2.3 Elementary Matrices; Finding A^{-1}

2.4 Equivalent Matrices

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ELEMENTARY MATRICES

Definition:

An $n \times n$ matrix is **row-elementary** if it can be obtained from the $n \times n$ identity matrix I_n by application of a single elementary row operation.
Denote the set of all row-elementary matrices by $\mathcal{E}_{\rho,n}$.

An $n \times n$ matrix is **column-elementary** if it can be obtained from the $n \times n$ identity matrix I_n by application of a single elementary column operation.
The set of column-elementary matrices is $\mathcal{E}^T_{\rho,n} = (\mathcal{E}_{\rho,n})^T$.

A matrix is **elementary** if it is either row or column elementary.
The set of all elementary matrices is $\mathcal{E}_n = \mathcal{E}_{\rho,n} \cup \mathcal{E}^T_{\rho,n}$.

Each set of elementary matrices is sorted into one of the three types I,II and III according to the elementary operation involved.

Elementary Operations as multiplications with elementary matrices:

Type I: $1 \leq i \neq j, k \leq n,\ A \in M_n,\ B \in M_{n \times r},\ 1 \leq q \leq r$.

If $\widetilde{A}$ and $\widetilde{AB}$ denote the matrices obtained by applying the type I operation $row_i \Leftrightarrow row_j$ to A and AB, respectively, then

$$(\widetilde{AB})_{k\,q} = \begin{cases} (AB)_{k\,q} = \sum_{p=1}^{n} A_{k\,p} B_{p\,q}, & \text{if } i \neq k \neq j; \\ (AB)_{j\,q} = \sum_{p=1}^{n} A_{j\,p} B_{p\,q}, & \text{if } k = i; \\ (AB)_{i\,q} = \sum_{p=1}^{n} A_{i\,p} B_{p\,q}, & \text{if } k = j. \end{cases} \tag{1}$$

$$(\tilde{A}B)_{k\,q} = \sum_{p=1}^{n} \tilde{A}_{k\,p} B_{p\,q} = \begin{cases} \sum_{p=1}^{n} A_{k\,p} B_{p\,q}, & \text{if } i \neq k \neq j; \\ \sum_{p=1}^{n} A_{j\,p} B_{p\,q}, & \text{if } k = i; \\ \sum_{p=1}^{n} A_{i\,p} B_{p\,q}, & \text{if } k = j. \end{cases} \tag{2}$$

Comparison of equations (1) and (2) yields for type I operations

$$\widetilde{AB} \;=\; \tilde{A}B\,.$$

Type II: $1 \leq i, k \leq n,\ c \in \mathbb{R},\ A \in M_n,\ B \in M_{n\times r},\ 1 \leq q \leq r\,.$
If $\tilde{A}$ and $\widetilde{AB}$ denote the matrices obtained by applying the type II operation $row_i \Rightarrow c \cdot row_i$ to A and AB, respectively, then

$$(\widetilde{AB})_{k\,q} = \begin{cases} (AB)_{k\,q} = \sum_{p=1}^{n} A_{k\,p}B_{p\,q}, & \text{if } i \neq k \neq j; \\ c \cdot (AB)_{i\,q} = c \cdot \sum_{p=1}^{n} A_{i\,p}B_{p\,q}, & \text{if } k = i. \end{cases} \tag{3}$$

$$(\tilde{A}B)_{k\,q} = \sum_{p=1}^{n} \tilde{A}_{k\,p}B_{p\,q} = \begin{cases} \sum_{p=1}^{n} A_{k\,p}B_{p\,q}, & \text{if } i \neq k \neq j; \\ \sum_{p=1}^{n} c \cdot A_{i\,p}B_{p\,q}, & \text{if } k = i. \end{cases} \tag{4}$$

Comparison of equations (3) and (4) yields for type II operations

$$\widetilde{AB} \;=\; \tilde{A}B\,.$$

Type III: $1 \le i \ne j, k \le n$, $c \in \mathbb{R}$, $A \in M_n$, $B \in M_{n\times r}$, $1 \le q \le r$. If $\widetilde{A}$ and $\widetilde{AB}$ denote the matrices obtained by applying the type III operation $row_i \Leftrightarrow row_i + c \cdot row_j$ to A and AB, respectively, then

$$(\widetilde{AB})_{k\,q} = \begin{cases} (AB)_{k\,q} = \sum_{p=1}^{n} A_{k\,p} B_{p\,q}, & \text{if } k \ne i; \\ (AB)_{i\,q} = \sum_{p=1}^{n} A_{i\,p} B_{p\,q} + c \sum_{p=1}^{n} A_{j\,p} B_{p\,q}, & \text{if } k = i. \end{cases} \tag{5}$$

$$(\widetilde{A}B)_{k\,q} = \sum_{p=1}^{n} \widetilde{A}_{k\,p} B_{p\,q} = \begin{cases} \sum_{p=1}^{n} A_{k\,p} B_{p\,q}, & \text{if } k \ne i; \\ \sum_{p=1}^{n} (A_{i\,p} + cA_{j\,p}) B_{p\,q}, & \text{if } k = i. \end{cases} \tag{6}$$

Comparison of equations (5) and (6) yields for type III operations

$$\widetilde{AB} \;=\; \widetilde{A}B\,.$$

SUMMARY

If $\widetilde{A}$ and $\widetilde{AB}$ denote the matrices obtained by applying the same elementary row operation to A and AB, respectively, then

$$\widetilde{AB} \;=\; \tilde{A}B\,. \tag{7}$$

Elementary row operations as multiplications:

In the special case, where $A = I_n$ and $E = \widetilde{I_n} \in \mathcal{E}_{\rho\,,n}$ is the elementary matrix obtained by applying a certain elementary row operation to I_n, equation (7) establishes that the result of that elementary row operation applied to B can be computed as the **matrix product** of E and B:

$$\widetilde{B} = \widetilde{I_n B} = \widetilde{I_n} B = EB \tag{8}$$

All elementary row operations acting on elements of $M_{n\times r}$ are thus "internalized" to matrix algebra in the form of left-multiplication by elements of $\mathcal{E}_{\rho\,,n} \subset M_n$. In particular, if $n = r$ "everything happens" in M_n.

Let $< \mathcal{E}_{\rho\,,n} >$ be the set of all **finite** products of elements from $\mathcal{E}_{\rho\,,n}$.
Similarly $< \mathcal{E}^{I}_{\rho\,,n} >$ (resp. $< \mathcal{E}^{II}_{\rho\,,n} >$, $< \mathcal{E}^{III}_{\rho\,,n} >$) is the set of all **finite** products of elements from $\mathcal{E}_{\rho\,,n}$ of type I (resp. type II, type III).

Theorem

$A, B \in M_{m\times n}$ are **row equivalent** if and only if there exists $C \in\ < \mathcal{E}_{\rho\,,n} >$ such that

$$B = CA\,. \tag{9}$$

Theorem

Let $E \in \mathcal{E}_{\rho\,,n}$.
Then E^{-1} exists and is in $\mathcal{E}_{\rho\,,n}$ of the same type as E .

(Type I) If E interchanges $row_i \Leftrightarrow row_j$ where $i \neq j$,
then $E^{-1} = E$.
(Type II) If E multiplies row_i by $c \neq 0$,
then E^{-1} multiplies row_i by $c^{-1} \neq 0$.
(Type III) If E adds $c \cdot row_j$ to row_i where $i \neq j$,
then E^{-1} adds $(-c) \cdot row_j$ to row_i .

Nonsingular Matrices

The set of all **nonsingular** matrices in M_n is called the **general linear group** (in dimension n) and is denoted by GL_n.

Theorem

$$GL_n = < \mathcal{E}_{\rho,n} > \ . \tag{10}$$

PROOF:

By the previous theorem $< \mathcal{E}_{\rho,n} > \subseteq GL_n$.

Let $A \in GL_n$ have the RREF B, so that there exists $F \in < \mathcal{E}_{\rho,n} >$ with $A = FB$. Because A is nonsingular $ker A = \{0\}$ and there are no zero rows in B. It follows that there are n leading ones in B and $B = I_n$. Hence $A = FB = BI_n = F$ and $GL_n \subseteq < \mathcal{E}_{\rho,n} >$.

Thus, $GL_n = < \mathcal{E}_{\rho,n} >$.   QED

Existence and uniqueness of solutions to $Ax = b$

If $A \in GL_n$ the equation $Ax = b$ **has** the **unique** solution $x = A^{-1}b$ for arbitrary $b \in M_{n\times 1}$.

Conversely, supposing that the equation $Ax = 0$ has a unique solution it follows that $ker A = \{0\}$ and that the RREF of A has no zero rows. Hence $A \in < \mathcal{E}_{\rho,n} > = GL_n$.

A is **nonsingular**

if and only if

the equation $Ax = b$ **has** a **unique** solution for every $b \in M_{n\times 1}$.

SUMMARY

The following are equivalent statements regarding $A \in M_n$:

1. A is nonsingular.

2. $Ax = 0$ has only the trivial solution.

3. The RREF of A is I_n.

4. The equation $Ax = b$ **has** a **unique** solution for every $b \in M_{n\times 1}$.

5. A is in $< \mathcal{E}_n >$.

Singular square matrices

The following are equivalent statements regarding $A \in M_n$:

1. A is singular.
2. $Ax = 0$ has a solution $x \neq 0$.
3. The RREF of A has $row_n = 0$.
4. The equation $Ax = b$ either does not have a solution or the solution is not unique.
5. A is not in $< \mathcal{E}_n >$.

Computation of A^{-1} by row reduction:

For a given $A \in M_n$ let B be its RREF. Then there exists $F \in < \mathcal{E}_{\rho,n} >$ with $FA = B$. Either $B = I_n$ and $A^{-1} = F$ or $B \neq I_n$ and A^{-1} does not exist.

Multiplying the matrix $[A \mid I_n] \in M_{n\times 2n}$ on the left by $F \in < \mathcal{E}_{\rho,n} >$ results in the matrix $[B \mid F]$. If A is nonsingular then the "right half" of this matrix is A^{-1}.

Also, if $F = E_k E_{k-1} \dots E_2 E_1$ with $E_s \in \mathcal{E}_{\rho,n}$, $1 \le s \le k$, then $F^{-1} = E_1^{-1} E_2^{-1} \dots E_{k-1}^{-1} E_k^{-1}$ and $A = F^{-1} B$.

Theorem

If A, $B \in M_n$ satisfy either $AB = I_n$ or $BA = I_n$ then $B = A^{-1}$ and $B^{-1} = A$.

Proof:

We may assume that $AB = I_n$ since the conclusion is symmetric in A and B and an interchange of A and B will lead to this hypothesis. **If A were singular** it could be written in the form $A = FC$ with $F \in < \mathcal{E}_{\rho,n} >$ and C in RREF and having its last row equal to the zero row. Then CB has its last row equal to the zero row and there exists a nonzero $0 \neq x \in M_{n \times 1}$ with $CBx = 0$. Then $0 \neq x = I_n x = ABx = (FC)Bx = F(CB)x = F0 = 0$. This contradiction proves that the hypothesis of A being singular is untenable and A^{-1} exists. Then $A^{-1} = A^{-1} I_n = A^{-1}(AB) = (A^{-1}A)B = I_n B = B$. QED

Equivalent matrices
Definition:
Two $m \times n$ matrices A and B are **equivalent** if there exist $F \in <\mathcal{E}_{\rho,n}>$ and $G \in <\mathcal{E}^T_{\rho,n}>$ such that

$$A = FBG\,. \tag{11}$$

Remark: This is an **equivalence relation** in the sense that it has the following three properties

Refexivity: Any A is equivalent to itself.

Symmetry: If A is equivalent to B, then B is equivalent to A.

Transitivity: If A is equivalent to B, and B is equivalent to C, then A is equivalent to C.

Theorem

Any nonzero $m \times n$ matrix A is equivalent to a matrix of the form

$$\begin{bmatrix} I_r & 0_{r,\,n-r} \\ 0_{m-r,\,r} & 0_{m-r,\,n-r} \end{bmatrix} \quad \text{where } 1 \le r \le \min(m,n)\,. \tag{12}$$

Theorem

Two $m \times n$ matrices A and B are equivalent if and only if there exist $P \in GL(m)$ and $Q \in GL(n)$ such that

$$A = PBQ\,. \tag{13}$$

Theorem

An $n \times n$ matrix A is nonsingular if and only if it is equivalent to I_n.