8.8 Improper Integrals

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Type I improper integrals

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx \; + \; \int_c^\infty f(x)\,dx \text{ for any } c \in \mathbb{R}\,.$$

Type II improper integrals

1. If $f(x)$ is continuous on $(a,b]$ and not continuous at a^+, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

2. If $f(x)$ is continuous on $[a,b)$ and not continuous at b^-, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

3. For $a < c < b$, if $f(x)$ is continuous on $[a,c) \cup (c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx\,.$$

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

Limit Comparison Test

Let $f > 0$ and $g > 0$ be continuous on $[a, \infty)$.

If there exists L with $0 < L < \infty$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both **converge** or both **diverge**.

$\boxed{\int_0^1 x^q\,dx}$

Let $0 < \epsilon < 1$. If $q \neq -1$ then

$$\int_\epsilon^1 x^q\,dx = \left.\frac{x^{1+q}}{1+q}\right|_\epsilon^1 = \frac{1^{1+q} - \epsilon^{1+q}}{1+q}$$

If $q = -1$ then $\int_\epsilon^1 x^q\,dx = \ln\frac{1}{\epsilon} = -\ln\epsilon$.

<u>CONCLUSION</u>: For $q \leq -1$ the integral $\int_0^1 x^q\,dx$ is improper of type II and divergent. It is a normal integral for $q > -1$.

$$\int_0^1 x^q\,dx = \begin{cases} (1+q)^{-1}, & \text{if } q > -1; \\ \infty, & \text{if } q \leq -1. \end{cases}$$

$\boxed{\int_1^{\infty} x^q\,dx}$

Let $1 < M$. If $q \neq -1$ then

$$\int_1^M x^q\,dx = \left.\frac{x^{1+q}}{1+q}\right|_1^M = \frac{M^{1+q} - 1^{1+q}}{1+q}$$

If $q = -1$ then $\int_1^M x^q\,dx = \ln \frac{M}{1} = \ln M$.

<u>CONCLUSION</u>: The integral $\int_1^{\infty} x^q\,dx$ is improper of type I. It is divergent for $q \geq -1$ and convergent for $q < -1$.

$$\int_1^{\infty} x^q\,dx = \begin{cases} \infty, & \text{if } q \geq -1; \\ |1+q|^{-1}, & \text{if } q < -1. \end{cases}$$

Exercise 66: $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\, dx \neq \lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}\, dx$

Since the integrand is **odd** one finds for $b > 0$

$$\int_{-b}^{b} \frac{2x}{x^2+1}\, dx = 0 \text{ so that } \lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}\, dx = 0\,.$$

Since

$$\lim_{x\to\infty} \frac{1/x}{2x/(x^2+1)} = \frac{1}{2} \quad \text{and} \quad \int_{1}^{\infty} \frac{1}{x}\, dx = \infty$$

the Limit Comparison Test implies that $\int_{1}^{\infty} \frac{2x}{x^2+1}\, dx = \infty\,.$
Similarly, $\int_{-\infty}^{1} \frac{2x}{x^2+1}\, dx = -\infty\,.$
Thus, the improper integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\, dx$ is divergent, hence $\neq 0\,.$

Exercise 65 a: $\int_1^2 \frac{dx}{x(\ln x)^p}$

Let $0 < \epsilon < 1$. The substitution $u = \ln x$ and $du = \frac{dx}{x}$ yields

$$\int_{1+\epsilon}^{2} \frac{dx}{x(\ln x)^p} = \int_{\ln(1+\epsilon)}^{\ln 2} u^{-p}\, du\,.$$

Since $\lim_{\epsilon \to 0+} \ln(1+\epsilon) = 0$ the results above imply that the improper integral converges only when $-p > -1$ or $p < 1$.

Exercise 65 b: $\int_2^\infty \frac{dx}{x(\ln x)^p}$

Let $2 < M$. The substitution $u = \ln x$ and $du = \frac{dx}{x}$ yields

$$\int_2^M \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln M} u^{-p}\, du\,.$$

Since $\lim_{M\to\infty} \ln M = \infty$ the results above imply that the improper integral converges only when $-p < -1$ or $p > 1$.