10.2 Classifying Conic Sections by Eccentricity

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Eccentricity and Directrix

Given two real quantities $a > 0$ and $e > 0$ with $e \neq 1$, define the auxiliary quantities $c = a \cdot e$ and $d = \frac{a}{e}$.
Define the **focus** as the point $F(c, 0)$ and the **directrix** as the vertical line D with equation $x = d$.

Consider the set $\mathcal{C} = \mathcal{C}(a, e)$ defined as the set of all points $P(x, y)$ in the plane which satisfy the condition

$$PF \;=\; e \cdot PD\,. \tag{1}$$

The points of $\mathcal{C}(a,e)$ are described by

$$\begin{aligned} (PF)^2 &= e^2 \cdot (PD)^2 \\ (x-c)^2 + (y-0)^2 &= e^2 \cdot (x-d)^2 \\ x^2 - 2cx + c^2 + y^2 &= e^2 \left(x^2 - 2dx + d^2 \right) \\ \left(1-e^2\right) x^2 + y^2 + 2(de^2 - c)x &= e^2 d^2 - c^2 \,. \end{aligned}$$

Since $d\,e^2 = c$ and $d\,e = a$ we find that the points of $\mathcal{C}(a,e)$ are characterized by the equation

$$\boxed{\left(1-e^2\right) x^2 + y^2 = a^2 - c^2 \,.}$$

Elliptic case:

If $0 < e < 1$ it follows that $0 < c < a < d$.

Defining $b > 0$ by $b^2 = a^2 - c^2$ the last equation above implies that $\left(1 - e^2\right) x^2 + y^2 = b^2$.

Since $\frac{b^2}{1-e^2} = a^2$ we obtain the standard equation of an **ellipse**

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.}$$

Hyperbolic case:

If $1 < e$ it follows that $0 < d < a < c$.

Defining $b > 0$ by $b^2 = -a^2 + c^2$ the last equation above implies that $\left(1 - e^2\right) x^2 + y^2 = -b^2$.

Since $\frac{-b^2}{1-e^2} = a^2$ we obtain the standard equation of a **hyperbola**

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.}$$

Summary:

Recall that the **parabola** was defined in terms of a **focus** $F(p, 0)$ where $p > 0$ and the **directrix** D with equation $x = -p$ in terms of the condition

$$\boxed{PF = 1 \cdot PD\,.}$$

Hence a **common** definition for the standard ellipse, parabola, hyperbola is provided by the **focus-directrix** equation

$$\boxed{PF \;=\; e \cdot PD\,,}$$

where for fixed $e > 0$ the set of points in the plane satisfying this condition is

an **ellipse**, if $0 < e < 1$;
a **parabola**, if $e = 1$;
a **hyperbola**, if $e > 1$.