Chapter 12
Section 5
Lines and Planes
in Space

Example 1

Show that the line through the points $(0,1,1)$ and $(1,-1,6)$ is perpendicular to the line through the points $(-4,2,1)$ and $(-1,6,2)$.

Vector equation for the first line:

$$\begin{aligned}\mathbf{r}_1(t) &\doteq < 0,1,1 > + t\ (< 1,-1,6 > - < 0,1,1 >) \\ &= < 0,1,1 > + t\ < 1,-2,5 >\end{aligned}$$

Vector equation for the second line:

$$\begin{aligned}\mathbf{r}_2(s) &\doteq < -4,2,1 > + s\ (< -1,6,2 > - < -4,2,1 >) \\ &= < -4,2,1 > + s\ < 3,4,1 >\end{aligned}$$

$$\begin{aligned}\boxed{\cos\theta} &= \frac{< 1,-2,5 > \cdot < 3,4,1 >}{|< 1,-2,5 >|\ |< 3,4,1 >|} \\ &= \frac{(1)(3)+(-2)(4)+(5)(1)}{\sqrt{1^2+(-2)^2+5^2}\ \sqrt{3^2+4^2+1^2}} \\ &= \frac{0}{\sqrt{30}\,\sqrt{26}} = \boxed{0}\end{aligned}$$

Remark: These two lines are *skew*.

Example 2

(a) Find parametric equations for the line through $(5, 1, 0)$ that is perpendicular to the plane $2x - y + z = 1$

A normal vector to the plane is:

$$\mathbf{n} =< 2, -1, 1 >$$
$$\mathbf{r}(t) =< 5, 1, 0 > + t < 2, -1, 1 >$$

(b) In what points does this line intersect the coordinate planes?

xy-plane: $0 \doteq 0 + t\,1$

$$t = 0 \Rightarrow \mathbf{r}(0) =< 5, 1, 0 >$$

yz-plane: $0 \doteq 5 + t\,2$

$$t = \frac{-5}{2} \Rightarrow \mathbf{r}(\frac{-5}{2}) =< 0, \frac{7}{2}, \frac{-5}{2} >$$

zx-plane: $0 \doteq 1 + t\,(-1)$

$$t = 1 \Rightarrow \mathbf{r}(1) =< 7, 0, 1 >$$

Example 3

Parallelism, intersection for:

$$
\begin{aligned}
L_1: &\quad \frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{4} \\
\mathbf{r}_1(t) &\quad =< 1, 0, 1 > + t \; < 2, 1, 4 > \\
&\quad =< 1 + 2t, t, 1 + 4t > \\
L_2: &\quad \frac{x}{1} = \frac{y+2}{2} = \frac{z+2}{3} \\
\mathbf{r}_2(s) &\quad =< 0, -2, -2 > + s \; < 1, 2, 3 > \\
&\quad =< s, -2 + 2s, -2 + 3s >
\end{aligned}
$$

$$
\begin{aligned}
&< 2, 1, 4 > \times < 1, 2, 3 > \\
&= < -5, -2, 3 > \neq \mathbf{0} \;\Rightarrow\; \boxed{L_1 \nparallel L_2}
\end{aligned}
$$

$$\mathbf{r}_1(t) \doteq \mathbf{r}_2(s)$$

$$
\begin{aligned}
&< 1 + 2t, t, 1 + 4t > \\
&\doteq < s, -2 + 2s, -2 + 3s >
\end{aligned}
$$

$$
\begin{aligned}
1 + 2t &= s \\
t &= -2 + 2s \\
1 + 4t &= -2 + 3s
\end{aligned}
$$

Solving the first two equations:

$$t = 0, \quad s = 1$$

Checking the third equation:

$$1 + 4(0) = -2 + 3(1) \quad \text{(satisfied)}$$

Consequently:

$$\boxed{L_1 \cap L_2} \;=\; \boxed{\{\mathbf{r}_1(0)\} = \{\mathbf{r}_2(1)\} = \{< 1, 0, 1 >\}}$$

Example 4

Plane through $(2,1,-3)$, $(5,-1,4)$, $(2,-2,4)$

$$\overrightarrow{(2,1,-3)\,(5,-1,4)} = <3,-2,7>$$
$$\overrightarrow{(2,1,-3)\,(2,-2,4)} = <0,-3,7>$$
$$\mathbf{n} \doteq <3,-2,7> \times <0,-3,7>$$
$$= <7,-21,-9>$$

Equation for plane:

$$(\mathbf{r}- <2,1,-3>) \cdot \mathbf{n} = 0$$
$$(x-2)(7) + (y-1)(-21) + (z+3)(-9) = 0$$
$$7x - 21y - 9z$$
$$+ ((-2)(7) + (-1)(-21) + (3)(-9)) = 0$$
$$\boxed{7x - 21y - 9z = 20}$$

Example 5

Plane through the point $(-1, 0, 1)$
and the line $x = 5t,\ y = 1 + t,\ z = -t$

$$\mathbf{r}(t) = < 0, 1, 0 > + t < 5, 1, -1 >$$

$$\mathbf{r}(0) = < 0, 1, 0 >$$

$$\overrightarrow{(-1, 0, 1)\,(0, 1, 0)} = < 1, 1, -1 >$$

$$\mathbf{n} \doteq < 1, 1, -1 > \times < 5, 1, -1 >$$

$$= < 0, -4, -4 >$$

Equation for plane:

$$(\mathbf{r} - < -1, 0, 1 >) \cdot \mathbf{n} = 0$$

$$(x - (-1))(0) + (y - 0)(-4) + (z - 1)(-4) = 0$$

$$0x - 4y - 4z$$

$$-((-1)(0) + (0)(-4) + (1)(-4)) = 0$$

$$-4y - 4z + 4 = 0$$

$$\boxed{y + z = 1}$$

Example 6

Intersection of line and plane:

$$\text{Line: } x = 1 - t,\ y = t,\ z = 1 + t$$
$$\text{Plane: } z = 1 - 2x + y$$

Substitute line in plane equation:

$$(1 + t) = 1 - 2(1 - t) + (t)$$
$$0 = -1 - t + 1 - 2 + 2t + t$$
$$2 = 2t$$
$$t = 1$$

$$\text{Line} \cap \text{Plane} = < 1 - 1, 1, 1 + 1 >= \boxed{< 0, 1, 2 >}$$

Example 7

Direction numbers for intersection of planes:

Plane 1: $x + y + z = 1$

Plane 2: $x + z = 0$

$$\mathbf{n}_1 = < 1, 1, 1 >$$
$$\mathbf{n}_2 = < 1, 0, 1 >$$

Line direction numbers:

$$\mathbf{n}_1 \times \mathbf{n}_2 = < 1, 1, 1 > \times < 1, 0, 1 >$$
$$= \boxed{< 1, 0, -1 >}$$

Unit vector:

$$\mathbf{u} = < \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} >$$

Example 8

Intersection of planes:

Plane 1: $x - 2y + z = 1$

Plane 2: $2x + y + z = 1$

$$\begin{aligned} \mathbf{n}_1 &= < 1, -2, 1 > \\ \mathbf{n}_2 &= < 2, 1, 1 > \end{aligned}$$

Line direction numbers:

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= < 1, -2, 1 > \times < 2, 1, 1 > \\ &= \boxed{< -3, 1, 5 >} \end{aligned}$$

Common point: $(0, 0, 1)$

Symmetric equations:

$$\frac{x - 0}{-3} = \frac{y - 0}{1} = \frac{z - 1}{5}$$

$$\boxed{-\frac{x}{3} = y = \frac{z - 1}{5}}$$

Example 9

Plane of points equidistant from $(1,1,0)$, $(0,1,1)$

Midpoint $= \frac{1}{2}(< 1,1,0 > + < 0,1,1 >) =< \frac{1}{2}, 1, \frac{1}{2} >$

Normal $=< 1,1,0 > - < 0,1,1 >=< 1,0,-1 >$

Equation:

$$\begin{aligned}(\mathbf{r}- < \frac{1}{2}, 1, \frac{1}{2} >)\cdot < 1,0,-1 > &= 0\\ (x-\frac{1}{2})(1)+(y-1)(0)+(z-\frac{1}{2})(-1) &= 0\\ (x-\frac{1}{2})-(z-\frac{1}{2}) &= 0\\ x-z &= 0\\ x &= z\end{aligned}$$

The plane has the equation $x = z$

Example 10

Find an equation for the plane with
x-intercept a, y-intercept b, z-intercept c.

Given points in plane:

$$\begin{aligned}
P_x &= (a, 0, 0) \\
P_y &= (0, b, 0) \\
P_z &= (0, 0, c) \\
\overrightarrow{P_z\,P_x} &=< a, 0, -c > \\
\overrightarrow{P_z\,P_y} &=< 0, b, -c >
\end{aligned}$$

Normal $\mathbf{n} = \overrightarrow{P_z\,P_x} \times \overrightarrow{P_z\,P_y}$

$$\mathbf{n} =< bc, ca, ab >$$

Equation:

$$\begin{aligned}
0 &= (\mathbf{r}- < 0, 0, c >) \cdot \mathbf{n} \\
0 &=< x, y, z - c > \cdot < bc, ca, ab > \\
0 &= bcx + cay + ab(z - c) \\
abc &= bcx + cay + abz
\end{aligned}$$

If $a \cdot b \cdot c \neq 0$:

$$\mathbf{n} = (abc) \cdot < \frac{1}{a}, \frac{1}{b}, \frac{1}{c} >$$

$$1 = \left(\frac{bc}{abc}\right) x + \left(\frac{ca}{abc}\right) y + \left(\frac{ab}{abc}\right) z$$

$$1 = ax + by + cz$$

Example 11

Find parametric equations for the line L through $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t,\ y = 1 - t,\ z = 2t$.

Principal task = Find direction $\mathbf{v}$ of L:

Let $\mathbf{v} =< a, b, c >\neq \mathbf{0}$

$$
\begin{array}{ll}
 & \mathbf{v}\ \parallel\ \text{(Plane)} \\
\therefore & \mathbf{v} \perp \text{(Normal(Plane))} \\
\therefore & \mathbf{v} \perp\ < 1, 1, 1 > \\
 & \mathbf{v} \perp \text{Direction(Line))} \\
\therefore & \mathbf{v} \perp\ < 1, -1, 2 > \\
\therefore & \mathbf{v}\ \propto < 1, 1, 1 > \times < 1, -1, 2 >
\end{array}
$$

Let $\mathbf{v} =< 1, 1, 1 > \times < 1, -1, 2 >=< 3, -1, -2 >$

Vector equation of L :

$$\mathbf{r}(t)\ =< 0, 1, 2 > +\ t < 3, -1, -2 >$$

Parametric equations of L :

$$\boxed{x = 3t,\ y = 1 - t,\ z = 2 - 2t}$$

Example 12

Find equations of the planes parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

The distance D between parallel planes $ax+by+cz+d_1 = 0$ and $ax+by+cz+d_2 = 0$ is

$$\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

$$< a, b, c > = < 1, 2, -2 >$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

$$D = \frac{|d_2 - (-1)|}{3} \doteq 2$$

$$|d_2 - (-1)| \doteq 6$$

$$\therefore \quad d_2 \doteq (-1) \pm 6 = \begin{cases} 5 \\ -7 \end{cases}$$

$$\boxed{x + 2y - 2z + 5 = 0} \quad \boxed{x + 2y - 2z - 7 = 0}$$

Example 13

$$\text{Line } L_1 : x = y = z$$
$$\text{Direction } \mathbf{v}_1 = < 1, 1, 1 >$$

$$\text{Line } L_2 : x + 1 = \frac{y}{2} = \frac{z}{3}$$
$$\text{Direction } \mathbf{v}_2 = < 1, 2, 3 >$$

$$P \in (L_1 \bigcap L_2) \Leftrightarrow P(x, x, x) \,\text{has}\, x + 1 = \frac{x}{2} = \frac{x}{3}$$

No solution! $\therefore$ L_1 and L_2 are skew.

Cross product: $\mathbf{n} \doteq \mathbf{v}_1 \times \mathbf{v}_1 =< 1, -2, 1 >$

P_1 : $P_1 \ni L_1$ and $P_1 \parallel L_2$

P_2 : $P_2 \ni L_2$ and $P_2 \parallel L_1$

Then: $Distance(L_1, L_2) = Distance(P_1, P_2)$

$$L_1 \ni (1,1,1)$$

$$P_1: \; <1,-2,1> \cdot <x,y,z> \\ - <1,-2,1> \cdot <1,1,1> = 0 \\ \boxed{x - 2y + z + 0 = 0}$$

$$L_2 \ni (0,2,3)$$

$$P_1: \; <1,-2,1> \cdot <x,y,z> \\ - <1,-2,1> \cdot <0,2,3> = 0 \\ \boxed{x - 2y + z + 1 = 0}$$

Distance formula:

$$D = \frac{|1-0|}{\sqrt{(1)^2 + (-2)^2 + (1)^2}} = \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$$

Example 14 Geometric descriptions

(a) $x + y + z = c$, c real:
Family of planes orthogonal to the line $x = y = z$.

(b) $x + y + cz = 1$, c real:
Family of planes containing the line $x + y = 1, z = 0$.
Plane is vertical, if $c = 0$.
Else, plane has z-intercept $\frac{1}{c}$.

(c) $y \cos\theta + z \sin\theta = 1$, θ real:
Family of planes parallel to x-axis, orthogonal to $< 0, \cos\theta, \sin\theta >$, containing the point $P(0, \cos\theta, \sin\theta)$.

Alternatively:
Family of planes parallel to x-axis, tangent to the cylinder $y^2 + z^2 = 1$.
For given θ the plane contains the point $P(0, \cos\theta, \sin\theta)$.