

Noise Tradeoffs, Stationary Information Flow, and Structural Balance in Unit-Birth Networks

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Abstract

In 2019, Paulsson and collaborators conjectured that stochastic biochemical control networks exhibit fundamental limits on how much intrinsic noise can be simultaneously suppressed across multiple components. While supported by simulations and partial analytic results, a general proof remained unavailable. Recently, Ripsman, Kell, and Hilfinger proposed a formal proof strategy for unit-birth models based on a stationary information-theoretic decomposition. The purpose of this paper is to provide the corresponding rigorous mathematical justification.

We consider continuous-time Markov chains on $\mathbb{Z}_{\geq 0}^N$ in which each component is degraded linearly and is produced in unit births at a state-dependent rate that may depend on the other components, but not directly on itself. Noise in component i is measured by the Fano factor F_{X_i} , the ratio of stationary variance to stationary mean, with the Poisson value 1 as the natural baseline. Our first contribution is to isolate explicit hypotheses on moments, mean birth rates, and growth of the total jump rate under which the formal information-flow identities can be rigorously justified on the infinite state space. Under these hypotheses, and following the proof outline of Ripsman, Kell, and Hilfinger, we prove that the resulting tradeoff

$$\sum_{i=1}^N \frac{F_{X_i} - 1}{\tau_i} \geq 0$$

holds for *any* network topology and *any* allowable choice of rate functions. Thus uniform sub-Poissonian noise suppression is impossible: if $F_{X_i} < 1$ for some i , then $F_{X_j} > 1$ for some $j \neq i$.

Our second contribution is to make these hypotheses checkable: a uniform positive lower bound on the birth rates together with at-most-linear total growth dominated by the weakest degradation rate suffices, which we prove with the use of a Foster–Lyapunov function.

Our third contribution is a structural strengthening. Under a signed monotonicity condition on the rate functions, satisfied in particular by structurally balanced signed interaction networks, we prove that the stationary distribution is associated with respect to the corresponding signed partial order. This upgrades the global tradeoff to the termwise bound $F_{X_i} \geq 1$ for every i . Consequently, within the signed-monotone subclass, sub-Poissonian noise requires a frustrated interaction topology.

1 Introduction

Stochastic reaction networks are a standard modeling framework for biochemical systems in which molecular species are present in low copy numbers [12, 15, 23, 10, 17, 19]. In such systems, individual

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production and degradation events can generate substantial fluctuations in molecular abundances, and these fluctuations may propagate through regulatory networks. Such fluctuations can affect cellular switching, signaling, timing, and the reliability of downstream responses [15, 17, 19].

To discuss whether regulatory networks can suppress these fluctuations, one first needs a baseline. For a single species produced at a constant rate and degraded linearly, the stationary distribution is Poisson [3]; in particular, its variance equals its mean. Thus the Fano factor

$$F_X := \frac{\text{Var}(X)}{\mathbb{E}[X]}$$

equals one. This Poisson value is the natural noise baseline for the models considered here. Following the terminology used in the noise-suppression literature, we will say that a component is sub-Poissonian if $F_X < 1$ and super-Poissonian if $F_X > 1$; Yan, Hilfinger, Vinnicombe, and Paulsson, for example, refer to components below the Poisson baseline as exhibiting sub-Poisson fluctuations [25].

Biological networks, however, are rarely collections of independent birth-death processes. One species may regulate another through transcriptional regulation, signaling pathways, sequestration, enzymatic activity, or other interaction mechanisms [15, 21, 24, 18, 5]. It is therefore natural to ask whether a sufficiently well-designed regulatory network could suppress fluctuations in many components simultaneously. For example, if X_1 happens to fluctuate upward, one could imagine that this increases the production of a second component X_2 , which in turn represses the production of X_1 and thereby pushes it back toward its mean. More complicated networks could use many such interactions, and one might imagine that the collective effect is to reduce the noise level of every component.

A natural question is whether such uniform suppression is possible. That is, given the freedom to choose any topology and any nonlinear rate functions, can one design an interaction network whose components all have Fano factors below the Poisson baseline? Paulsson and collaborators conjectured that, for a broad class of models, the answer is no [25]. Their conjecture allows for various production mechanisms, including bursty production, while restricting to linear degradation. The present paper treats the unit-birth case, in which each production event increments a single component by one. In biochemical terms, the variables X_i may be interpreted as protein copy numbers or molecular species abundances: each component is degraded linearly, while its production rate may be regulated arbitrarily by the other components, but not directly by itself. Thus the model excludes direct self-feedback in production, while still allowing arbitrary indirect feedback through the rest of the network.

The conjectured principle is striking because of its generality: it is not a statement about a particular topology, a particular choice of rate functions, or a perturbative regime. Rather, it says that no matter how the components regulate one another’s production rates, the network cannot push every component below the Poisson noise baseline at stationarity. Equivalently, if one component has $F_{X_i} < 1$, then at least one other component must have $F_{X_j} > 1$. See Figure 1(a) for a visual representation in the two-component case.

Until recently, no general proof strategy existed for this conjecture, even for restricted settings. A recent formal proof strategy of Ripsman, Kell, and Hilfinger [20] identifies the correct mechanism behind this tradeoff. Their argument decomposes stationary information-theoretic quantities, including mutual information rates, into directional information flows associated with the different components. At a formal level, these flows satisfy a global cancellation identity, and this cancellation combines with componentwise information inequalities to yield the desired noise tradeoff.

One purpose of the present paper is to provide a rigorous mathematical foundation for this argument in the unit-birth model. The main difficulty is that the natural state space is infinite.

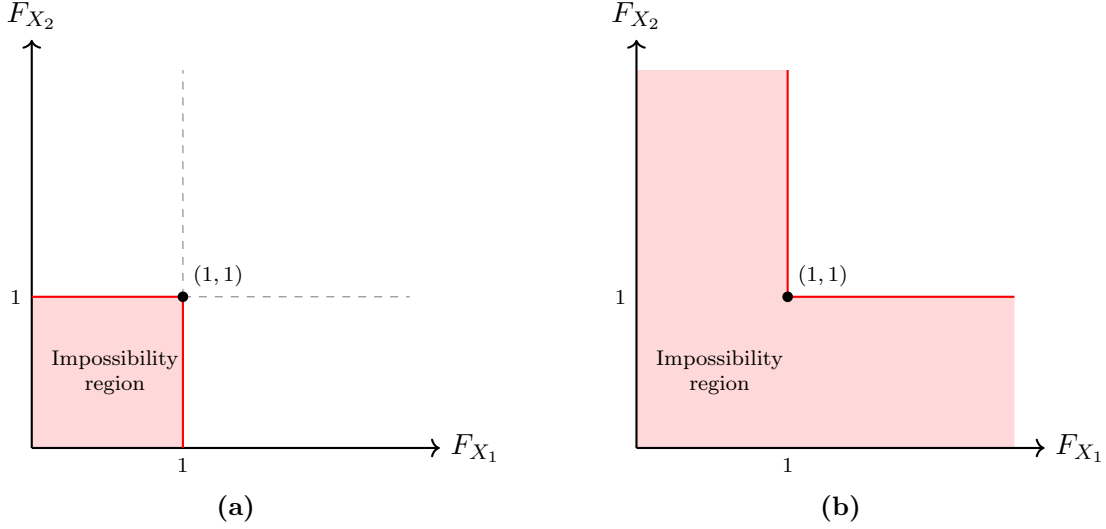


Figure 1: Impossibility regions in (F_{X_1}, F_{X_2}) space for two-component unit-birth networks. (a) Illustrates the impossibility of simultaneous sub-Poissonian noise suppression: for any network topology and any allowable rate functions, the two Fano factors cannot both lie below the Poisson baseline 1 (Theorem 1). The theorem in fact establishes the stronger weighted tradeoff $(F_{X_1} - 1)/\tau_1 + (F_{X_2} - 1)/\tau_2 \geq 0$, which forbids a half-plane bounded by a slanted line through $(1, 1)$ whose slope depends on the timescale ratio τ_1/τ_2 ; the unit square shown here is the topology- and timescale-independent piece that holds for any choice of τ_i . Hilfinger et al. [20] plot the equivalent quantity $CV_i/\sqrt{1/\langle X_i \rangle} = \sqrt{F_{X_i}}$; the corresponding regions are the same. (b) Under the additional signed monotonicity hypothesis (Assumption 5.1), each Fano factor must individually satisfy $F_{X_i} \geq 1$ (Theorem 3); the impossibility region enlarges accordingly.

Consequently, the entropy-like quantities appearing in the information-flow decomposition involve logarithms of stationary probabilities and conditional stationary probabilities, and the formal manipulations require interchanging infinite sums, applying generators to unbounded log-potentials, and justifying stationary generator identities. For example, the log-potential $g(x) = -\log \pi(x)$ is unbounded, and the stationary distribution π enters our arguments only through the standing hypotheses rather than via any explicit formula, so the integrability of expressions involving g requires careful verification.

Our first contribution, therefore, is to isolate explicit hypotheses under which the information-flow argument is valid. The most important moment assumption is a finite second-moment condition, supplemented by natural regularity conditions such as irreducible positive recurrence, full support of the stationary distribution, finite positive mean birth rates, and mild total-rate growth. Under these hypotheses we rigorously justify the formal identities used by Ripsman, Kell, and Hilfinger. In particular, we prove the exact Fano-covariance identity

$$F_{X_i} = 1 + \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\mathbb{E}_\pi[f_i(X_{-i})]},$$

where f_i is the production rate of component i and X_{-i} denotes the vector X with its i th coordinate removed, derive the componentwise information inequality, establish the stationary global

cancellation identity, and conclude the tradeoff

$$\sum_{i=1}^N \frac{F_{X_i} - 1}{\tau_i} \geq 0.$$

This is the first impossibility result of the paper: within this framework, uniform sub-Poissonian noise suppression is impossible. If one component has sub-Poissonian noise, then the deficit must be balanced by super-Poissonian noise in at least one other component.

Our second contribution is to make the hypotheses checkable. We give concrete sufficient conditions on the birth-rate functions that imply the standing assumptions used in the proof. In particular, a uniform positive lower bound on the birth rates, together with at-most-linear growth of the total birth rate with linear coefficient dominated by the weakest degradation rate, yields nonexplosion, positive recurrence, finite moments, and the integrability needed for the information-flow identities. This separates the abstract stationary identities from the model-level conditions that guarantee they apply.

Our third contribution is a new structural theorem identifying a broad topological regime in which the global tradeoff strengthens to a termwise bound. Specifically, we show that for networks satisfying a signed monotonicity condition, the global inequality upgrades to the following:

$$F_{X_i} \geq 1 \quad \text{for every } i.$$

A broad graph-theoretic class satisfying this condition is given by structurally balanced signed interaction networks, namely networks whose signed interaction graph contains no cycle with an odd number of negative edges. See Figure 2 for examples. In this case, a suitable signed order makes the stochastic dynamics order-preserving. We prove that the stationary distribution is associated with respect to this order, in the sense of Esary et al. and of Lindqvist [11, 14], and use the exact Fano-covariance identity to show that each covariance

$$\text{Cov}_\pi(f_i(X_{-i}), X_i)$$

is nonnegative. See Figure 1(b) for the corresponding impossibility region.

This gives a *second* impossibility result, strictly stronger than the first: in a structurally balanced network, even componentwise sub-Poissonian noise suppression is impossible. Thus, within the class of models whose interactions admit a global sign description, any attempt to construct a component with $F_{X_i} < 1$ must use a frustrated interaction structure, meaning a signed interaction graph that is not structurally balanced. In this sense, frustration is not merely one way to obtain noise suppression; it is a necessary mechanism for sub-Poissonian noise in this class. This places a hard structural limit on what regulatory architecture can achieve.

The remainder of the paper is organized as follows. Section 2 introduces the unit-birth model, the standing hypotheses, and the main tradeoff theorem, and then gives checkable sufficient conditions under which those hypotheses hold. Section 3 proves the technical generator and stationary identities needed for the information-flow calculation. Some of these are general countable-state Markov chain tools, while the final subsection applies them to the unit-birth logarithmic potentials, information flows, and marginal flux balance. Section 4 provides the proof of the main result. Specifically, it establishes the componentwise information inequalities, the cancellation of stationary information flows, and the resulting global Fano-factor tradeoff. Section 5 proves the termwise bound under signed monotonicity and relates this condition to structurally balanced interaction graphs and monotone-systems theory.

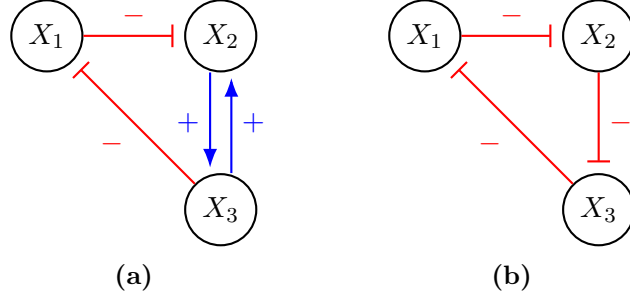


Figure 2: Examples of signed interaction graphs for $N = 3$ component networks. Activating edges are drawn in blue with an arrowhead and labeled $+$; inhibiting edges are drawn in red with a T-bar terminator and labeled $-$. (a) A structurally balanced network: every cycle contains an even number of negative edges and Theorem 3 gives the termwise bound $F_{X_i} \geq 1$ for every i . (b) The repressilator of Elowitz and Leibler [9], in which each species represses the next. The signed interaction graph is a directed 3-cycle with three negative edges; it is not structurally balanced, and the signed monotonicity hypothesis fails.

2 The unit-birth model: hypotheses, main theorem, and sufficient conditions

In this section we introduce the model, fix notation, and state the standing hypotheses under which the stationary identities below are justified. We then state the main “noise tradeoff” theorem for unit-birth models, and give a concrete sufficient condition showing that the standing hypotheses hold for a broad class of unit-birth models.

2.1 The unit-birth model

Before defining the model, we fix a few pieces of notation. For a continuous-time Markov chain on a countable state space S , we write $q(x, y)$ for the transition rate from x to y . For $x \in \mathbb{R}^N$, we write x_i for the i th coordinate and x_{-i} for the vector obtained by deleting the i th coordinate.

We will require that any model found in this paper satisfies the following regularity conditions that can be found as Condition 1 in [2].

Condition 1. We require the following.

- For $x, y \in S$ with $x \neq y$, we have $0 \leq q(x, y) < \infty$.
- For each $y \in S$, we have $\sum_{x \in S \setminus \{y\}} q(y, x) < \infty$.
- For each $x \in S$, we have $\sup_{y \in S \setminus \{x\}} q(y, x) < \infty$.

We now turn to our model of interest. Fix an integer $N \geq 2$. Let X be a continuous-time Markov chain on

$$S := \mathbb{Z}_{\geq 0}^N$$

with generator L acting on functions $\varphi : S \rightarrow \mathbb{R}$ by

$$(L\varphi)(x) := \sum_{i=1}^N f_i(x_{-i})(\varphi(x + e_i) - \varphi(x)) + \sum_{i=1}^N \frac{x_i}{\tau_i}(\varphi(x - e_i) - \varphi(x)), \quad (1)$$

where $\tau_i > 0$, and e_i is the i th coordinate vector, and with the convention that the death summand is omitted when $x_i = 0$. Thus coordinate i undergoes births at rate $f_i(x_{-i})$ (depending only on the other coordinates) and deaths at rate x_i/τ_i , with $f_i : \mathbb{Z}_{\geq 0}^{N-1} \rightarrow \mathbb{R}_{\geq 0}$. Because the model has only single-coordinate birth and death transitions, Condition 1 is automatic whenever the rates $f_i(v)$ are finite. For later use, we write

$$L = \sum_{i=1}^N L_i,$$

where

$$(L_i \varphi)(x) := f_i(x_{-i})(\varphi(x + e_i) - \varphi(x)) + \frac{x_i}{\tau_i}(\varphi(x - e_i) - \varphi(x)), \quad (2)$$

with the death term omitted when $x_i = 0$.

For a generic Markov chain with transition rates $q(x, y)$, we will denote the total jump rate out of state x as

$$\Lambda(x) := \sum_{y \neq x} q(x, y). \quad (3)$$

Hence, for our model defined above, we have

$$\Lambda(x) = \sum_{i=1}^N f_i(x_{-i}) + \sum_{i=1}^N \frac{x_i}{\tau_i}.$$

Also define

$$s(x) := \|x\|_1 = \sum_{j=1}^N x_j. \quad (4)$$

Let π denote a stationary distribution for X (when it exists), and write $\mathbb{E}_\pi[\cdot]$ for expectation under π . For a fixed $i \in \{1, \dots, N\}$, we will often have need to separate out the i th component. We therefore perform the following slight abuse of notation. For a given $x \in \mathbb{Z}_{\geq 0}^N$ with $x_i = n$ and $x_{-i} = v$, we will often write $\pi(x) = \pi(n, v)$. The meaning should always be clear in context.

Let π_i denote the marginal law of X_i and, for each n with $\pi_i(n) > 0$, let π_n^i denote the conditional law of X_{-i} given $X_i = n$:

$$\pi_i(n) := \sum_{v \in \mathbb{Z}_{\geq 0}^{N-1}} \pi(n, v), \quad \pi_n^i(v) := P_\pi(X_{-i} = v \mid X_i = n) = \frac{\pi(n, v)}{\pi_i(n)}. \quad (5)$$

We will also use the logarithmic potentials

$$g(n, v) := -\log \pi(n, v), \quad g_i(n, v) := -\log \pi_i(n), \quad h_i(n, v) := -\log \pi_n^i(v). \quad (6)$$

Under Assumption 2.1, the stationary distribution has full support, and hence these quantities are finite for every $(n, v) \in S$. Moreover, since

$$\pi(n, v) = \pi_i(n) \pi_n^i(v),$$

we have, for each $i \in \{1, \dots, N\}$, the decomposition

$$g(n, v) = g_i(n, v) + h_i(n, v). \quad (7)$$

Define

$$m_i(n) := \mathbb{E}_{\pi_n^i}[f_i(X_{-i})], \quad \mu_i := \mathbb{E}_\pi[f_i(X_{-i})] = \mathbb{E}_\pi[m_i(X_i)]. \quad (8)$$

Finally, define the Fano factor

$$F_{X_i} := \frac{\text{Var}_\pi(X_i)}{\mathbb{E}_\pi[X_i]}.$$

2.2 Standing hypotheses

The following standing hypotheses will be used throughout the paper. They give a convenient rigorous framework in which the stationary identities and information-flow decompositions developed by Hilfinger et al. [20] can be proved. The hypotheses are tailored to the unit-birth setting, but several of the technical lemmas below are stated in a form that applies more generally to countable-state continuous-time Markov chains satisfying the corresponding regularity and integrability conditions.

Assumption 2.1 (Standing hypotheses). Assuming Condition 1 is satisfied, we also assume the following:

(A0) *Irreducibility and stationarity.* The CTMC X is irreducible on S and admits a stationary distribution π .

(A1) *Finite second moments.* $\mathbb{E}_\pi[X_i^2] < \infty$ for each i . Equivalently, $\mathbb{E}_\pi[s(X)^2] < \infty$.

(A2) *Finite, positive mean birth rates.* For each i ,

$$\mu_i = \mathbb{E}_\pi[f_i(X_{-i})] \in (0, \infty).$$

(A3) *At-most-linear total rate growth.* There exists a constant $C_\Lambda < \infty$ such that

$$\Lambda(x) \leq C_\Lambda(1 + s(x)) \quad \forall x \in S.$$

Two simple consequences of Assumption 2.1 will be used repeatedly below. First, since the chain is irreducible and admits a stationary distribution under (A0), the stationary distribution has full support: $\pi(x) > 0$ for every $x \in S$. Thus the conditional laws and logarithmic potentials appearing below are well defined. Second, although nonexplosion is not included as a standing hypothesis, it follows from (A0), (A1), and (A3) by Lemma 1.

2.3 Statement of the main theorem

We now state the main theorem. Recall the marginal law π_i , the conditional laws π_n^i , the mean birth rates μ_i , and the Fano factors F_{X_i} defined above in (5) and (8). By the full-support consequence of (A0), discussed above, $\pi_i(n) > 0$ for every $n \geq 0$, so the conditional laws π_n^i are well defined for every $n \geq 0$. When focusing on coordinate i , we continue to write $x = (n, v)$, where $n = x_i$ and $v = x_{-i}$.

Following the probability-current formulation of Ripsman, Kell, and Hilfinger [20], we define the stationary probability current across the i -birth edge $(n, v) \rightarrow (n+1, v)$ by

$$J_i(n+1, n; v) := \pi(n, v)f_i(v) - \pi(n+1, v)\frac{n+1}{\tau_i}. \quad (9)$$

Note that if the chain satisfies detailed balance, then each J_i vanishes. Although we do not assume detailed balance, stationarity does imply a weaker marginal balance across each adjacent pair of levels of X_i . Namely,

$$\sum_v J_i(n+1, n; v) = 0.$$

Equivalently,

$$\pi_i(n)m_i(n) = \frac{n+1}{\tau_i}\pi_i(n+1). \quad (10)$$

Thus the net probability flux between the marginal levels $X_i = n$ and $X_i = n + 1$ vanishes, even though the individual conditional fluxes $J_i(n + 1, n; v)$ need not vanish. The marginal balance (10) will be proved in Lemma 6 below and will be used in the proof of part (3) of Theorem 1 below.

Also following [20], we define the single information flow as follows.

Definition 1 (Single information flow). Assume that the series below is absolutely convergent. Define

$$\begin{aligned} (\dot{I})_{X_i} &:= \sum_{n \geq 0} \sum_{v \in \mathbb{Z}_{\geq 0}^{N-1}} J_i(n + 1, n; v) \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)} \\ &= \sum_{n \geq 0} \sum_v \left(\pi(n, v) f_i(v) - \pi(n + 1, v) \frac{n + 1}{\tau_i} \right) (h_i(n, v) - h_i(n + 1, v)), \end{aligned} \quad (11)$$

where the second equality utilizes the notation from (6).

Lemma 5 below shows that under Assumption 2.1 the series is indeed absolutely convergent, and gives a useful representation.

We now present the main theorem for the unit-birth model (1).

Theorem 1 (Noise tradeoff for the unit-birth model). *Consider the unit-birth CTMC on $S = \mathbb{Z}_{\geq 0}^N$ with generator (1). Suppose Assumption 2.1 holds, and let π be its stationary distribution. Then the following statements hold.*

(1) For each i ,

$$\mathbb{E}_\pi[X_i] = \tau_i \mu_i, \quad (12)$$

and

$$F_{X_i} = 1 + \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\mu_i}. \quad (13)$$

(2) The single information flow $(\dot{I})_{X_i}$ defined in (11) is well defined, and for each i ,

$$\frac{F_{X_i} - 1}{\tau_i} \geq (\dot{I})_{X_i}. \quad (14)$$

(3) We have zero total information flow:

$$\sum_{i=1}^N (\dot{I})_{X_i} = 0. \quad (15)$$

Consequently, we have the global tradeoff

$$\sum_{i=1}^N \frac{F_{X_i} - 1}{\tau_i} \geq \sum_{i=1}^N (\dot{I})_{X_i} = 0. \quad (16)$$

In particular, it is impossible to have $F_{X_i} \leq 1$ for all i with strict inequality for at least one index: if $F_{X_k} < 1$ for some k , then there exists $j \neq k$ such that $F_{X_j} > 1$.

2.4 Sufficient conditions for Assumption 2.1

We record a concrete sufficient condition for Assumption 2.1. The class is broad: the individual birth-rate functions may be state dependent and unbounded, but their total growth must be at most linear and not strong enough to overwhelm the weakest linear death rate. The uniform lower bound ensures that all upward moves remain accessible.

Theorem 2 (Lower bounds and controlled growth imply Assumption 2.1). *Consider the continuous time Markov chain on $S = \mathbb{Z}_{\geq 0}^N$ with generator (1). Assume there exists $\varepsilon \in (0, 1)$ such that*

$$\varepsilon \leq f_i(v) \quad \text{for all } i \in \{1, \dots, N\} \text{ and all } v \in \mathbb{Z}_{\geq 0}^{N-1}. \quad (17)$$

Assume further that there exist constants $A, B \in [0, \infty)$ such that, for all $x \in S$,

$$\sum_{i=1}^N f_i(x_{-i}) \leq A + B s(x), \quad s(x) = x_1 + \dots + x_N, \quad (18)$$

and that

$$B < \frac{1}{\tau_{\max}}, \quad \tau_{\max} := \max_{1 \leq i \leq N} \tau_i. \quad (19)$$

Then Condition 1 holds, and the corresponding CTMC satisfies Assumption 2.1.

Proof. First note that Condition 1 holds. The rates $q(x, y)$ are finite because the birth rates are nonnegative and satisfy

$$\sum_{i=1}^N f_i(x_{-i}) \leq A + B s(x) < \infty,$$

and the death rates x_i/τ_i are finite. The total rate out of x is finite for the same reason. Finally, for fixed x , there are at most $2N$ states y with $q(y, x) > 0$, namely $y = x - e_i$ or $y = x + e_i$ when these states belong to S , and each corresponding rate is finite. Hence

$$\sup_{y \neq x} q(y, x) < \infty.$$

Thus Condition 1 is satisfied.

We now verify items (A0)-(A3) of Assumption 2.1.

(A0) *Irreducibility and stationarity, and (A1) finite second moments.*

By (17), from any state x and for every coordinate i , the birth transition $x \rightarrow x + e_i$ occurs at strictly positive rate. Also, whenever $x_i > 0$, the death transition $x \rightarrow x - e_i$ occurs at rate $x_i/\tau_i > 0$. Hence any two states $x, y \in S$ can be connected by a finite sequence of $\pm e_i$ moves, each having strictly positive rate. Therefore the chain is irreducible on S .

Let

$$V(x) := (1 + s(x))^2.$$

A birth increases s by 1, so it changes V by

$$(1 + s(x) + 1)^2 - (1 + s(x))^2 = 2s(x) + 3.$$

A death decreases s by 1, so it changes V by

$$(1 + s(x) - 1)^2 - (1 + s(x))^2 = -2s(x) - 1.$$

Therefore

$$(LV)(x) = (2s(x) + 3) \sum_{i=1}^N f_i(x_{-i}) - (2s(x) + 1) \sum_{i=1}^N \frac{x_i}{\tau_i}.$$

Using (18) and

$$\sum_{i=1}^N \frac{x_i}{\tau_i} \geq \frac{s(x)}{\tau_{\max}},$$

we get

$$\begin{aligned} (LV)(x) &\leq (2s(x) + 3)(A + Bs(x)) - (2s(x) + 1) \frac{s(x)}{\tau_{\max}} \\ &= \left(2B - \frac{2}{\tau_{\max}}\right) s(x)^2 + \left(2A + 3B - \frac{1}{\tau_{\max}}\right) s(x) + 3A. \end{aligned}$$

By (19), the coefficient of $s(x)^2$ is strictly negative. Since $V(x) = (1 + s(x))^2$, there exist constants $c > 0$, $b < \infty$, and $R < \infty$ such that

$$(LV)(x) \leq -cV(x) + b\mathbf{1}_{\{s \leq R\}}(x) \quad \forall x \in S.$$

The set $\{x \in S : s(x) \leq R\}$ is finite. Hence the standard Foster–Lyapunov criterion for countable-state CTMCs (see [8]) implies nonexplosion and positive recurrence, and therefore existence of a stationary distribution π ; moreover,

$$\mathbb{E}_{\pi}[V(X)] < \infty.$$

Together with the irreducibility proved above, this proves (A0). Since $V(X) = (1 + s(X))^2$, it also gives

$$\mathbb{E}_{\pi}[s(X)^2] < \infty,$$

which is equivalent to (A1).

(A2)*Finite, positive mean birth rates.* The lower bound (17) gives

$$\mu_i = \mathbb{E}_{\pi}[f_i(X_{-i})] \geq \varepsilon > 0.$$

For finiteness, (18) implies

$$f_i(x_{-i}) \leq \sum_{j=1}^N f_j(x_{-j}) \leq A + B s(x).$$

Therefore

$$\mu_i \leq A + B \mathbb{E}_{\pi}[s(X)] < \infty,$$

where $\mathbb{E}_{\pi}[s(X)] < \infty$ follows from (A1). Thus

$$\mu_i \in (0, \infty)$$

for each i , proving (A2).

(A3)*At-most-linear total rate growth.* For $x \in S$, the total transition rate is

$$\Lambda(x) = \sum_{i=1}^N f_i(x_{-i}) + \sum_{i=1}^N \frac{x_i}{\tau_i}.$$

Using (18) and

$$\sum_{i=1}^N \frac{x_i}{\tau_i} \leq \frac{s(x)}{\tau_{\min}}, \quad \tau_{\min} := \min_{1 \leq i \leq N} \tau_i,$$

we obtain

$$\Lambda(x) \leq A + B s(x) + \frac{s(x)}{\tau_{\min}} \leq C_{\Lambda}(1 + s(x)),$$

for example with $C_{\Lambda} := A + B + \tau_{\min}^{-1}$. This proves (A3).

Thus (A0)–(A3) hold, and so Assumption 2.1 holds. $\square$

3 Stationary Identities and Logarithmic Integrability

The proof of the main theorem relies on stationary generator identities for unbounded logarithmic test functions. This section collects the estimates needed to justify those identities on the infinite state space. We first record general stationarity tools for countable-state continuous-time Markov chains, and then apply them to the logarithmic potentials introduced above.

We first record that Assumption 2.1 gives the integrability of the total exit rate needed below.

Lemma 1 (Λ -integrability and nonexplosion). *Consider the unit-birth CTMC on $S = \mathbb{Z}_{\geq 0}^N$ with transition rates $q(x, y)$ and total exit rate*

$$\Lambda(x) := \sum_{y \neq x} q(x, y).$$

Suppose Assumption 2.1 holds. Then

$$\sum_{x \in S} \pi(x) \Lambda(x) = \mathbb{E}_{\pi}[\Lambda(X)] < \infty. \quad (20)$$

In particular, by [2, Theorem 2], the CTMC is nonexplosive.

Proof. By (A3),

$$\mathbb{E}_{\pi}[\Lambda(X)] \leq C_{\Lambda}(1 + \mathbb{E}_{\pi}[s(X)]).$$

By Jensen's inequality and (A1),

$$\mathbb{E}_{\pi}[s(X)] \leq (\mathbb{E}_{\pi}[s(X)^2])^{1/2} < \infty.$$

Therefore $\mathbb{E}_{\pi}[\Lambda(X)] < \infty$, proving (20). The nonexplosion conclusion follows from [2, Theorem 2]. $\square$

The arguments below repeatedly use the stationarity identity

$$\mathbb{E}_{\pi}[(L\varphi)(X)] = 0$$

for test functions that are not necessarily bounded. The next lemma gives natural hypotheses under which this identity is valid. Although the ingredients in the proof are standard, we have not found a reference that states the particular form needed here. We therefore include the short proof, both for completeness and to make clear exactly what must be verified for the logarithmic test functions appearing in the information-flow identities. We note that the lemma is a general statement for continuous-time Markov chains on countable state spaces and is not specific to the unit-birth model. For the unit-birth model, the stationarity, full-support, and Λ -integrability hypotheses are supplied by Assumption 2.1 and Lemma 1; the remaining task in later applications is to verify the displayed $L^1(\pi)$ jump condition (21) for the particular test function under consideration.

Lemma 2 (Stationarity identity for $L^1(\pi)$ test functions). *Consider a CTMC on a countable state space S with transition rates $q(x, y)$ and total exit rate $\Lambda(x) = \sum_{y \neq x} q(x, y)$. Suppose π is a stationary distribution with full support and*

$$\mathbb{E}_\pi[\Lambda(X)] < \infty.$$

Let $\varphi : S \rightarrow \mathbb{R}$ satisfy

$$\sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) |\varphi(y) - \varphi(x)| < \infty. \quad (21)$$

Then, for every $x \in S$, the series

$$\sum_{y \neq x} q(x, y) |\varphi(y) - \varphi(x)|$$

is finite, and hence

$$(L\varphi)(x) := \sum_{y \neq x} q(x, y) (\varphi(y) - \varphi(x))$$

is absolutely convergent and well defined. Moreover, $L\varphi \in L^1(\pi)$ and

$$\mathbb{E}_\pi[(L\varphi)(X)] = 0.$$

Proof. Set

$$h(x) := \sum_{y \neq x} q(x, y) |\varphi(y) - \varphi(x)|.$$

By the assumed $L^1(\pi)$ jump condition (21), $h \in L^1(\pi)$. Since π has full support, it follows that $h(x) < \infty$ for every $x \in S$. Hence, for every $x \in S$, the series defining

$$(L\varphi)(x) := \sum_{y \neq x} q(x, y) (\varphi(y) - \varphi(x))$$

is absolutely convergent and well defined.

We now prove the desired stationarity identity. The argument has two steps: first we prove the result for bounded truncations of φ , and then we pass to the limit in $L^1(\pi)$.

For $M \in \mathbb{N}$, let

$$\varphi^{(M)} := (-M) \vee (\varphi \wedge M).$$

Then $|\varphi^{(M)}| \leq M$, and hence

$$|(L\varphi^{(M)})(x)| \leq \sum_{y \neq x} q(x, y) |\varphi^{(M)}(y) - \varphi^{(M)}(x)| \leq 2M\Lambda(x).$$

Since $\mathbb{E}_\pi[\Lambda(X)] < \infty$, this shows that $L\varphi^{(M)} \in L^1(\pi)$.

We claim that

$$\mathbb{E}_\pi[(L\varphi^{(M)})(X)] = 0.$$

Indeed, since $|\varphi^{(M)}| \leq M$ and $\mathbb{E}_\pi[\Lambda(X)] < \infty$, the sums below are absolutely convergent, and therefore

$$\begin{aligned} \mathbb{E}_\pi[(L\varphi^{(M)})(X)] &= \sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) (\varphi^{(M)}(y) - \varphi^{(M)}(x)) \\ &= \sum_{x \in S} \sum_{y \neq x} \pi(x) q(x, y) \varphi^{(M)}(y) - \sum_{x \in S} \pi(x) \Lambda(x) \varphi^{(M)}(x). \end{aligned}$$

Interchanging the order of summation in the first term gives

$$\sum_{x \in S} \sum_{y \neq x} \pi(x) q(x, y) \varphi^{(M)}(y) = \sum_{y \in S} \varphi^{(M)}(y) \sum_{x \neq y} \pi(x) q(x, y).$$

Since π is stationary and $\mathbb{E}_\pi[\Lambda(X)] < \infty$, the stationary forward equation holds entry-by-entry:

$$\sum_{x \neq y} \pi(x) q(x, y) = \pi(y) \Lambda(y), \quad y \in S,$$

see, for example, Norris [16, Section 3.5, p. 117]. Thus the two terms above are equal, and so

$$\mathbb{E}_\pi[(L\varphi^{(M)})(X)] = 0 \quad \text{for every } M \in \mathbb{N}. \quad (22)$$

It remains to pass to the limit. For $x, y \in S$ with $y \neq x$, define

$$D_M(x, y) := (\varphi^{(M)}(y) - \varphi^{(M)}(x)) - (\varphi(y) - \varphi(x)).$$

For each fixed x and $y \neq x$, we have $D_M(x, y) \rightarrow 0$ as $M \rightarrow \infty$. Moreover, since the truncation map $r \mapsto (-M) \vee (r \wedge M)$ is 1-Lipschitz, we have $|\varphi^{(M)}(y) - \varphi^{(M)}(x)| \leq |\varphi(y) - \varphi(x)|$ and so

$$|D_M(x, y)| \leq 2|\varphi(y) - \varphi(x)|, \quad \text{for all } x \text{ and } y \neq x.$$

Hence

$$q(x, y)|D_M(x, y)| \leq 2q(x, y)|\varphi(y) - \varphi(x)|.$$

For fixed x , the right-hand side is summable in y , since

$$\sum_{y \neq x} 2q(x, y)|\varphi(y) - \varphi(x)| = 2h(x) < \infty.$$

Therefore dominated convergence, applied to the counting measure sum over $y \neq x$, gives

$$(L\varphi^{(M)})(x) - (L\varphi)(x) = \sum_{y \neq x} q(x, y) D_M(x, y) \rightarrow 0 \quad \text{for every } x \in S, \text{ as } M \rightarrow \infty.$$

The same estimate also gives, for every $x \in S$,

$$|(L\varphi^{(M)})(x) - (L\varphi)(x)| \leq \sum_{y \neq x} q(x, y) |D_M(x, y)| \leq 2h(x).$$

Since $h \in L^1(\pi)$, dominated convergence with respect to π yields

$$L\varphi^{(M)} - L\varphi \rightarrow 0 \quad \text{in } L^1(\pi). \quad (23)$$

In particular, $L\varphi \in L^1(\pi)$. Combining (22) and (23), we obtain

$$|\mathbb{E}_\pi[(L\varphi)(X)]| \leq \mathbb{E}_\pi \left[\left| (L\varphi)(X) - (L\varphi^{(M)})(X) \right| \right] \rightarrow 0.$$

Thus $\mathbb{E}_\pi[(L\varphi)(X)] = 0$, as claimed. $\square$

3.1 Logarithmic integrability estimates

The next estimates are stated for the lattice state space $S = \mathbb{Z}_{\geq 0}^N$, but they do not use the unit-birth structure except through the moment and total-rate growth assumptions. In particular, the weighted entropy estimate is a statement about the stationary law π , while the subsequent generator estimate applies to any CTMC on S whose total exit rate Λ satisfies the corresponding integrability bounds. In the unit-birth model, these hypotheses are supplied by Assumption 2.1.

Lemma 3 (Weighted entropy tail and Λ -weighted integrability). *Let π be a probability distribution on $S = \mathbb{Z}_{\geq 0}^N$ with full support satisfying $\mathbb{E}_\pi[s(X)^2] < \infty$, where $s(x) = \|x\|_1$. Then*

$$\sum_{x \in S} \pi(x) (1 + s(x)) |\log \pi(x)| < \infty. \quad (24)$$

Moreover, if $\Lambda : S \rightarrow [0, \infty)$ satisfies

$$\Lambda(x) \leq C_\Lambda (1 + s(x)) \quad \forall x \in S, \quad (25)$$

then

$$\sum_{x \in S} \pi(x) \Lambda(x) (1 + |\log \pi(x)|) < \infty. \quad (26)$$

Proof. Note that under (25), the inequality (26) follows immediately from (24). Hence it suffices to prove (24).

Fix $\rho \in (0, 1)$ and define a reference probability measure on S by

$$Q_\rho(x) := (1 - \rho)^N \rho^{s(x)}, \quad x \in S.$$

Let $\beta = \mathbb{E}_\pi[1 + s(X)]$, which is finite by $\mathbb{E}_\pi[s(X)^2] < \infty$ and Jensen's inequality, and define the “tilted” probability measure

$$\pi^*(x) := \frac{(1 + s(x))\pi(x)}{\beta}.$$

Since $Q_\rho(x) > 0$ for all x , we have $\pi^* \ll Q_\rho$, and hence $D_{\text{KL}}(\pi^* \| Q_\rho) \geq 0$, i.e.

$$-\sum_x \pi^*(x) \log \pi^*(x) \leq -\sum_x \pi^*(x) \log Q_\rho(x).$$

Now $\log Q_\rho(x) = N \log(1 - \rho) + s(x) \log \rho$, so

$$-\sum_x \pi^*(x) \log Q_\rho(x) = -N \log(1 - \rho) - (\log \rho) \mathbb{E}_{\pi^*}[s(X)].$$

Moreover,

$$\mathbb{E}_{\pi^*}[s(X)] = \frac{\mathbb{E}_\pi[(1 + s(X))s(X)]}{\beta} \leq \frac{\mathbb{E}_\pi[(1 + s(X))^2]}{\beta} \leq \frac{\mathbb{E}_\pi[2(1 + s(X)^2)]}{\beta} < \infty,$$

where finiteness follows from the assumed finite second moment. Therefore

$$H(\pi^*) := -\sum_x \pi^*(x) \log \pi^*(x) < \infty.$$

Next, note that

$$\log \pi^*(x) = \log \pi(x) + \log(1 + s(x)) - \log \beta$$

which we rearrange and use the triangle inequality to obtain

$$|\log \pi(x)| \leq |\log \pi^*(x)| + \log(1 + s(x)) + |\log \beta|.$$

Multiplying by $(1 + s(x))\pi(x)$ and summing over x yields

$$\begin{aligned} \sum_x \pi(x)(1 + s(x)) |\log \pi(x)| &\leq \sum_x \pi(x)(1 + s(x)) |\log \pi^*(x)| + \sum_x \pi(x)(1 + s(x)) \log(1 + s(x)) \\ &\quad + |\log \beta| \sum_x \pi(x)(1 + s(x)) \\ &= \beta \sum_x \pi^*(x) |\log \pi^*(x)| + \sum_x \pi(x)(1 + s(x)) \log(1 + s(x)) + |\log \beta| \beta. \end{aligned}$$

Since $H(\pi^*) < \infty$ implies $\sum_x \pi^*(x) |\log \pi^*(x)| < \infty$, the first term is finite. For the second term, use the elementary bound $(1 + r) \log(1 + r) \leq (1 + r^2)$ for $r \geq 0$ and the assumed finite second moment to conclude finiteness. This proves (24). $\square$

We next apply the preceding estimate to the logarithmic potential $g(x) = -\log \pi(x)$. This is the first place where the general stationarity Lemma 2 above is used: the goal is to verify its $L^1(\pi)$ jump condition (21) for g , and hence justify the identity $\mathbb{E}_\pi[(Lg)(X)] = 0$.

Lemma 4 (Integrability for $g = -\log \pi$ and stationarity of Lg). *Consider a CTMC on $S = \mathbb{Z}_{\geq 0}^N$ for which Assumption 2.1 holds. Let $\Lambda(x) := \sum_{y \neq x} q(x, y)$ and define $g : S \rightarrow \mathbb{R}$ by*

$$g(x) := -\log \pi(x).$$

Then

$$\sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) (|g(y)| + |g(x)|) < \infty. \quad (27)$$

In particular, $Lg \in L^1(\pi)$ and

$$\mathbb{E}_\pi[(Lg)(X)] = 0.$$

Proof. By Lemma 1, we have $\sum_{x \in S} \pi(x) \Lambda(x) < \infty$. Moreover, by Lemma 3 and the linear growth assumption (A3),

$$\sum_{x \in S} \pi(x) \Lambda(x) |g(x)| = \sum_{x \in S} \pi(x) \Lambda(x) |\log \pi(x)| < \infty. \quad (28)$$

Write

$$T := \sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) (|g(y)| + |g(x)|) = T_1 + T_2,$$

where

$$T_1 := \sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) |g(x)| = \sum_{x \in S} \pi(x) \Lambda(x) |g(x)|$$

and

$$T_2 := \sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) |g(y)|.$$

By (28), $T_1 < \infty$. Since all terms are nonnegative, Tonelli's theorem permits swapping sums in T_2 :

$$T_2 = \sum_{y \in S} |g(y)| \sum_{x \in S, x \neq y} \pi(x) q(x, y).$$

Because π is stationary, for each $y \in S$,

$$\sum_{x \in S, x \neq y} \pi(x) q(x, y) = \pi(y) \sum_{z \in S, z \neq y} q(y, z) = \pi(y) \Lambda(y),$$

and hence

$$T_2 = \sum_{y \in S} \pi(y) \Lambda(y) |g(y)| = T_1 < \infty.$$

Therefore $T = T_1 + T_2 < \infty$, proving (27).

Next, since $|g(y) - g(x)| \leq |g(y)| + |g(x)|$, we have

$$\sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) |g(y) - g(x)| \leq T < \infty.$$

Thus the hypotheses of Lemma 2 (with $\varphi = g$) hold, and we conclude $Lg \in L^1(\pi)$ and $\mathbb{E}_\pi[(Lg)(X)] = 0$. $\square$

3.2 Logarithmic potentials, single information flow, and marginal flux balance

Section 2.3 introduced the fiber currents $J_i(n+1, n; v)$ and the single information flow $(\dot{I})_{X_i}$. We now prove the two unit-birth identities promised there. First, we show that the series defining $(\dot{I})_{X_i}$ is absolutely convergent and equals the coordinate-generator quantity $-\mathbb{E}_\pi[(L_i h_i)(X)]$. Second, we prove the marginal flux balance (10), equivalently $\sum_v J_i(n+1, n; v) = 0$. These identities form the technical bridge between the information-flow expression and the generator formalism used in the proof of Theorem 1.

Lemma 5 (Single information flow: absolute convergence and generator representation). *Fix $i \in \{1, \dots, N\}$. For the unit-birth model with generator (1), suppose Assumption 2.1 holds. Then the series defining $(\dot{I})_{X_i}$ in (11) is absolutely convergent.*

More precisely, writing

$$(\dot{I})_{X_i} = \sum_{n \geq 0} \sum_v \left(\pi(n, v) f_i(v) - \pi(n+1, v) \frac{n+1}{\tau_i} \right) (h_i(n, v) - h_i(n+1, v)),$$

each of the four nonnegative sums

$$\begin{aligned} & \sum_{n,v} \pi(n, v) f_i(v) h_i(n+1, v), & \sum_{n,v} \pi(n, v) f_i(v) h_i(n, v), \\ & \sum_{n,v} \pi(n+1, v) \frac{n+1}{\tau_i} h_i(n+1, v), & \sum_{n,v} \pi(n+1, v) \frac{n+1}{\tau_i} h_i(n, v) \end{aligned} \tag{29}$$

is finite. Consequently, $L_i h_i \in L^1(\pi)$ and

$$(\dot{I})_{X_i} = -\mathbb{E}_\pi[(L_i h_i)(X)]. \tag{30}$$

Proof. We first record the pointwise comparison between h_i and g . Since

$$\pi_n^i(v) = \frac{\pi(n, v)}{\pi_i(n)} \quad \text{and} \quad \pi_i(n) \leq 1,$$

we have $\pi_n^i(v) \geq \pi(n, v)$. Therefore

$$0 \leq h_i(n, v) \leq g(n, v) \quad \text{for all } (n, v) \in S. \quad (31)$$

By Lemma 4,

$$\sum_{x \in S} \pi(x) \sum_{y \neq x} q(x, y) (|g(y)| + |g(x)|) < \infty. \quad (32)$$

We now write the relevant pieces of this bound in the present (n, v) notation. The i -birth transitions are

$$(n, v) \longrightarrow (n+1, v) \quad \text{with rate } f_i(v),$$

and their contribution to (32) is

$$\sum_{n, v} \pi(n, v) f_i(v) (|g(n+1, v)| + |g(n, v)|) < \infty.$$

Similarly, the i -death transitions may be indexed as

$$(n+1, v) \longrightarrow (n, v) \quad \text{with rate } \frac{n+1}{\tau_i},$$

and their contribution to (32) is

$$\sum_{n, v} \pi(n+1, v) \frac{n+1}{\tau_i} (|g(n, v)| + |g(n+1, v)|) < \infty.$$

Since $g \geq 0$ and $0 \leq h_i \leq g$ by (31), these two finite bounds show that the four sums in (29) are finite. It follows that the series defining $(\dot{I})_{X_i}$ is absolutely convergent, since

$$\sum_{n \geq 0} \sum_v \left| \pi(n, v) f_i(v) - \pi(n+1, v) \frac{n+1}{\tau_i} \right| |h_i(n, v) - h_i(n+1, v)|$$

is bounded above by the sum of the four nonnegative sums appearing in (29).

The same four bounds also imply

$$\sum_{n, v} \pi(n, v) f_i(v) |h_i(n+1, v) - h_i(n, v)| + \sum_{n \geq 1, v} \pi(n, v) \frac{n}{\tau_i} |h_i(n-1, v) - h_i(n, v)| < \infty.$$

Thus $L_i h_i \in L^1(\pi)$, and the following termwise computation is justified:

$$\begin{aligned} \mathbb{E}_\pi[(L_i h_i)(X)] &= \sum_{n, v} \pi(n, v) f_i(v) (h_i(n+1, v) - h_i(n, v)) \\ &\quad + \sum_{n \geq 1, v} \pi(n, v) \frac{n}{\tau_i} (h_i(n-1, v) - h_i(n, v)). \end{aligned}$$

Reindexing the death term by replacing n with $n+1$ gives

$$\sum_{n \geq 1, v} \pi(n, v) \frac{n}{\tau_i} (h_i(n-1, v) - h_i(n, v)) = \sum_{n \geq 0, v} \pi(n+1, v) \frac{n+1}{\tau_i} (h_i(n, v) - h_i(n+1, v)).$$

Combining the paired birth and death terms gives

$$\mathbb{E}_\pi[(L_i h_i)(X)] = - \sum_{n \geq 0} \sum_v \left(\pi(n, v) f_i(v) - \pi(n+1, v) \frac{n+1}{\tau_i} \right) (h_i(n, v) - h_i(n+1, v)).$$

By Definition 1, the sum on the right-hand side is $(\dot{I})_{X_i}$. Hence $\mathbb{E}_\pi[(L_i h_i)(X)] = -(\dot{I})_{X_i}$ as claimed. $\square$

We also prove the marginal flux balance stated in Section 2.3. This is weaker than detailed balance: the individual fiber currents $J_i(n+1, n; v)$ need not vanish, but their sum over each fiber does.

Lemma 6 (Marginal flux balance). *Consider the unit-birth CTMC on $S = \mathbb{Z}_{\geq 0}^N$ with generator (1). Suppose Assumption 2.1 holds, and let π be its stationary distribution. Then, for every $i \in \{1, \dots, N\}$ and $n \geq 0$,*

$$\pi_i(n) m_i(n) = \frac{n+1}{\tau_i} \pi_i(n+1). \quad (33)$$

Equivalently,

$$\sum_v J_i(n+1, n; v) = 0.$$

Proof. Fix i and $n \geq 0$, and let

$$\varphi(x) := \mathbf{1}_{\{x_i \leq n\}}.$$

Since φ is bounded, the $L^1(\pi)$ jump condition in Lemma 2 is bounded by $2\mathbb{E}_\pi[\Lambda(X)] < \infty$. Thus Lemma 2 applies, and hence

$$\mathbb{E}_\pi[(L\varphi)(X)] = 0.$$

Only i -births from level n and i -deaths from level $n+1$ change φ , so

$$(L\varphi)(x) = -f_i(x_{-i})\mathbf{1}_{\{x_i=n\}} + \frac{n+1}{\tau_i}\mathbf{1}_{\{x_i=n+1\}}.$$

Taking expectation under π gives

$$0 = -\sum_v \pi(n, v) f_i(v) + \frac{n+1}{\tau_i} \sum_v \pi(n+1, v),$$

which is exactly (33). The equivalent form $\sum_v J_i(n+1, n; v) = 0$ follows directly from the definition of $J_i(n+1, n; v)$. $\square$

4 Proof of Theorem 1

We prove Theorem 1 according to its numbered claims. The technical estimates in Section 3 justify the stationary generator identities for the unbounded test functions used below. Throughout this section we work under Assumption 2.1.

4.1 Part (1): stationary mean and Fano-covariance identity

Proof of Theorem 1 part (1). Fix $i \in \{1, \dots, N\}$ and write $X \sim \pi$.

First let $\varphi(x) = x_i$. Since only jumps in the i th coordinate change φ , we have

$$(L\varphi)(x) = f_i(x_{-i}) - \frac{x_i}{\tau_i}.$$

Moreover,

$$\sum_{y \neq x} q(x, y) |\varphi(y) - \varphi(x)| = f_i(x_{-i}) + \frac{x_i}{\tau_i}.$$

Taking π -expectations, this is finite by (A2) and (A1). Hence Lemma 2 applies and gives

$$0 = \mathbb{E}_\pi[(L\varphi)(X)] = \mathbb{E}_\pi[f_i(X_{-i})] - \frac{1}{\tau_i} \mathbb{E}_\pi[X_i] = \mu_i - \frac{1}{\tau_i} \mathbb{E}_\pi[X_i].$$

Therefore

$$\mathbb{E}_\pi[X_i] = \tau_i \mu_i, \quad (34)$$

which proves (12).

Next let $\varphi(x) = x_i^2$. Again only jumps in the i th coordinate contribute, and

$$\begin{aligned} (L\varphi)(x) &= f_i(x_{-i})((x_i + 1)^2 - x_i^2) + \frac{x_i}{\tau_i}((x_i - 1)^2 - x_i^2) \\ &= f_i(x_{-i})(2x_i + 1) + \frac{x_i}{\tau_i}(-2x_i + 1). \end{aligned}$$

To justify stationarity, note that

$$|(x_i + 1)^2 - x_i^2| = 2x_i + 1, \quad |(x_i - 1)^2 - x_i^2| \leq 2x_i + 1,$$

and therefore

$$\sum_{y \neq x} q(x, y) |\varphi(y) - \varphi(x)| \leq (2x_i + 1) \left(f_i(x_{-i}) + \frac{x_i}{\tau_i} \right) \leq (2x_i + 1) \Lambda(x).$$

Using (A3) and $x_i \leq s(x)$,

$$(2x_i + 1) \Lambda(x) \leq 2C_\Lambda(1 + s(x))(1 + x_i) \leq 2C_\Lambda(1 + s(x))^2 \leq 4C_\Lambda(1 + s(x)^2).$$

This is π -integrable by (A1), so Lemma 2 applies. Thus

$$0 = \mathbb{E}_\pi[(L\varphi)(X)] = \mathbb{E}_\pi[f_i(X_{-i})(2X_i + 1)] + \mathbb{E}_\pi\left[\frac{X_i}{\tau_i}(-2X_i + 1)\right].$$

Equivalently,

$$0 = 2\mathbb{E}_\pi[f_i(X_{-i})X_i] + \mu_i - \frac{2}{\tau_i}\mathbb{E}_\pi[X_i^2] + \frac{1}{\tau_i}\mathbb{E}_\pi[X_i].$$

Using (34), this becomes

$$\mathbb{E}_\pi[X_i^2] = \tau_i \mathbb{E}_\pi[f_i(X_{-i})X_i] + \mathbb{E}_\pi[X_i].$$

Subtracting $\mathbb{E}_\pi[X_i]^2 = \tau_i \mathbb{E}_\pi[f_i(X_{-i})] \mathbb{E}_\pi[X_i]$ from both sides gives

$$\text{Var}_\pi(X_i) = \tau_i \text{Cov}_\pi(f_i(X_{-i}), X_i) + \mathbb{E}_\pi[X_i].$$

Finally, dividing by $\mathbb{E}_\pi[X_i] = \tau_i \mu_i > 0$ yields

$$F_{X_i} = 1 + \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\mu_i},$$

which is (13). □

4.2 Part (2): single-component information-flow inequality

The argument in this subsection follows the formal log-sum estimate of Ripsman, Kell, and Hilfinger [20]. The new point here is that, by the results of Section 3, all sums and reindexings below are justified under Assumption 2.1.

Proof of Theorem 1 part (2). Fix $i \in \{1, \dots, N\}$. By Lemma 5, the series defining $(\dot{I})_{X_i}$ is absolutely convergent. Write $(\dot{I})_{X_i} = A + B$, where

$$A := \sum_{n \geq 0} \sum_v \pi(n, v) f_i(v) \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)},$$

$$B := - \sum_{n \geq 0} \sum_v \pi(n+1, v) \frac{n+1}{\tau_i} \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)}.$$

We first show that the second term is nonpositive. Fix $n \geq 0$ and set

$$b_n := \pi_i(n+1) \frac{n+1}{\tau_i}.$$

Since π has full support, $b_n > 0$. Then

$$\sum_v \pi(n+1, v) \frac{n+1}{\tau_i} \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)} = b_n \sum_v \pi_{n+1}^i(v) \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)} = b_n D_{\text{KL}}(\pi_{n+1}^i \| \pi_n^i) \geq 0.$$

Hence $B \leq 0$, and therefore

$$(\dot{I})_{X_i} \leq A. \tag{35}$$

We next bound A by applying the log-sum inequality level by level. Fix $n \geq 0$ and define

$$b_v := \pi(n, v) f_i(v) = \pi_i(n) \pi_n^i(v) f_i(v), \quad a_v := \pi_i(n) \pi_{n+1}^i(v) f_i(v).$$

If $f_i(v) = 0$, then $a_v = b_v = 0$, and the corresponding term is interpreted as zero. If $f_i(v) > 0$, then

$$\frac{a_v}{b_v} = \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)}.$$

Thus, writing

$$\hat{A}_n := \sum_v \pi(n, v) f_i(v) \log \frac{\pi_{n+1}^i(v)}{\pi_n^i(v)} = \sum_v b_v \log \frac{a_v}{b_v},$$

so that $A = \sum_{n \geq 0} \hat{A}_n$, the log-sum inequality gives

$$\hat{A}_n \leq \left(\sum_v b_v \right) \log \frac{\sum_v a_v}{\sum_v b_v}.$$

Since

$$\sum_v b_v = \pi_i(n) m_i(n), \quad \sum_v a_v = \pi_i(n) m_i(n+1),$$

with each term positive by (33), we obtain

$$\hat{A}_n \leq \pi_i(n) m_i(n) \log \frac{m_i(n+1)}{m_i(n)}.$$

Summing over n gives

$$A \leq \sum_{n \geq 0} \pi_i(n) m_i(n) \log \frac{m_i(n+1)}{m_i(n)}. \tag{36}$$

We now estimate the right-hand side of (36). Multiplying and dividing by μ_i , we write

$$\log \frac{m_i(n+1)}{m_i(n)} = \log \frac{m_i(n+1)}{\mu_i} - \log \frac{m_i(n)}{\mu_i}.$$

Using $\log r \leq r - 1$ and $-\log r \leq r^{-1} - 1$ for $r > 0$ gives

$$\begin{aligned} & \pi_i(n) m_i(n) \left(\log \frac{m_i(n+1)}{\mu_i} - \log \frac{m_i(n)}{\mu_i} \right) \\ & \leq \pi_i(n) \left(\frac{m_i(n) m_i(n+1)}{\mu_i} - m_i(n) \right) + \pi_i(n) (\mu_i - m_i(n)). \end{aligned}$$

Summing over n and using

$$\sum_{n \geq 0} \pi_i(n) m_i(n) = \mu_i$$

yields

$$A \leq \frac{1}{\mu_i} \sum_{n \geq 0} \pi_i(n) m_i(n) m_i(n+1) - \mu_i.$$

By the marginal flux balance identity (33),

$$\pi_i(n) m_i(n) = \frac{n+1}{\tau_i} \pi_i(n+1),$$

and hence

$$\begin{aligned} \sum_{n \geq 0} \pi_i(n) m_i(n) m_i(n+1) &= \sum_{n \geq 0} \frac{n+1}{\tau_i} \pi_i(n+1) m_i(n+1) \\ &= \frac{1}{\tau_i} \sum_{n \geq 1} n \pi_i(n) m_i(n) \\ &= \frac{1}{\tau_i} \mathbb{E}_\pi [X_i m_i(X_i)]. \end{aligned}$$

Therefore

$$A \leq \frac{1}{\tau_i \mu_i} \mathbb{E}_\pi [X_i m_i(X_i)] - \mu_i.$$

Combining this with (35) gives

$$(\dot{I})_{X_i} \leq \frac{1}{\tau_i \mu_i} \mathbb{E}_\pi [X_i m_i(X_i)] - \mu_i.$$

Using $\mathbb{E}_\pi [X_i] = \tau_i \mu_i$ from part (1) and $\mathbb{E}_\pi [m_i(X_i)] = \mu_i$, the right-hand side becomes

$$\frac{\text{Cov}_\pi(X_i, m_i(X_i))}{\tau_i \mu_i}.$$

Finally, since

$$m_i(X_i) = \mathbb{E}_\pi [f_i(X_{-i}) \mid X_i],$$

we have

$$\text{Cov}_\pi(X_i, m_i(X_i)) = \text{Cov}_\pi(X_i, f_i(X_{-i})).$$

Thus

$$(\dot{I})_{X_i} \leq \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\tau_i \mu_i}.$$

By the Fano–covariance identity from part (1),

$$F_{X_i} - 1 = \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\mu_i}.$$

Therefore

$$(\dot{I})_{X_i} \leq \frac{F_{X_i} - 1}{\tau_i},$$

which is exactly (14). □

4.3 Part (3): zero total information flow

This cancellation is the rigorous generator version of the formal entropy decomposition used by Ripsman, Kell, and Hilfinger in [20]: the global logarithmic potential is split into marginal and conditional pieces, the marginal contribution vanishes by marginal flux balance, and the conditional contributions are precisely the single information flows.

Proof of Theorem 1 part (3). Recall the logarithmic potentials from (6). For each i ,

$$g(x) = g_i(x) + h_i(x), \quad x \in S.$$

By Lemma 4,

$$Lg \in L^1(\pi) \quad \text{and} \quad \mathbb{E}_\pi[(Lg)(X)] = 0.$$

Recalling L_i from (2), we next show that

$$\mathbb{E}_\pi[(L_i g_i)(X)] = 0 \quad \text{for each } i.$$

Fix i . When writing $x = (n, v)$, we use the shorthand $g_i(n) := -\log \pi_i(n)$. Since $\pi_i(n) \geq \pi(n, v)$, we have

$$0 \leq g_i(n) \leq g(n, v) \quad \text{for all } (n, v) \in S.$$

Restricting the integrability bound from Lemma 4 to the i –birth and i –death transitions therefore gives

$$\sum_{n,v} \pi(n, v) f_i(v) |g_i(n+1) - g_i(n)| + \sum_{n \geq 1, v} \pi(n, v) \frac{n}{\tau_i} |g_i(n-1) - g_i(n)| < \infty.$$

Thus $L_i g_i \in L^1(\pi)$, and the following termwise computation is justified:

$$\begin{aligned} \mathbb{E}_\pi[(L_i g_i)(X)] &= \sum_{n,v} \pi(n, v) f_i(v) (g_i(n+1) - g_i(n)) \\ &\quad + \sum_{n \geq 1, v} \pi(n, v) \frac{n}{\tau_i} (g_i(n-1) - g_i(n)) \\ &= \sum_{n \geq 0} \pi_i(n) m_i(n) (g_i(n+1) - g_i(n)) \\ &\quad + \sum_{n \geq 1} \pi_i(n) \frac{n}{\tau_i} (g_i(n-1) - g_i(n)). \end{aligned}$$

Reindexing the second sum by replacing n with $n + 1$ gives

$$\sum_{n \geq 1} \pi_i(n) \frac{n}{\tau_i} (g_i(n-1) - g_i(n)) = \sum_{n \geq 0} \pi_i(n+1) \frac{n+1}{\tau_i} (g_i(n) - g_i(n+1)).$$

Combining the two sums,

$$\mathbb{E}_\pi[(L_i g_i)(X)] = \sum_{n \geq 0} \left(\pi_i(n) m_i(n) - \frac{n+1}{\tau_i} \pi_i(n+1) \right) (g_i(n+1) - g_i(n)).$$

By the marginal flux balance Lemma 6, the coefficient in parentheses is zero for every n . Hence

$$\mathbb{E}_\pi[(L_i g_i)(X)] = 0.$$

Finally, since $L = \sum_{i=1}^N L_i$ and $g = g_i + h_i$ for each i (from (7)), we have the pointwise identity

$$Lg = \sum_{i=1}^N (L_i g) = \sum_{i=1}^N L_i g_i + \sum_{i=1}^N L_i h_i.$$

All terms are integrable by the preceding paragraph, Lemma 4, and Lemma 5. Taking expectations gives

$$0 = \mathbb{E}_\pi[(Lg)(X)] = \sum_{i=1}^N \mathbb{E}_\pi[(L_i g_i)(X)] + \sum_{i=1}^N \mathbb{E}_\pi[(L_i h_i)(X)].$$

The first sum is zero, and therefore

$$\sum_{i=1}^N \mathbb{E}_\pi[(L_i h_i)(X)] = 0.$$

Using Lemma 5,

$$(\dot{I})_{X_i} = -\mathbb{E}_\pi[(L_i h_i)(X)],$$

we conclude that

$$\sum_{i=1}^N (\dot{I})_{X_i} = 0. \quad \square$$

5 Termwise Bounds under Signed Monotonicity

The global tradeoff (16) shows that

$$\sum_{i=1}^N \frac{F_{X_i} - 1}{\tau_i} \geq 0.$$

In this section we isolate a natural structural regime in which one has the stronger *termwise* conclusion

$$F_{X_i} \geq 1 \quad \text{for every } i.$$

The hypothesis is most cleanly stated in terms of a partial order obtained by reversing the order on a subset of coordinates. Structurally balanced signed interaction graphs provide the natural graph-theoretic realization of the hypothesis, but the theorem is stated directly in the order-theoretic language used in the proof.

5.1 Signed ordering and our main assumption

Fix a signature vector

$$\sigma = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N.$$

Define a partial order $\preceq_\sigma$ on $S = \mathbb{Z}_{\geq 0}^N$ by

$$x \preceq_\sigma y \iff \sigma_i x_i \leq \sigma_i y_i \quad \text{for all } i = 1, \dots, N. \quad (37)$$

Thus on coordinates with $\sigma_i = 1$ this is the usual order, whereas on coordinates with $\sigma_i = -1$ the order is reversed.

Our main structural hypothesis for this section is the following signed monotonicity condition.

Assumption 5.1 (Signed monotonicity). There exists $\sigma \in \{-1, 1\}^N$ such that for each $i \in \{1, \dots, N\}$, whenever $x \preceq_\sigma y$, one has

$$\sigma_i f_i(x_{-i}) \leq \sigma_i f_i(y_{-i}). \quad (38)$$

Equivalently, for each fixed i , the map $x \mapsto f_i(x_{-i})$ is $\preceq_\sigma$ -increasing when $\sigma_i = 1$ and $\preceq_\sigma$ -decreasing when $\sigma_i = -1$.

We now describe a standard graph-theoretic sufficient condition for Assumption 5.1. Consider the signed interaction graph with vertex set

$$\{X_1, \dots, X_N\},$$

where the vertex X_i represents the i th species (equivalently, the counting process for that species). For each ordered pair $j \neq i$, draw an edge $X_j \rightarrow X_i$ if and only if f_i depends nontrivially on x_j . Assume that whenever such a dependence is present, the map

$$x_j \mapsto f_i(x_{-i})$$

is globally monotone when the remaining coordinates are held fixed. Label the edge $X_j \rightarrow X_i$ by $+$ if this map is nondecreasing and by $-$ if it is nonincreasing.

A signed interaction graph is called *structurally balanced* if there exists a partition of the vertex set into two classes,

$$\{X_1, \dots, X_N\} = V_1 \cup V_2,$$

such that every positive edge has both endpoints in the same class, while every negative edge has its endpoints in different classes.

Proposition 1 (Graph criterion for signed monotonicity). *Assume that for each ordered pair $j \neq i$ such that f_i depends nontrivially on x_j , the map $x_j \mapsto f_i(x_{-i})$ is globally either nondecreasing or nonincreasing. Then the following are equivalent:*

1. *every cycle in the underlying signed graph, obtained by ignoring edge orientation, has positive parity, that is, contains an even number of negative edges;*
2. *the signed interaction graph is structurally balanced;*
3. *there exists $\sigma \in \{-1, 1\}^N$ such that Assumption 5.1 holds.*

In particular, every structurally balanced interaction graph yields a signature σ for which (38) holds.

Here and below, *cycle parity* is understood in the underlying signed graph obtained by forgetting edge orientation.

Proof. The equivalence of (1) and (2) follows from Harary's balance theorem for signed graphs [13]. We prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2).

Assume (2). That is, we assume that the signed interaction graph is structurally balanced and we denote the classes by V_1 and V_2 . We must now construct a $\sigma \in \{-1, 1\}^N$ such that Assumption 5.1 holds. Define $\sigma_i = 1$ when $X_i \in V_1$ and $\sigma_i = -1$ when $X_i \in V_2$. Fix i . If $X_j \rightarrow X_i$ is a positive edge, then X_j and X_i lie in the same class, so $\sigma_j = \sigma_i$; if $X_j \rightarrow X_i$ is a negative edge, then they lie in different classes, so $\sigma_j = -\sigma_i$. Thus, for each j on which f_i depends, the map $x_j \mapsto \sigma_i f_i(x_{-i})$ is nondecreasing when the remaining coordinates are held fixed. Since this holds for every relevant j , it follows by varying coordinates one at a time that

$$x \preceq_\sigma y \implies \sigma_i f_i(x_{-i}) \leq \sigma_i f_i(y_{-i}),$$

which is exactly (38).

Conversely, assume (3), that there exists a $\sigma \in \{-1, 1\}^N$ such that Assumption 5.1 holds. We must show that the signed interaction graph is structurally balanced. Let

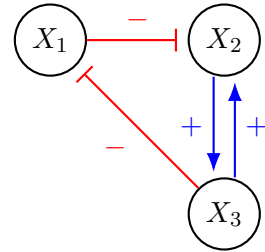
$$V_1 := \{X_i : \sigma_i = 1\}, \quad V_2 := \{X_i : \sigma_i = -1\}.$$

If $X_j \rightarrow X_i$ is a positive edge, then f_i is nondecreasing in x_j . Choose x, y differing only in coordinate j , with $x \preceq_\sigma y$. Then Assumption 5.1 is compatible with this positive dependence only when $\sigma_j = \sigma_i$. Similarly, if $X_j \rightarrow X_i$ is a negative edge, then f_i is nonincreasing in x_j , and the same comparison shows that Assumption 5.1 is possible only when $\sigma_j = -\sigma_i$. Hence positive edges lie within classes and negative edges go across classes, so the graph is structurally balanced. $\square$

If the interaction graph is connected, which is the case we consider in this paper, then the choice of σ is unique up to global sign flip. In particular, replacing σ by $-\sigma$ reverses the order $\preceq_\sigma$ but leaves Assumption 5.1 unchanged.

Example 1 (A 3-node structurally balanced system). Consider a system with $N = 3$ variables and the following monotone dependencies:

- $x_1 \xrightarrow{-} f_2$: X_1 negatively affects f_2 .
- $x_3 \xrightarrow{+} f_2$: X_3 positively affects f_2 .
- $x_2 \xrightarrow{+} f_3$: X_2 positively affects f_3 .
- $x_3 \xrightarrow{-} f_1$: X_3 negatively affects f_1 .



By choosing the partition $V_1 = \{X_1\}$ and $V_2 = \{X_2, X_3\}$, we obtain the signature vector $\sigma = (1, -1, -1)$. This configuration is valid because the only negative interactions occur across classes (between X_1 and $\{X_2, X_3\}$), while the positive interactions between X_2 and X_3 occur within the class V_2 .

To illustrate (38), consider f_2 . Since $\sigma = (1, -1, -1)$ and $\sigma_2 = -1$, we must show that

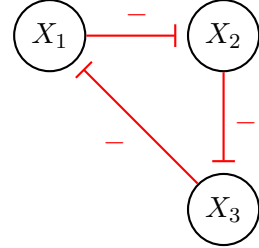
$$f_2(y_1, y_3) \leq f_2(x_1, x_3) \quad (\text{since } \sigma_2 = -1)$$

whenever $x_1 \leq y_1$ (since $\sigma_1 = 1$) and $y_3 \leq x_3$ (since $\sigma_3 = -1$). This follows immediately because f_2 is nonincreasing in x_1 and nondecreasing in x_3 , as indicated by the signs in the interaction graph above. $\triangle$

We next record a canonical example in the opposite direction: the repressilator, introduced by Elowitz and Leibler [9], whose interaction graph is not structurally balanced and therefore does not satisfy the hypothesis of Theorem 3 below.

Example 2 (The repressilator). Consider the 3-species repressilator, in which each species represses the next:

- $x_1 \xrightarrow{-} f_2,$
- $x_2 \xrightarrow{-} f_3,$
- $x_3 \xrightarrow{-} f_1.$



Thus the signed interaction graph is a directed 3-cycle with three negative edges. In particular, the unique cycle has negative parity, since it contains an odd number of negative edges. By Proposition 1, the graph is not structurally balanced, and there is no signature $\sigma \in \{-1, 1\}^N$ for which Assumption 5.1 holds.

Accordingly, the termwise conclusion of Theorem 3 below is not available from the signed-monotonicity argument in this case. $\triangle$

We end this subsection by briefly connecting our setup to the theory of *monotone systems*. For a signature vector $\sigma \in \{-1, 1\}^N$, monotone-systems theory studies the orthant cone

$$K_\sigma := \{z \in \mathbb{R}^N : \sigma_i z_i \geq 0 \text{ for all } i\},$$

and the induced order

$$x \preceq_\sigma y \iff y - x \in K_\sigma \iff \sigma_i x_i \leq \sigma_i y_i \text{ for all } i.$$

Thus, for example, if $\sigma = (1, -1, -1)$, then

$$x \preceq_\sigma y \iff x_1 \leq y_1, \quad x_2 \geq y_2, \quad x_3 \geq y_3.$$

In this language, Assumption 5.1 says precisely that, for each i , the signed birth-rate function $\sigma_i f_i$ is increasing with respect to the order $\preceq_\sigma$. This places us in the same sign-structured framework used in the monotone-systems literature; see, for example, Smith's general treatment of monotone dynamical systems and the biochemical-network formulations of Angeli, De Leenheer, and Sontag [22, 4, 7]. For the present unit-birth model, however, the point is not to invoke monotone-systems results directly, but to adapt the same signed-order viewpoint to a stochastic setting. Here the signature σ identifies an order under which the dynamics are order-preserving and the stationary law can be shown to be associated. This stationary association is the key new step that yields the termwise bound $F_{X_i} \geq 1$.

This perspective is also complementary to stochastic comparison results for reaction networks, such as those of Campos et al. [6]. There as well, sign and ordering information are used to obtain comparison results for the dynamics, much as in the monotone-systems literature. Our goal here is different: we use the signature σ from Assumption 5.1 to deduce a property of the stationary distribution itself, namely association with respect to $\preceq_\sigma$.

5.2 Main theorem, and proof outline, for networks with signed monotonicity

Proposition 1 identifies a broad graph-theoretic regime in which Assumption 5.1 holds. We now show that Assumption 5.1 itself is sufficient to strengthen the global tradeoff to a termwise bound on each component.

Theorem 3 (Termwise bound under signed monotonicity). *Assume Assumption 2.1 and Assumption 5.1. Then, for every $i \in \{1, \dots, N\}$, we have*

$$\text{Cov}_\pi(f_i(X_{-i}), X_i) \geq 0.$$

In particular, for every $i \in \{1, \dots, N\}$,

$$F_{X_i} \geq 1. \tag{39}$$

We make two observations about the implications of Theorem 3. First, under Assumption 5.1, the global tradeoff

$$\sum_{i=1}^N \frac{F_{X_i} - 1}{\tau_i} \geq 0$$

upgrades to the termwise conclusion $F_{X_i} \geq 1$ for every i . In particular, no individual component can suppress its Fano factor below the Poisson baseline in this signed-monotone regime.

Second, we will call a signed interaction graph *frustrated* if it is not structurally balanced, equivalently, if the underlying signed graph contains a cycle with negative parity. With this terminology, Theorem 3 implies that, within the class of networks whose interactions admit a global sign description, frustration is a necessary condition for sub-Poissonian noise in at least one component. In other words, structurally balanced networks cannot achieve $F_{X_i} < 1$, so any such noise suppression must come from a frustrated interaction structure.

5.2.1 Proof preparation and outline

To prepare for the proof, let $(P_t)_{t \geq 0}$ denote the Markov semigroup of the chain:

$$(P_t \phi)(x) := \mathbb{E}_x[\phi(X(t))], \quad x \in S,$$

for bounded measurable $\phi : S \rightarrow \mathbb{R}$.

We will also use the following standard notion of positive dependence, adapted here to the signed order $\preceq_\sigma$.

Definition 2 (Association with respect to $\preceq_\sigma$). A probability measure μ on S is said to be *associated* with respect to $\preceq_\sigma$ if for all bounded $\preceq_\sigma$ -increasing functions $u, v : S \rightarrow \mathbb{R}$,

$$\text{Cov}_\mu(u, v) \geq 0.$$

This is the measure-theoretic version of the notion of association on a partially ordered space studied by Lindqvist [14]. It is also closely related to the classical notion introduced by Esary, Proschan, and Walkup [11], who define a random vector $X = (X_1, \dots, X_n)$ to be associated if

$$\text{Cov}(f(X), g(X)) \geq 0$$

for all coordinatewise nondecreasing functions f and g for which the covariance exists. This is equivalent to saying that the law μ of X has the property that

$$\text{Cov}_\mu(f, g) \geq 0$$

for all coordinatewise nondecreasing test functions f and g for which the covariance exists. In the present setting, we work with the same positive dependence idea, but formulated directly for a probability measure on the partially ordered state space $(S, \preceq_\sigma)$ and restricted to bounded $\preceq_\sigma$ -increasing test functions. This bounded formulation is sufficient for the semigroup and truncation arguments used below.

The key point is that, under Assumption 5.1, the signed order $\preceq_\sigma$ is preserved by the dynamics strongly enough to force association of the stationary law. Once this is established, the desired covariance bound follows by applying association to the observables X_i and $f_i(X_{-i})$ (or their negatives when $\sigma_i = -1$).

The proof occupies the remainder of the section and proceeds in four steps:

1. We show that the semigroup $(P_t)_{t \geq 0}$ is monotone with respect to $\preceq_\sigma$, meaning that $P_t \phi$ is $\preceq_\sigma$ -increasing whenever ϕ is bounded and $\preceq_\sigma$ -increasing. We also prove the analogous monotonicity statement for a finite-box truncation of the chain, which will be used in Step 2.
2. We prove that for every $x \in S$ and every $t \geq 0$, the transition law $P_t(x, \cdot)$ is associated with respect to $\preceq_\sigma$.
3. We use stationarity and ergodicity to deduce that the stationary law π is itself associated with respect to $\preceq_\sigma$.
4. We apply this association property to obtain

$$\text{Cov}_\pi(f_i(X_{-i}), X_i) \geq 0,$$

and then conclude (39) from the Fano-covariance identity established earlier.

5.3 Proof of Theorem 3

We now prove Theorem 3. The proof is organized according to the four steps outlined above. Step 1 establishes the order-preserving property of the dynamics at the level of the semigroup. Step 2 uses this monotonicity, together with a finite-box approximation, to prove association of the transition laws $P_t(x, \cdot)$. Step 3 passes from association of transition laws to association of the stationary distribution π . Step 4 applies stationary association to the observables $f_i(X_{-i})$ and X_i , yielding the covariance bound in the theorem and hence the termwise Fano bound.

5.3.1 Step 1: Monotonicity of the semigroup

Proposition 2 (Monotonicity of the semigroup P_t). *Assume Assumption 2.1 and Assumption 5.1. Then, for every $t \geq 0$, the semigroup P_t is monotone with respect to $\preceq_\sigma$: if $\phi : S \rightarrow \mathbb{R}$ is bounded and $\preceq_\sigma$ -increasing, then $P_t \phi$ is also $\preceq_\sigma$ -increasing.*

Proof. Fix $x, y \in S$ with $x \preceq_\sigma y$. We first explain what we need to construct. If there is a coupling of the chains started from x and y such that

$$X^x(t) \preceq_\sigma X^y(t) \quad \text{for all } t \geq 0$$

almost surely, then for every bounded $\preceq_\sigma$ -increasing function ϕ ,

$$\phi(X^x(t)) \leq \phi(X^y(t)) \quad \text{almost surely.}$$

Taking expectations gives

$$(P_t \phi)(x) \leq (P_t \phi)(y),$$

and hence $P_t \phi$ is $\preceq_\sigma$ -increasing. Thus it remains only to construct such an order-preserving coupling.

We use the standard split coupling, written in random-time-change form [1]. For each i , define

$$\begin{aligned} b_i^0(z, w) &:= \min\{f_i(z_{-i}), f_i(w_{-i})\}, \\ b_i^x(z, w) &:= f_i(z_{-i}) - \min\{f_i(z_{-i}), f_i(w_{-i})\}, \\ b_i^y(z, w) &:= f_i(w_{-i}) - \min\{f_i(z_{-i}), f_i(w_{-i})\}, \end{aligned}$$

and

$$\begin{aligned} d_i^0(z, w) &:= \min\{z_i/\tau_i, w_i/\tau_i\}, \\ d_i^x(z, w) &:= \frac{z_i}{\tau_i} - \min\{z_i/\tau_i, w_i/\tau_i\}, \\ d_i^y(z, w) &:= \frac{w_i}{\tau_i} - \min\{z_i/\tau_i, w_i/\tau_i\}. \end{aligned}$$

Define $(X^x(t), X^y(t))$ by

$$\begin{aligned} X^x(t) &= x + \sum_i Y_i^{b,0} \left(\int_0^t b_i^0(X^x(s), X^y(s)) ds \right) e_i + \sum_i Y_i^{b,x} \left(\int_0^t b_i^x(X^x(s), X^y(s)) ds \right) e_i \\ &\quad - \sum_i Y_i^{d,0} \left(\int_0^t d_i^0(X^x(s), X^y(s)) ds \right) e_i - \sum_i Y_i^{d,x} \left(\int_0^t d_i^x(X^x(s), X^y(s)) ds \right) e_i, \\ X^y(t) &= y + \sum_i Y_i^{b,0} \left(\int_0^t b_i^0(X^x(s), X^y(s)) ds \right) e_i + \sum_i Y_i^{b,y} \left(\int_0^t b_i^y(X^x(s), X^y(s)) ds \right) e_i \\ &\quad - \sum_i Y_i^{d,0} \left(\int_0^t d_i^0(X^x(s), X^y(s)) ds \right) e_i - \sum_i Y_i^{d,y} \left(\int_0^t d_i^y(X^x(s), X^y(s)) ds \right) e_i, \end{aligned}$$

where the e_i are the standard unit vectors in $\mathbb{R}^N$ and the Y_i^* are independent unit-rate Poisson processes. Since

$$b_i^0 + b_i^x = f_i(z_{-i}), \quad b_i^0 + b_i^y = f_i(w_{-i}),$$

and

$$d_i^0 + d_i^x = \frac{z_i}{\tau_i}, \quad d_i^0 + d_i^y = \frac{w_i}{\tau_i},$$

the two marginals have the correct transition rates.

It remains to check that the coupling preserves the signed order. By nonexplosion, it suffices to check this at a single jump time, which we denote by t . Suppose that

$$X^x(t-) \preceq_\sigma X^y(t-)$$

just before the jump, and consider a jump in coordinate i . We consider the various cases.

First, common births and common deaths preserve the order. Indeed, a common birth adds e_i to both chains, while a common death subtracts e_i from both chains; in either case the signed comparison between the two states is unchanged.

Second, consider unmatched deaths. If $\sigma_i = 1$, then $X_i^x(t-) \leq X_i^y(t-)$, so an unmatched death can occur only in the y -chain. Since the state space is integer-valued, this moves the two

coordinates closer together without reversing the inequality. If $\sigma_i = -1$, then $X_i^x(t-) \geq X_i^y(t-)$, so an unmatched death can occur only in the x -chain, and again the two coordinates move closer together without reversing the signed order.

Third, consider unmatched births when the two i th coordinates are not equal. If $\sigma_i = 1$, then $X_i^x(t-) < X_i^y(t-)$. An unmatched birth in the x -chain gives

$$X_i^x(t-) + 1 \leq X_i^y(t-),$$

while an unmatched birth in the y -chain only increases the upper coordinate. If $\sigma_i = -1$, then $X_i^x(t-) > X_i^y(t-)$. An unmatched birth in the x -chain only increases the upper coordinate in the signed order, while an unmatched birth in the y -chain gives

$$X_i^y(t-) + 1 \leq X_i^x(t-).$$

Thus unmatched births also preserve the signed order whenever the two coordinates are not equal.

It remains only to consider unmatched births when

$$X_i^x(t-) = X_i^y(t-).$$

Here Assumption 5.1 rules out exactly the order-violating unmatched birth. If $\sigma_i = 1$, then

$$f_i(X_{-i}^x(t-)) \leq f_i(X_{-i}^y(t-)),$$

so $b_i^x(X^x(t-), X^y(t-)) = 0$ and an unmatched birth in the x -chain is impossible. If $\sigma_i = -1$, then

$$f_i(X_{-i}^x(t-)) \geq f_i(X_{-i}^y(t-)),$$

so $b_i^y(X^x(t-), X^y(t-)) = 0$ and an unmatched birth in the y -chain is impossible. In both cases, the only possible unmatched birth is order-preserving.

Therefore every jump of the coupled process preserves $X^x \preceq_\sigma X^y$. Since the order holds at time 0, it holds for all $t \geq 0$ almost surely. As observed at the start of the proof, this implies that P_t maps bounded $\preceq_\sigma$ -increasing functions to bounded $\preceq_\sigma$ -increasing functions. $\square$

For use in the next step, we also introduce the finite-box version of the process. For $M \geq 1$, let

$$\Lambda_M := \{0, 1, \dots, M\}^N,$$

and let $L^{(M)}$ be the generator on Λ_M obtained by suppressing births at the boundary:

$$L^{(M)}\phi(z) = \sum_{i=1}^N f_i(z_{-i}) \mathbf{1}_{\{z_i < M\}} (\phi(z + e_i) - \phi(z)) + \sum_{i=1}^N \frac{z_i}{\tau_i} (\phi(z - e_i) - \phi(z)), \quad (40)$$

with the convention that the death term is zero when $z_i = 0$. Let $(P_t^{(M)})_{t \geq 0}$ denote the corresponding finite-state semigroup.

Corollary 1 (Finite-box monotonicity). *Assume Assumption 2.1 and Assumption 5.1. Then, for every $M \geq 1$ and every $t \geq 0$, the semigroup $P_t^{(M)}$ is monotone with respect to $\preceq_\sigma$: if $\phi : \Lambda_M \rightarrow \mathbb{R}$ is $\preceq_\sigma$ -increasing, then $P_t^{(M)}\phi$ is $\preceq_\sigma$ -increasing.*

Proof. The proof is the same split-coupling argument as above, with the birth rates replaced by

$$f_i(z_{-i}) \mathbf{1}_{\{z_i < M\}}.$$

The boundary factor causes no new difficulty. If the two i th coordinates are equal, then the boundary indicators are equal, so the equal-coordinate case is exactly the same as for the full chain. If the two i th coordinates are unequal, then any allowed unmatched birth is order-preserving for the same integer-valued reason as above, while suppressing a boundary birth only removes a possible jump. Thus the finite-box coupling also preserves $\preceq_\sigma$, and the monotonicity of $P_t^{(M)}$ follows. $\square$

5.3.2 Step 2: Association of the transition laws

We next prove the association statement for the transition laws.

Proposition 3 (Association of the transition laws). *Assume Assumption 2.1 and Assumption 5.1. Then, for every $x \in S$ and every $t \geq 0$, the transition law $P_t(x, \cdot)$ is associated with respect to $\preceq_\sigma$.*

Proof. Fix $x \in S$, $t \geq 0$, and bounded $\preceq_\sigma$ -increasing functions $u, v : S \rightarrow \mathbb{R}$. To prove that $P_t(x, \cdot)$ is associated with respect to $\preceq_\sigma$, it suffices to show that

$$P_t(uv)(x) \geq (P_t u)(x)(P_t v)(x),$$

since this inequality is exactly

$$\text{Cov}_{P_t(x, \cdot)}(u, v) \geq 0.$$

We first prove a finite-box version of this inequality. Fix M large enough that $x \in \Lambda_M$. We work with the finite-state semigroup $(P_t^{(M)})_{t \geq 0}$ generated by $L^{(M)}$ in (40). Throughout this finite-box argument, when $P_t^{(M)}$ is applied to a function originally defined on S , we mean its restriction to Λ_M .

For functions $a, b : \Lambda_M \rightarrow \mathbb{R}$, define

$$\Gamma_M(a, b) := L^{(M)}(ab) - aL^{(M)}b - bL^{(M)}a.$$

Writing

$$L^{(M)}h(z) = \sum_{w \neq z} q_M(z, w)(h(w) - h(z)),$$

where $q_M(z, w)$ are the transition rates of the finite-box chain, we obtain

$$\begin{aligned} \Gamma_M(a, b)(z) &= (L^{(M)}(ab))(z) - a(z)(L^{(M)}b)(z) - b(z)(L^{(M)}a)(z) \\ &= \sum_{w \neq z} q_M(z, w) \left[a(w)b(w) - a(z)b(z) \right. \\ &\quad \left. - a(z)(b(w) - b(z)) - b(z)(a(w) - a(z)) \right] \\ &= \sum_{w \neq z} q_M(z, w)(a(w) - a(z))(b(w) - b(z)). \end{aligned}$$

For $0 \leq s \leq t$, set

$$a_s := P_{t-s}^{(M)}u, \quad b_s := P_{t-s}^{(M)}v, \quad F_s := a_s b_s.$$

Note that $a_t = u$ and $b_t = v$. Define

$$H(s) := P_s^{(M)}F_s(x) = P_s^{(M)}\left[(P_{t-s}^{(M)}u)(P_{t-s}^{(M)}v)\right](x).$$

Since Λ_M is finite, the forward and backward equations hold without domain issues. In particular, for any differentiable family $F_s : \Lambda_M \rightarrow \mathbb{R}$,

$$\frac{d}{ds}P_s^{(M)}F_s = P_s^{(M)}L^{(M)}F_s + P_s^{(M)}\dot{F}_s.$$

Also,

$$\dot{a}_s = -L^{(M)}a_s, \quad \dot{b}_s = -L^{(M)}b_s,$$

and hence

$$\dot{F}_s = -(L^{(M)}a_s)b_s - a_s(L^{(M)}b_s).$$

Applying the preceding differentiation identity to $H(s) = P_s^{(M)}F_s(x)$ gives

$$\begin{aligned} H'(s) &= P_s^{(M)} \left[L^{(M)}(a_s b_s) - (L^{(M)}a_s)b_s - a_s(L^{(M)}b_s) \right](x) \\ &= P_s^{(M)} \Gamma_M(a_s, b_s)(x). \end{aligned}$$

By Corollary 1, $P_r^{(M)}$ is monotone with respect to $\preceq_\sigma$ for every $r \geq 0$. Therefore a_s and b_s are $\preceq_\sigma$ -increasing. If $q_M(z, w) > 0$, then w is obtained from z by changing a single coordinate by $+1$ or -1 , so z and w are comparable under $\preceq_\sigma$. It follows that

$$(a_s(w) - a_s(z))(b_s(w) - b_s(z)) \geq 0$$

for every allowed jump $z \rightarrow w$. Hence

$$\Gamma_M(a_s, b_s)(z) \geq 0 \quad \text{for every } z \in \Lambda_M.$$

Since $P_s^{(M)}$ preserves nonnegative functions, we have $H'(s) \geq 0$ for all $s \in [0, t]$. Thus $H(t) \geq H(0)$, that is,

$$P_t^{(M)}(uv)(x) \geq (P_t^{(M)}u)(x)(P_t^{(M)}v)(x),$$

where we utilized that $H(t) = P_t^{(M)}(uv)(x)$ while $H(0) = (P_t^{(M)}u)(x)(P_t^{(M)}v)(x)$.

We now pass to the full chain. Let τ_M be the first time the full chain, started from x , exits Λ_M . Couple the full chain, denoted X , and the finite-box chain, denoted $X^{(M)}$, also started from x , using the same driving processes up to time τ_M . On the event $\{\tau_M > t\}$, the two chains agree at time t .

By nonexplosion, the full chain makes only finitely many jumps on $[0, t]$ almost surely. Hence, for each fixed sample path, $\tau_M > t$ for all sufficiently large M , and therefore

$$\mathbb{P}_x(\tau_M \leq t) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Therefore, under this coupling, $X^{(M)}(t) \rightarrow X(t)$ almost surely as $M \rightarrow \infty$. Since u , v , and uv are bounded, bounded convergence gives

$$P_t^{(M)}u(x) \rightarrow P_t u(x), \quad P_t^{(M)}v(x) \rightarrow P_t v(x), \quad P_t^{(M)}(uv)(x) \rightarrow P_t(uv)(x),$$

as $M \rightarrow \infty$. Hence, letting $M \rightarrow \infty$ in the finite-box inequality gives

$$P_t(uv)(x) \geq (P_t u)(x)(P_t v)(x).$$

As noted at the start of the proof, this is precisely association of $P_t(x, \cdot)$ with respect to $\preceq_\sigma$. $\square$

5.3.3 Step 3: Association of the stationary law

We now pass from association of the transition laws to association of the stationary distribution.

Proposition 4 (Association of the stationary law). *Assume Assumption 2.1 and Assumption 5.1. Then the stationary law π is associated with respect to $\preceq_\sigma$.*

Proof. Let $u, v : S \rightarrow \mathbb{R}$ be bounded and $\preceq_\sigma$ -increasing. By Proposition 3, for every $x \in S$ and every $t \geq 0$,

$$P_t(uv)(x) \geq (P_t u)(x)(P_t v)(x).$$

Integrating with respect to π gives

$$\sum_{x \in S} \pi(x) P_t(uv)(x) \geq \sum_{x \in S} \pi(x) (P_t u)(x) (P_t v)(x).$$

By stationarity of π , the left-hand side is

$$\sum_{x \in S} \pi(x) P_t(uv)(x) = \mathbb{E}_\pi[u(X)v(X)],$$

where $X \sim \pi$. Thus

$$\mathbb{E}_\pi[u(X)v(X)] \geq \sum_{x \in S} \pi(x) (P_t u)(x) (P_t v)(x).$$

Since the chain is irreducible and positive recurrent under the standing hypotheses, the ergodic theorem for countable-state continuous-time Markov chains gives

$$P_t g(x) \rightarrow \mathbb{E}_\pi[g(X)] \quad \text{as } t \rightarrow \infty$$

for every bounded $g : S \rightarrow \mathbb{R}$. Applying this to u and v , and using bounded convergence with respect to π , we obtain

$$\sum_{x \in S} \pi(x) (P_t u)(x) (P_t v)(x) \rightarrow \mathbb{E}_\pi[u(X)] \mathbb{E}_\pi[v(X)].$$

Letting $t \rightarrow \infty$ therefore yields

$$\mathbb{E}_\pi[u(X)v(X)] \geq \mathbb{E}_\pi[u(X)] \mathbb{E}_\pi[v(X)].$$

Since this holds for all bounded $\preceq_\sigma$ -increasing u and v , π is associated with respect to $\preceq_\sigma$. $\square$

5.3.4 Step 4: The covariance bound and conclusion

We now finish the proof of Theorem 3.

Proof of Theorem 3. Fix $i \in \{1, \dots, N\}$. Let $X \sim \pi$. We must show that

$$\text{Cov}_\pi(f_i(X_{-i}), X_i) \geq 0.$$

For $M \geq 1$, define the bounded truncations

$$U_M(x) := f_i(x_{-i}) \wedge M, \quad V_M(x) := x_i \wedge M.$$

We claim that

$$\text{Cov}_\pi(U_M(X), V_M(X)) \geq 0 \quad \text{for every } M \geq 1. \tag{41}$$

Suppose first that $\sigma_i = 1$. Then the coordinate map $x \mapsto x_i$ is $\preceq_\sigma$ -increasing, and Assumption 5.1 implies that $x \mapsto f_i(x_{-i})$ is also $\preceq_\sigma$ -increasing. Since $z \mapsto z \wedge M$ is nondecreasing, both U_M and V_M are bounded $\preceq_\sigma$ -increasing functions. Hence (41) follows from Proposition 4.

Suppose now that $\sigma_i = -1$. Then both $x \mapsto x_i$ and $x \mapsto f_i(x_{-i})$ are $\preceq_\sigma$ -decreasing. Therefore $-U_M$ and $-V_M$ are bounded $\preceq_\sigma$ -increasing functions. Applying Proposition 4 gives

$$\text{Cov}_\pi(-U_M(X), -V_M(X)) \geq 0.$$

Since

$$\text{Cov}_\pi(-U_M(X), -V_M(X)) = \text{Cov}_\pi(U_M(X), V_M(X)),$$

(41) holds in this case as well.

It remains to let $M \rightarrow \infty$. By Assumption 2.1, in particular (A1) and (A3), we have

$$\mathbb{E}_\pi[f_i(X_{-i})^2] < \infty, \quad \mathbb{E}_\pi[X_i^2] < \infty.$$

Here we use $f_i(x_{-i}) \leq \Lambda(x)$ and the at-most-linear growth bound on Λ . Thus $f_i(X_{-i})X_i$ is integrable by Cauchy-Schwarz. Since

$$U_M(X) \nearrow f_i(X_{-i}), \quad V_M(X) \nearrow X_i, \quad U_M(X)V_M(X) \nearrow f_i(X_{-i})X_i,$$

the monotone convergence theorem gives

$$\mathbb{E}_\pi[U_M(X)] \rightarrow \mathbb{E}_\pi[f_i(X_{-i})], \quad \mathbb{E}_\pi[V_M(X)] \rightarrow \mathbb{E}_\pi[X_i],$$

and

$$\mathbb{E}_\pi[U_M(X)V_M(X)] \rightarrow \mathbb{E}_\pi[f_i(X_{-i})X_i].$$

Therefore, letting $M \rightarrow \infty$ in (41), we obtain

$$\text{Cov}_\pi(f_i(X_{-i}), X_i) \geq 0.$$

Finally, by the Fano-covariance identity (13),

$$F_{X_i} = 1 + \frac{\text{Cov}_\pi(f_i(X_{-i}), X_i)}{\mu_i}.$$

Since $\mu_i \in (0, \infty)$ by Assumption 2.1, the covariance bound implies

$$F_{X_i} \geq 1.$$

This proves the theorem. □

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