satisfy the equation $g(x) = 0$?

For Exercises 37–40, use appropriate technology to sketch the graph of the function f defined by the given formula on the given interval.

37 $f(x) = 2x^3 - 9x^2 + 12x - 3$ on the interval $[\frac{1}{2}, \frac{5}{2}]$

38 $f(x) = 0.6x^5 - 7.5x^4 + 35x^3 - 75x^2 + 72x - 20$ on the interval $[\frac{1}{2}, \frac{9}{2}]$

39 $f(t) = \dfrac{t^2 + 1}{t^5 + 2}$ on the interval $[-\frac{1}{2}, 2]$

40 $f(t) = \dfrac{8t^3 - 5}{t^4 + 2}$ on the interval $[-1, 3]$

For Exercises 41–46, assume g and h are the functions completely defined by the tables below:

x	$g(x)$
−3	−1
−1	1
1	2.5
3	−2

x	$h(x)$
−4	2
−2	−3
2	−1.5
3	1

41 What is the domain of g?

42 What is the domain of h?

43 What is the range of g?

44 What is the range of h?

45 Draw the graph of g.

46 Draw the graph of h.

PROBLEMS

47 Sketch the graph of a function whose domain equals the interval $[1, 3]$ and whose range equals the interval $[-2, 4]$.

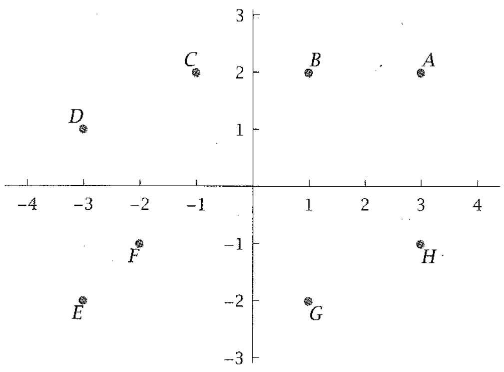
48 Sketch the graph of a function whose domain is the interval $[0, 4]$ and whose range is the set of two numbers $\{2, 3\}$.

49 Give an example of a line in the coordinate plane that is not the graph of any function.

50 Give an example of a set consisting of two points in the coordinate plane that is not the graph of any function.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–8, give the coordinates of the specified point using the figure below:



1 A

SOLUTION To get to the point A starting at the origin, we must move 3 units right and 2 units up. Thus A has coordinates $(3, 2)$.

Numbers obtained from a figure should be considered approximations. Thus the actual coordinates of A might be $(3.01, 1.98)$.

3 C

SOLUTION To get to the point C starting at the origin, we must move 1 unit left and 2 units up. Thus C has coordinates $(-1, 2)$.

5 E

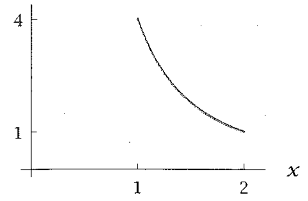
SOLUTION To get to the point E starting at the origin, we must move 3 units left and 2 units down. Thus E has coordinates $(-3, -2)$.

7 G

SOLUTION To get to the point G starting at the origin, we must move 1 unit right and 2 units down. Thus G has coordinates $(1, -2)$.

EXERCISES

For Exercises 1–14, assume f is the function defined on the interval $[1,2]$ by the formula $f(x) = \frac{4}{x^2}$. Thus the domain of f is the interval $[1,2]$, the range of f is the interval $[1,4]$, and the graph of f is shown here.



The graph of f.

For each function g described below:

(a) ***Sketch the graph of g.***

(b) ***Find the domain of g (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of g).***

(c) ***Give a formula for g.***

(d) ***Find the range of g (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of g).***

1 The graph of g is obtained by shifting the graph of f up 1 unit.

2 The graph of g is obtained by shifting the graph of f up 3 units.

3 The graph of g is obtained by shifting the graph of f down 3 units.

4 The graph of g is obtained by shifting the graph of f down 2 units.

5 The graph of g is obtained by vertically stretching the graph of f by a factor of 2.

6 The graph of g is obtained by vertically stretching the graph of f by a factor of 3.

7 The graph of g is obtained by shifting the graph of f left 3 units.

8 The graph of g is obtained by shifting the graph of f left 4 units.

9 The graph of g is obtained by shifting the graph of f right 1 unit.

10 The graph of g is obtained by shifting the graph of f right 3 units.

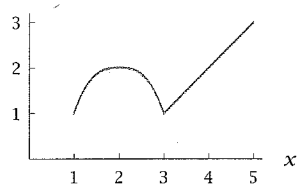
11 The graph of g is obtained by horizontally stretching the graph of f by a factor of 2.

12 The graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$.

13 The graph of g is obtained by flipping the graph of f across the horizontal axis.

14 The graph of g is obtained by flipping the graph of f across the vertical axis.

For Exercises 15–50, assume f is a function whose domain is the interval $[1,5]$, whose range is the interval $[1,3]$, and whose graph is the figure below.



The graph of f.

For each given function g:

(a) ***Find the domain of g.***

(b) ***Find the range of g.***

(c) ***Sketch the graph of g.***

15 $g(x) = f(x) + 1$

16 $g(x) = f(x) + 3$

17 $g(x) = f(x) - 3$

18 $g(x) = f(x) - 5$

19 $g(x) = 2f(x)$

20 $g(x) = \frac{1}{2}f(x)$

21 $g(x) = f(x+2)$

22 $g(x) = f(x+3)$

23 $g(x) = f(x-1)$

24 $g(x) = f(x-2)$

25 $g(x) = f(2x)$

26 $g(x) = f(3x)$

27 $g(x) = f(\frac{x}{2})$

28 $g(x) = f(\frac{5x}{8})$

29 $g(x) = 2f(x) + 1$

30 $g(x) = 3f(x) + 2$

31 $g(x) = \frac{1}{2}f(x) - 1$

32 $g(x) = \frac{2}{3}f(x) - 2$

33 $g(x) = 3 - f(x)$

34 $g(x) = 2 - f(x)$

35 $g(x) = -f(x-1)$

36 $g(x) = -f(x-3)$

37 $g(x) = f(x+1) + 2$

38 $g(x) = f(x+2) + 1$

39 $g(x) = f(2x) + 1$

40 $g(x) = f(3x) + 2$

41 $g(x) = f(2x+1)$

42 $g(x) = f(3x+2)$

43 $g(x) = 2f(\frac{x}{2} + 1)$

44 $g(x) = 3f(\frac{2x}{5} + 2)$

45 $g(x) = 2f(\frac{x}{2} + 1) - 3$

46 $g(x) = 3f(\frac{2x}{5} + 2) + 1$

47 $g(x) = 2f(\frac{x}{2} + 3)$

48 $g(x) = 3f(\frac{2x}{5} - 2)$

49 $g(x) = 6 - 2f(\frac{x}{2} + 3)$

50 $g(x) = 1 - 3f(\frac{2x}{5} - 2)$

51 Suppose g is an even function whose domain is $[-2,-1] \cup [1,2]$ and whose graph on the interval $[1,2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of g on $[-2,-1] \cup [1,2]$.

52 Suppose g is an even function whose domain is $[-5,-1] \cup [1,5]$ and whose graph on the interval $[1,5]$ is the graph used in the instructions for Exercises 15–50. Sketch the graph of g on $[-5,-1] \cup [1,5]$.

53 Suppose h is an odd function whose domain is $[-2,-1] \cup [1,2]$ and whose graph on the interval $[1,2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of h on $[-2,-1] \cup [1,2]$.

54 Suppose h is an odd function whose domain is $[-5,-1] \cup [1,5]$ and whose graph on the interval $[1,5]$ is the graph used in the instructions for Exercises 15–50. Sketch the graph of h on $[-5,-1] \cup [1,5]$.

For Exercises 55–58, suppose f is a function whose domain is the interval $[-5,5]$ and

$$f(x) = \frac{x}{x+3}$$

for every x in the interval $[0,5]$.

55 Suppose f is an even function. Evaluate $f(-2)$.

56 Suppose f is an even function. Evaluate $f(-3)$.

57 Suppose f is an odd function. Evaluate $f(-2)$.

58 Suppose f is an odd function. Evaluate $f(-3)$.

PROBLEMS

For Problems 59–62, suppose that to provide additional funds for higher education, the federal government adopts a new income tax plan that consists of the 2011 income tax plus an additional \$100 per taxpayer. Let g be the function such that $g(x)$ is the 2011 federal income tax for a single person with taxable income x dollars, and let h be the corresponding function for the new income tax plan.

59 Is h obtained from g by a vertical function transformation or by a horizontal function transformation?

60 Write a formula for $h(x)$ in terms of $g(x)$.

61 Using the explicit formula for $g(x)$ given in Example 2 in Section 1.1, give an explicit formula for $h(x)$.

62 Under the new income tax plan, what will be the income tax for a single person whose annual taxable income is \$50,000?

For Problems 63–66, suppose that to pump more money into the economy during a recession, the federal government adopts a new income tax plan that makes income taxes 90% of the 2011 income tax. Let g be the function such that $g(x)$ is the 2011 federal income tax for a single person with taxable income x dollars, and let h be the corresponding function for the new income tax plan.

63 Is h obtained from g by a vertical function transformation or by a horizontal function transformation?

64 Write a formula for $h(x)$ in terms of $g(x)$.

65 Using the explicit formula for $g(x)$ given in Example 2 in Section 1.1, give an explicit formula for $h(x)$.

66 Under the new income tax plan, what will be the income tax for a single person whose annual taxable income is \$60,000?

67 Find the only function whose domain is the set of real numbers and that is both even and odd.

68 Show that if f is an odd function such that 0 is in the domain of f, then $f(0) = 0$.

69 The result box following Example 2 could have been made more complete by including explicit information about the domain and range of the functions g and h. For example, the more complete result box might have looked like the one shown here:

> ***Shifting a graph up or down***
>
> Suppose f is a function and $a > 0$. Define functions g and h by
>
> $$g(x) = f(x) + a \quad \text{and} \quad h(x) = f(x) - a.$$
>
> Then
>
> - g and h have the same domain as f;
> - the range of g is obtained by adding a to every number in the range of f;
> - the range of h is obtained by subtracting a from every number in the range of f;
> - the graph of g is obtained by shifting the graph of f up a units;
> - the graph of h is obtained by shifting the graph of f down a units.

Construct similar complete result boxes, including explicit information about the domain and range of the functions g and h, for each of the other five result boxes in this section that deal with function transformations.

70 True or false: If f is an odd function whose domain is the set of real numbers and a function g is defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0, \end{cases}$$

then g is an even function. Explain your answer.

71 True or false: If f is an even function whose domain is the set of real numbers and a function g is defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0, \end{cases}$$

then g is an odd function. Explain your answer.

72 (a) True or false: Just as every integer is either even or odd, every function whose domain is the set of integers is either an even function or an odd function.

(b) Explain your answer to part (a). This means that if the answer is "true", then you should explain why every function whose domain is the set of integers is either an even function or an odd function; if the answer is "false", then you should give an example of a function whose domain is the set of integers but that is neither even nor odd.

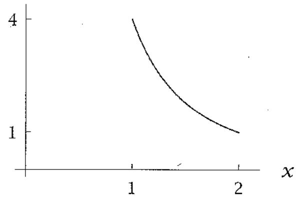
73 Show that the function f defined by $f(x) = mx + b$ is an odd function if and only if $b = 0$.

74 Show that the function f defined by $f(x) = mx + b$ is an even function if and only if $m = 0$.

75 Show that the function f defined by $f(x) = ax^2 + bx + c$ is an even function if and only if $b = 0$.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–14, assume f is the function defined on the interval $[1, 2]$ by the formula $f(x) = \frac{4}{x^2}$. Thus the domain of f is the interval $[1, 2]$, the range of f is the interval $[1, 4]$, and the graph of f is shown here.



The graph of f.

For each function g described below:

(a) ***Sketch the graph of g.***

(b) ***Find the domain of g (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of g).***

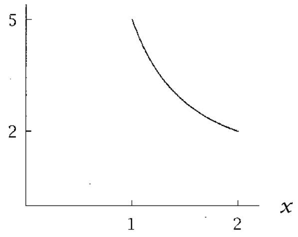
(c) ***Give a formula for g.***

(d) ***Find the range of g (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of g).***

1 The graph of g is obtained by shifting the graph of f up 1 unit.

SOLUTION

(a)



Shifting the graph of f up 1 unit gives this graph.

(b) The domain of g is the same as the domain of f. Thus the domain of g is the interval $[1, 2]$.

(c) Because the graph of g is obtained by shifting the graph of f up 1 unit, we have $g(x) = f(x) + 1$. Thus

$$g(x) = \frac{4}{x^2} + 1$$

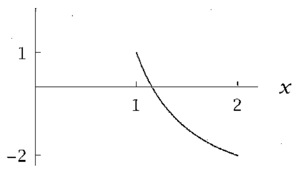
for each number x in the interval $[1, 2]$.

(d) The range of g is obtained by adding 1 to each number in the range of f. Thus the range of g is the interval $[2, 5]$.

3 The graph of g is obtained by shifting the graph of f down 3 units.

SOLUTION

(a)



Shifting the graph of f down 3 units gives this graph.

EXERCISES

For Exercises 1–10, evaluate the indicated expression assuming that f, g, and h are the functions completely defined by these tables:

x	$f(x)$
1	4
2	1
3	2
4	2

x	$g(x)$
1	2
2	4
3	1
4	3

x	$h(x)$
1	3
2	3
3	4
4	1

1 $(f \circ g)(1)$

2 $(f \circ g)(3)$

3 $(g \circ f)(1)$

4 $(g \circ f)(3)$

5 $(f \circ f)(2)$

6 $(f \circ f)(4)$

7 $(g \circ g)(4)$

8 $(g \circ g)(2)$

9 $(f \circ g \circ h)(2)$

10 $(h \circ g \circ f)(2)$

For Exercises 17–30, evaluate the indicated expression assuming that

$$f(x) = \sqrt{x}, \quad g(x) = \frac{x+1}{x+2}, \quad h(x) = |x-1|.$$

11 $(f + g)(3)$

12 $(g + h)(6)$

13 $(gh)(7)$

14 $(fh)(9)$

15 $\left(\frac{f}{h}\right)(10)$

16 $\left(\frac{h}{g}\right)(11)$

17 $(f \circ g)(4)$

18 $(f \circ g)(5)$

19 $(g \circ f)(4)$

20 $(g \circ f)(5)$

21 $(f \circ h)(-3)$

22 $(f \circ h)(-15)$

23 $(f \circ g \circ h)(0)$

24 $(h \circ g \circ f)(0)$

25 $(f \circ g)(0.23)$

26 $(f \circ g)(3.85)$

27 $(g \circ f)(0.23)$

28 $(g \circ f)(3.85)$

29 $(h \circ f)(0.3)$

30 $(h \circ f)(0.7)$

In Exercises 31–36, for the given functions f and g find formulas for (a) $f \circ g$ and (b) $g \circ f$. Simplify your results as much as possible.

31 $f(x) = x^2 + 1, \quad g(x) = \frac{1}{x}$

32 $f(x) = (x+1)^2, \quad g(x) = \frac{3}{x}$

33 $f(x) = \frac{x-1}{x+1}, \quad g(x) = x^2 + 2$

34 $f(x) = \frac{x+2}{x-3}, \quad g(x) = \frac{1}{x+1}$

35 $f(t) = \frac{t-1}{t^2+1}, \quad g(t) = \frac{t+3}{t+4}$

36 $f(t) = \frac{t-2}{t+3}, \quad g(t) = \frac{1}{(t+2)^2}$

37 Find a number b such that $f \circ g = g \circ f$, where $f(x) = 2x + b$ and $g(x) = 3x + 4$.

38 Find a number c such that $f \circ g = g \circ f$, where $f(x) = 5x - 2$ and $g(x) = cx - 3$.

39 Suppose

$$h(x) = \left(\frac{x^2+1}{x-1} - 1\right)^3.$$

(a) If $f(x) = x^3$, then find a function g such that $h = f \circ g$.

(b) If $f(x) = (x-1)^3$, then find a function g such that $h = f \circ g$.

40 Suppose

$$h(x) = \sqrt{\frac{1}{x^2+1} + 2}.$$

(a) If $f(x) = \sqrt{x}$, then find a function g such that $h = f \circ g$.

(b) If $f(x) = \sqrt{x+2}$, then find a function g such that $h = f \circ g$.

41 Suppose

$$h(t) = 2 + \sqrt{\frac{1}{t^2+1}}.$$

(a) If $g(t) = \frac{1}{t^2+1}$, then find a function f such that $h = f \circ g$.

(b) If $g(t) = t^2$, then find a function f such that $h = f \circ g$.

42 Suppose

$$h(t) = \left(\frac{t^2+1}{t-1} - 1\right)^3.$$

(a) If $g(t) = \frac{t^2+1}{t-1} - 1$, then find a function f such that $h = f \circ g$.

(b) If $g(t) = \frac{t^2+1}{t-1}$, then find a function f such that $h = f \circ g$.

In Exercises 43–46, find functions f and g, each simpler than the given function h, such that $h = f \circ g$.

43 $h(x) = (x^2 - 1)^2$

44 $h(x) = \sqrt{x^2 - 1}$

45 $h(x) = \frac{3}{2 + x^2}$

46 $h(x) = \frac{2}{3 + \sqrt{1+x}}$

In Exercises 47–48, find functions f, g, and h, each simpler than the given function T, such that $T = f \circ g \circ h$.

47 $T(x) = \frac{4}{5 + x^2}$

48 $T(x) = \sqrt{4 + x^2}$

For Exercises 49–54, suppose f is a function and a function g is defined by the given expression.

(a) *Write g as the composition of f and one or two linear functions.*

(b) *Describe how the graph of g is obtained from the graph of f.*

49 $g(x) = 3f(x) - 2$

50 $g(x) = -4f(x) - 7$

51 $g(x) = f(5x)$

52 $g(x) = f(-\frac{2}{3}x)$

53 $g(x) = 2f(3x) + 4$

54 $g(x) = -5f(-\frac{4}{3}x) - 8$

PROBLEMS

For Problems 55–59, suppose you are exchanging currency in the London airport. The currency exchange service there only makes transactions in which one of the two currencies is British pounds, but you want to exchange dollars for Euros. Thus you first need to exchange dollars for British pounds, then exchange British pounds for Euros. At the time you want to make the exchange, the function f for exchanging dollars for British pounds is given by the formula

$$f(d) = 0.66d - 1$$

and the function g for exchanging British pounds for Euros is given by the formula

$$g(p) = 1.23p - 2.$$

The subtraction of 1 or 2 in the number of British pounds or Euros that you receive is the fee charged by the currency exchange service for each transaction.

55 Is the function describing the exchange of dollars for Euros $f \circ g$ or $g \circ f$? Explain your answer in terms of which function is evaluated first when computing a value for a composition (the function on the left or the function on the right?).

56 Find a formula for the function given by your answer to Problem 55.

57 How many Euros would you receive for exchanging \$100 after going through this two-step exchange process?

58 How many Euros would you receive for exchanging \$200 after going through this two-step exchange process?

59 Which process gives you more Euros: exchanging \$100 for Euros twice or exchanging \$200 for Euros once?

60 Suppose $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are numbers. Show that $f \circ g = g \circ f$ if and only if $d(a-1) = b(c-1)$.

61 Suppose f and g are functions. Show that the composition $f \circ g$ has the same domain as g if and only if the range of g is contained in the domain of f.

62 Show that the sum of two even functions (with the same domain) is an even function.

63 Show that the product of two even functions (with the same domain) is an even function.

64 True or false: The product of an even function and an odd function (with the same domain) is an odd function. Explain your answer.

65 True or false: The sum of an even function and an odd function (with the same domain) is an odd function. Explain your answer.

66 Suppose g is an even function and f is any function. Show that $f \circ g$ is an even function.

67 Suppose f is an even function and g is an odd function. Show that $f \circ g$ is an even function.

68 Suppose f and g are both odd functions. Is the composition $f \circ g$ even, odd, or neither? Explain.

69 Show that if f, g, and h are functions, then

$$(f+g) \circ h = f \circ h + g \circ h.$$

70 Find functions f, g, and h such that

$$f \circ (g+h) \neq f \circ g + f \circ h.$$

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–10, evaluate the indicated expression assuming that f, g, and h are the functions completely defined by these tables:

x	$f(x)$
1	4
2	1
3	2
4	2

x	$g(x)$
1	2
2	4
3	1
4	3

x	$h(x)$
1	3
2	3
3	4
4	1

EXERCISES

For Exercises 1–8, check your answer by evaluating the appropriate function at your answer.

1 Suppose $f(x) = 4x + 6$. Evaluate $f^{-1}(5)$.

2 Suppose $f(x) = 7x - 5$. Evaluate $f^{-1}(-3)$.

3 Suppose $g(x) = \dfrac{x+2}{x+1}$. Evaluate $g^{-1}(3)$.

4 Suppose $g(x) = \dfrac{x-3}{x-4}$. Evaluate $g^{-1}(2)$.

5 Suppose $f(x) = 3x + 2$. Find a formula for f^{-1}.

6 Suppose $f(x) = 8x - 9$. Find a formula for f^{-1}.

7 Suppose $h(t) = \dfrac{1+t}{2-t}$. Find a formula for h^{-1}.

8 Suppose $h(t) = \dfrac{2-3t}{4+5t}$. Find a formula for h^{-1}.

9 Suppose $f(x) = 2 + \dfrac{x-5}{x+6}$.

(a) Evaluate $f^{-1}(4)$.

(b) Evaluate $[f(4)]^{-1}$.

(c) Evaluate $f(4^{-1})$.

10 Suppose $h(x) = 3 - \dfrac{x+4}{x-7}$.

(a) Evaluate $h^{-1}(9)$.

(b) Evaluate $[h(9)]^{-1}$.

(c) Evaluate $h(9^{-1})$.

11 Suppose $g(x) = x^2 + 4$, with the domain of g being the set of positive numbers. Evaluate $g^{-1}(7)$.

12 Suppose $g(x) = 3x^2 - 5$, with the domain of g being the set of positive numbers. Evaluate $g^{-1}(8)$.

13 Suppose $h(x) = 5x^2 + 7$, where the domain of h is the set of positive numbers. Find a formula for h^{-1}.

14 Suppose $h(x) = 3x^2 - 4$, where the domain of h is the set of positive numbers. Find a formula for h^{-1}.

For each of the functions f given in Exercises 15–24:

(a) ***Find the domain of f.***

(b) ***Find the range of f.***

(c) ***Find a formula for f^{-1}.***

(d) ***Find the domain of f^{-1}.***

(e) ***Find the range of f^{-1}.***

You can check your solutions to part (c) by verifying that $f^{-1} \circ f = I$ and $f \circ f^{-1} = I$ (recall that I is the function defined by $I(x) = x$).

15 $f(x) = 3x + 5$

16 $f(x) = 2x - 7$

17 $f(x) = \dfrac{1}{3x+2}$

18 $f(x) = \dfrac{4}{5x-3}$

19 $f(x) = \dfrac{2x}{x+3}$

20 $f(x) = \dfrac{3x-2}{4x+5}$

21 $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 4x & \text{if } x \geq 0 \end{cases}$

22 $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

23 $f(x) = x^2 + 8$, where the domain of f equals $(0, \infty)$.

24 $f(x) = 2x^2 + 5$, where the domain of f equals $(0, \infty)$.

25 Suppose $f(x) = x^5 + 2x^3$. Which of the numbers listed below equals $f^{-1}(8.10693)$?

1.1, 1.2, 1.3, 1.4

[For this particular function, it is not possible to find a formula for $f^{-1}(y)$.]

26 Suppose $f(x) = 3x^5 + 4x^3$. Which of the numbers listed below equals $f^{-1}(0.28672)$?

0.2, 0.3, 0.4, 0.5

[For this particular function, it is not possible to find a formula for $f^{-1}(y)$.]

For Exercises 27–28, use the U. S. 2011 federal income tax function for a single person as defined in Example 2 of Section 1.1.

27 What is the taxable income of a single person who paid \$10,000 in federal taxes for 2011?

28 What is the taxable income of a single person who paid \$20,000 in federal taxes for 2011?

29 Suppose $g(x) = x^7 + x^3$. Evaluate

$$(g^{-1}(4))^7 + (g^{-1}(4))^3 + 1.$$

30 Suppose $g(x) = 8x^9 + 7x^3$. Evaluate

$$8(g^{-1}(5))^9 + 7(g^{-1}(5))^3 - 3.$$

PROBLEMS

31 The exact number of meters in y yards is $f(y)$, where f is the function defined by

$$f(y) = 0.9144y.$$

(a) Find a formula for $f^{-1}(m)$.

(b) What is the meaning of $f^{-1}(m)$?

32 The exact number of kilometers in M miles is $f(M)$, where f is the function defined by

$$f(M) = 1.609344M.$$

(a) Find a formula for $f^{-1}(k)$.

(b) What is the meaning of $f^{-1}(k)$?

33 A temperature F degrees Fahrenheit corresponds to $g(F)$ degrees on the Kelvin temperature scale, where

$$g(F) = \tfrac{5}{9}F + 255.37.$$

(a) Find a formula for $g^{-1}(K)$.

(b) What is the meaning of $g^{-1}(K)$?

(c) Evaluate $g^{-1}(0)$. (This is absolute zero, the lowest possible temperature, because all molecular activity stops at 0 degrees Kelvin.)

34 Suppose g is the federal income tax function given by Example 2 of Section 1.1. What is the meaning of the function g^{-1}?

35 Suppose f is the function whose domain is the set of real numbers, with f defined on this domain by the formula

$$f(x) = |x + 6|.$$

Explain why f is not a one-to-one function.

36 Suppose g is the function whose domain is the interval $[-2, 2]$, with g defined on this domain by the formula

$$g(x) = (5x^2 + 3)^{7777}.$$

Explain why g is not a one-to-one function.

37 Show that if f is the function defined by $f(x) = mx + b$, where $m \neq 0$, then f is a one-to-one function.

38 Show that if f is the function defined by $f(x) = mx + b$, where $m \neq 0$, then the inverse function f^{-1} is defined by the formula $f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}$.

39 Consider the function h whose domain is the interval $[-4, 4]$, with h defined on this domain by the formula

$$h(x) = (2 + x)^2.$$

Does h have an inverse? If so, find it, along with its domain and range. If not, explain why not.

40 Consider the function h whose domain is the interval $[-3, 3]$, with h defined on this domain by the formula

$$h(x) = (3 + x)^2.$$

Does h have an inverse? If so, find it, along with its domain and range. If not, explain why not.

41 Suppose f is a one-to-one function. Explain why the inverse of the inverse of f equals f. In other words, explain why

$$(f^{-1})^{-1} = f.$$

42 The function f defined by

$$f(x) = x^5 + x^3$$

is one-to-one (here the domain of f is the set of real numbers). Compute $f^{-1}(y)$ for four different values of y of your choice.
[*For this particular function, it is not possible to find a formula for* $f^{-1}(y)$.]

43 Suppose f is a function whose domain equals $\{2, 4, 7, 8, 9\}$ and whose range equals $\{-3, 0, 2, 6, 7\}$. Explain why f is a one-to-one function.

44 Suppose f is a function whose domain equals $\{2, 4, 7, 8, 9\}$ and whose range equals $\{-3, 0, 2, 6\}$. Explain why f is not a one-to-one function.

45 Show that the composition of two one-to-one functions is a one-to-one function.

46 Give an example to show that the sum of two one-to-one functions is not necessarily a one-to-one function.

47 Give an example to show that the product of two one-to-one functions is not necessarily a one-to-one function.

48 Give an example of a function f such that the domain of f and the range of f both equal the set of integers, but f is not a one-to-one function.

49 Give an example of a one-to-one function whose domain equals the set of integers and whose range equals the set of positive integers.

The ideas used in the example above apply to any function defined by a table, as summarized below.

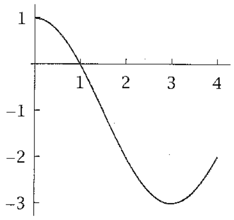
Inverse functions via tables

Suppose f is a function defined by a table. Then:

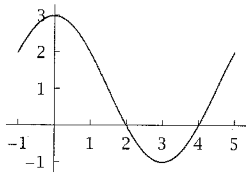
- f is one-to-one if and only if the table defining f has no repetitions in the second column.
- If f is one-to-one, then the table for f^{-1} is obtained by interchanging the columns of the table defining f.

EXERCISES

For Exercises 1–12, use the following graphs:



The graph of f.



The graph of g.

Here f has domain $[0, 4]$ and g has domain $[-1, 5]$.

1 What is the largest interval contained in the domain of f on which f is increasing?

2 What is the largest interval contained in the domain of g on which g is increasing?

3 Let F denote the function obtained from f by restricting the domain to the interval in Exercise 1. What is the domain of F^{-1}?

4 Let G denote the function obtained from g by restricting the domain to the interval in Exercise 2. What is the domain of G^{-1}?

5 With F as in Exercise 3, what is the range of F^{-1}?

6 With G as in Exercise 4, what is the range of G^{-1}?

7 What is the largest interval contained in the domain of f on which f is decreasing?

8 What is the largest interval contained in the domain of g on which g is decreasing?

9 Let H denote the function obtained from f by restricting the domain to the interval in Exercise 7. What is the domain of H^{-1}?

10 Let J denote the function obtained from g by restricting the domain to the interval in Exercise 8. What is the domain of J^{-1}?

11 With H as in Exercise 9, what is the range of H^{-1}?

12 With J as in Exercise 10, what is the range of J^{-1}?

For Exercises 13–36 suppose f and g are functions, each with domain of four numbers, with f and g defined by the tables below:

x	$f(x)$
1	4
2	5
3	2
4	3

x	$g(x)$
2	3
3	2
4	4
5	1

13 What is the domain of f?

14 What is the domain of g?

15 What is the range of f?

16 What is the range of g?

17 Sketch the graph of f.

18 Sketch the graph of g.

19 Give the table of values for f^{-1}.

20 Give the table of values for g^{-1}.

21 What is the domain of f^{-1}?

22 What is the domain of g^{-1}?

23 What is the range of f^{-1}?

24 What is the range of g^{-1}?

25 Sketch the graph of f^{-1}.

26 Sketch the graph of g^{-1}.

27 Give the table of values for $f^{-1} \circ f$.

28 Give the table of values for $g^{-1} \circ g$.

29 Give the table of values for $f \circ f^{-1}$.

30 Give the table of values for $g \circ g^{-1}$.

31 Give the table of values for $f \circ g$.

32 Give the table of values for $g \circ f$.

33 Give the table of values for $(f \circ g)^{-1}$.

34 Give the table of values for $(g \circ f)^{-1}$.

35 Give the table of values for $g^{-1} \circ f^{-1}$.

36 Give the table of values for $f^{-1} \circ g^{-1}$.

PROBLEMS

37 Suppose f is the function whose domain is the interval $[-2, 2]$, with f defined by the following formula:

$$f(x) = \begin{cases} -\frac{x}{3} & \text{if } -2 \le x < 0 \\ 2x & \text{if } 0 \le x \le 2. \end{cases}$$

(a) Sketch the graph of f.

(b) Explain why the graph of f shows that f is not a one-to-one function.

(c) Give an explicit example of two distinct numbers a and b such that $f(a) = f(b)$.

38 Draw the graph of a function that is increasing on the interval $[-2, 0]$ and decreasing on the interval $[0, 2]$.

39 Draw the graph of a function that is decreasing on the interval $[-2, 1]$ and increasing on the interval $[1, 5]$.

40 Give an example of an increasing function whose domain is the interval $[0, 1]$ but whose range does not equal the interval $[f(0), f(1)]$.

41 Show that the sum of two increasing functions is increasing.

42 Give an example of two increasing functions whose product is not increasing.
[*Hint:* There are no such examples where both functions are positive everywhere.]

43 Give an example of two decreasing functions whose product is increasing.

44 Show that the composition of two increasing functions is increasing.

45 Explain why it is important as a matter of social policy that the income tax function g in Example 2 of Section 1.1 be an increasing function.

46 Suppose the income tax function in Example 2 of Section 1.1 is changed so that

$$g(x) = 0.15x - 450 \quad \text{if } 8500 < x \le 34500,$$

with the other parts of the definition of g left unchanged. Show that if this change is made, then the income tax function g would no longer be an increasing function.

47 Suppose the income tax function in Example 2 of Section 1.1 is changed so that

$$g(x) = 0.14x - 425 \quad \text{if } 8500 < x \le 34500,$$

with the other parts of the definition of g left unchanged. Show that if this change is made, then the income tax function g would no longer be an increasing function.

48 Explain why an even function whose domain contains a nonzero number cannot be a one-to-one function.

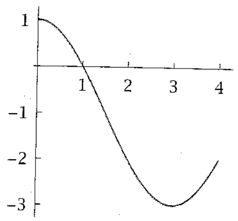
49 The solutions to Exercises 33 and 35 are the same, suggesting that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

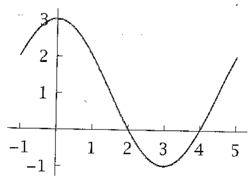
Explain why the equation above holds whenever f and g are one-to-one functions such that the range of g equals the domain of f.

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

For Exercises 1–12, use the following graphs:



The graph of f.



The graph of g.

Here f has domain $[0,4]$ and g has domain $[-1,5]$.

1 What is the largest interval contained in the domain of f on which f is increasing?

SOLUTION As can be seen from the graph, $[3, 4]$ is the largest interval on which f is increasing.

As usual when obtaining information solely from graphs, this answer (as well as the answers to the other parts of this exercise) should be considered an approximation. An expanded graph at a finer scale