PRACTICE EXAM

1. (Limits) Use the $\epsilon - \delta$ definition of a limit to show that $\lim_{x\to 3} 5x - 7 = 8$.

   Solution: Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{5}$. Then if $0 < |x-3| < \delta$, we have that

   $$|(5x-7)-8| = |5x-15| = 5|x-3| < 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

   Therefore, by the definition of limits, we have that $\lim_{x\to 3} 5x - 7 = 8$.

2. (Exponents) Compute

   $$\frac{4^{1000}\cdot(-\frac{1}{2})^{2015}}{(\frac{1}{2})^{20}}$$

   Solution:

   $$\frac{4^{1000}\cdot(-\frac{1}{2})^{2015}}{(\frac{1}{2})^{20}} = \frac{2^{2000}\cdot 2^{-2015}\cdot(-1)^{2015}}{2^{-20}} = -\frac{2^{-15}}{2^{-20}} = -2^5 = -32.$$

3. (Inequalities) Find all x such that $\frac{1}{2x+1} > \frac{3}{4x+5}$.

   Solution:

   $$\frac{1}{2x+1} - \frac{3}{4x+5} > 0,$$
   $$\frac{-2(x-1)}{(2x+1)(4x+5)} > 0,$$
   $$x \in (-\infty, -\frac{5}{4}) \cup (-\frac{1}{2}, 1)$$

4. (Inverse Functions/Quadratics) Let $g(x) = x^2 - 15x + 100$. Find $g^{-1}(50)$ in each of the following cases if it exists.

   (a) Domain of g is $(-\infty, 7)$.

   (b) Domain of g is $(11, 20)$.

   (c) Domain of g is $(-20, 20)$.

   Solution:

   $$x^2 - 15x + 100 = 50 \qquad \Rightarrow \qquad x = 5, 10$$

   (a) Only 5 belongs to the domain of g. Therefore, $g^{-1}(50) = 5$.

   (b) Neither 5 nor 10 belong to the domain of g. Therefore, $g^{-1}(50)$ does not exist.

   (c) Both 5 and 10 belong to the domain of g. Therefore, the horizontal line test fails and $g^{-1}(50)$ does not exist.

5. (Quadratics) Let $f(x) = x^2 - 6x + 8$.

   (a) Write $f(x)$ in the vertex form and state its vertex.
   (b) Find the largest interval containing 0 such that f is invertible.
   (c) Find f^{-1} for that domain.
   (d) Find the domain and the range of f^{-1}

   Solution:

   (a) Complete the square to find the vertex form for f:
   $f(x) = x^2 - 6x + 8 = x^2 - 6x + 9 - 1 = (x-3)^2 - 1$. So the vertex of f is $(3, -1)$.
   (b) In order for f to be invertible, it should be 1-1. So we need to find the largest interval containing 0 such that f is 1-1. The largest intervals on which f is 1-1 are $[3, \infty)$ and $(-\infty, 3]$. $(-\infty, 3]$ is the one containing 0.
   (c) $y = (x-3)^2 - 1 \;\Rightarrow\; y + 1 = (x-3)^2$
   $\Rightarrow\; \pm\sqrt{y+1} = x - 3$
   Because the domain of f is $(-\infty, 3]$, we should only consider $-\sqrt{y+1}$. So, $-\sqrt{y+1} = x - 3 \;\Rightarrow\; -\sqrt{y+1} + 3 = x$
   So, $f^{-1}(y) = -\sqrt{y+1} + 3$.
   (d) The range of f^{-1} is the domain of f: So it is $(-\infty, 3]$ and the domain of f^{-1} is the range of f: So it is $[-1, \infty]$.

6. (Absolute Values)

   (a) Find all x such that $|x-3| + |x+1| = 5$.
   (b) Using the geometrical interpretation of $|\cdot|$ explain why none of your answers were in the interval $[-1, 3]$.

   Solution:

   (a) Since $|x - a|$ represents the distance between a and x, we are looking for a point such that its distance to 3 plus its distance to -1 add to 5.

   First, we know that the distance between 3 and -1 is 4, then the number cannot be between -1 and 3 (there, the sum is always 4).

   Now, if $x \leq -1$ then its distance to 3 is going to be the distance it has to -1 plus the remaining distance it has to 3 in other words:

$$|x-3| + |x+1| = |x-3| + |x-(-1)| =$$
$$|x-(-1)| + |3-(-1)| + |x-(-1)| = 4 + 2|x-(-1)|.$$

   So we are looking for an x such that twice its distance to -1 plus 4 is 5, or such that twice its distance to -1 is $5 - 4 = 1$. Therefore, the number we're looking for has a distance of $\frac{1}{2}$ from -1 and it is less than it. It has to be $-1 - \frac{1}{2} = -\frac{3}{2}$.
   If $x \geq 3$ we can follow similar reasoning, so we will need a number bigger that 3 such that its distance to 3 is $\frac{1}{2}$. This number is $3 + \frac{1}{2} = \frac{7}{2}$.

(b) We know that the distance between 3 and -1 is 4, so $|x-3|+|x+1|$ cannot take the value 5 between -1 and 3, for every number in that interval, $|x-3|+|x+1|=4$

7. (Distance/Lines/Quadratics) What point on the graph of $y = 2x+1$ is closest to $(0,-4)$? Compute that distance.

   Solution: The closest point to $(0,-4)$ on the line $y = 2x+1$ is the one that is the intersection between $y = 2x+1$ and a perpendicular line to it going through $(0,-4)$.

   We know that the perpendicular line must have slope $\frac{-1}{2}$, so we have the equation

$$\frac{-1}{2} = \frac{y-(-4)}{x-0}$$

   solving for y we got $y = \frac{-1}{2}x - 4$.

   Now, to find the intersection, we set the line equations equal. Then

$$2x+1 = \frac{-1}{2}x - 4;\ 2x + \frac{1}{2}x = -4-1;\ \frac{5}{2}x = -5;\ x = -5 \cdot \frac{2}{5} = -2.$$

   Since $x = -2$ we have that $y = 2(-2)+1 = -3$.

   Finally, the distance between the two points is

$$\sqrt{(0-(-2))^2 + (-4-(-3))^2} = \sqrt{(2)^2+(-1)^2} = \sqrt{4+1} = \sqrt{5}$$