



Mathematical Analysis for a Class of Stochastic Copolymerization Processes

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Abstract

We study a stochastic model of a copolymerization process that has been extensively investigated in the physics literature. The main questions of interest include: (i) what are the criteria for transience, null recurrence, and positive recurrence in terms of the system parameters; (ii) in the transient regime, what are the limiting fractions of the different monomer types; and (iii) in the transient regime, what is the speed of growth of the polymer? Previous studies in the physics literature have addressed these questions using heuristic methods. Here, we utilize rigorous mathematical arguments to derive the results from the physics literature. Moreover, the techniques developed allow us to generalize to the copolymerization process with finitely many monomer types. We expect that the mathematical methods used and developed in this work will also enable the study of even more complex models in the future.

Keywords Continuous-time Markov chain · copolymerization · recurrence and transience · boundary process · tree-like state space · origin of life · stochastic modeling · polymer growth

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1 Introduction

All known forms of life are composed of cells, which contain long, self-replicating polymers that encode and transmit genetic information. Gaining a comprehensive understanding of the mathematical principles that govern polymer growth involving two or more monomer types (copolymerization) within a well-defined stochastic framework could therefore be essential for understanding the processes underlying the origin of life and the evolution of the genetic code (Gagrani and Baum 2025; Koonin and Novozhilov 2009; Nowak and Ohtsuki 2008). Despite considerable progress, a fully developed mathematical formalization of the biologically fundamental copolymer processes—such as DNA replication, wherein a copolymer grows and acquires information guided by another template copolymer—remains an open challenge (Gaspard 2020).

Andrieux and Gaspard (2008) were early adopters of a Markovian model of copolymerization, recognizing that the sequence of monomers in the polymer can be described by a continuous-time Markov chain. Thereafter, Esposito, et al. Esposito et al. (2010) analyzed the thermodynamic efficiency of copolymerization processes using a stochastic kinetic framework, deriving explicit expressions for limiting composition fractions and growth velocity. Their work was grounded in nonequilibrium thermodynamics and relied on entropic arguments, but it did not define the process as a Markov chain nor use formal probabilistic methods in the analysis. Similarly, in subsequent work, Gaspard and Andrieux (2014), and later Gaspard alone (Gaspard 2016), developed a framework for these processes and gave explicit expressions for the mean growth velocity and entropy production. While their results were derived analytically, the arguments remained largely heuristic from a mathematical standpoint, relying on thermodynamic consistency and detailed balance identities rather than formal probabilistic arguments.

Building on these developments, in the present work, we revisit this class of models from a mathematical perspective. By recasting the dynamics as a continuous-time Markov process on an infinite tree-like state space, we establish recurrence and transience criteria, and derive almost-sure laws for polymer growth and composition using the theory of Markov chains on trees with finitely many “cone types” (Woess 2009).

In this work, we study a simple copolymerization model in which a set of d monomers, which we will denote throughout via $\mathcal{M} = \{M_1, \dots, M_d\}$, attach to or detach from the tip of a polymer. This setup reflects a physical constraint: monomers cannot easily insert themselves into the middle of a tightly bound chain. It is also biologically relevant—for example, RNA polymerase extends RNA strands by adding nucleotides ($d = 4$) to the 3' end. Despite its simplicity, the model can exhibit quite interesting behavior, especially in the transient regime where the polymer will, with a probability of one, grow without bound.

We also assume that the binding and unbinding rates (affinities) for the different monomers are different, but fixed (i.e., do not depend upon the rest of the polymer chain). This framework can later be extended to address several biologically significant questions. For instance, incorporating sequence-dependent binding affinities allows the model to capture the behavior of template-based polymerization, such as RNA replication (Andrieux and Gaspard 2008). More broadly, a central question in

origins-of-life research is whether long polymers can emerge spontaneously or whether ecological interactions are necessary to sustain them (Gagrani and Baum 2025). Our model provides a principled null model for rigorously exploring such questions.

The organization of the remainder of the paper is as follows. In Section 2, we introduce the formal mathematical model for the process considered in this paper. Moreover, we more formally state the questions we will address. In Section 3, we establish conditions on the parameters of the model for when the model is transient, null recurrent, or positive recurrent. The results of this section are relatively straightforward. In Section 4, we characterize the asymptotic composition of the growing polymer chain in the transient regime. Specifically, for each monomer, M_i , we derive the almost sure limiting fraction of that monomer in the growing polymer, as $t \rightarrow \infty$. These fractions will be denoted via $\bar{\sigma}_i$, and are given as functions of the parameter set. In Section 5, we again consider the transient regime and characterize the rate of growth of the polymer. Specifically, we establish the existence of a deterministic value $v > 0$, which we derive as a function of the parameter set, such that the polymer length, denoted $|X(t)|$ below, satisfies

$$\lim_{t \rightarrow \infty} \frac{|X(t)|}{t} = v, \quad \text{almost surely.}$$

Section 5 is the largest part of this paper and contains the bulk of our novel results. In Section 6, we restrict to the case of only two monomers (i.e., $d = 2$), which was the setting of our motivating work (Esposito et al. 2010). By restricting our general results to this case, we are able to derive more explicit expressions and provide numerical simulations that help visualize the polymer growth behavior. This setting not only allows for closed-form analysis, but also serves as a useful comparison for our mathematical treatment with the thermodynamic treatment in Esposito et al. (2010).

Before proceeding, we explicitly note that throughout this paper we assume a basic knowledge of Markov chains at the level of, for example, the text by Norris (1998).

2 Mathematical Model

As mentioned in the introduction, we consider a copolymerization process with finitely many monomer types, $\mathcal{M} = \{M_1, \dots, M_d\}$, with $d \geq 1$. A polymer is then defined as a finite sequence of monomers. Hence, the state space of our model is the set of all finite sequences of monomers, including the polymer consisting of zero monomers, which we denote by o and refer to as the “root”. Thus, if, for example, $d = 3$, the set of polymers includes o , M_1 , M_2 , M_3 , M_1M_2 , M_2M_2 , M_3M_1 , $M_2M_2M_2$, $M_1M_3M_2$, and so forth. We will denote the state space by $\mathcal{T}$. Our resulting continuous-time Markov chain (CTMC) will be denoted by X so that $X(t) \in \mathcal{T}$ is the state of the process at time t .

We turn to the possible transitions of the process. The polymer itself may change in only one of two ways:

- (i) by having a single monomer of some type, M_i , $i \in \{1, \dots, d\}$, attach to the end of the current polymer, or

(ii) by having the monomer at the end of the polymer detach.

In the first case, if a polymer, denoted $x \in \mathcal{T}$, has a monomer M_i appended to it, then the new polymer is denoted $xM_i \in \mathcal{T}$. For example, if $x = M_2M_2M_1M_1M_2$, then $xM_3 = M_2M_2M_1M_1M_2M_3$. Conversely, if the next event is a detachment, then xM_i would transition to x .

We now specify the rates of the various transition types. For attachments, as mentioned in the introduction, we assume that the rate at which a monomer M_i is appended to a polymer x depends only on the monomer type, not on the polymer itself. We denote these attachment rates by $k_i^+ \in \mathbb{R}_{>0}$, for $i \in \{1, \dots, d\}$. Thus, denoting the transition rates for the process via $q : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}_{>0}$, for any $x \in \mathcal{T}$ and $i \in \{1, \dots, d\}$,

$$q(x, xM_i) = \lim_{h \rightarrow 0^+} \frac{P(X(t+h) = xM_i \mid X(t) = x)}{h} = k_i^+, \quad \text{for all } t \geq 0.$$

Similarly, the detachment rates depend only on the identity of the last monomer in the polymer. That is, for appropriate values $k_i^- \in \mathbb{R}_{>0}$, we have

$$q(xM_i, x) = k_i^-, \quad \text{for } x \in \mathcal{T}.$$

Note that the total rate $q(x)$ out of a state x determines the parameter for the exponential holding time at state x . In particular, $q(o) = \sum_{y \neq o} q(o, y) = \sum_{i=1}^d k_i^+$ and, for any $x \in \mathcal{T}$, $q(xM_j) = \sum_{y \neq xM_j} q(xM_j, y) = k_j^- + \sum_{i=1}^d k_i^+$. Note that these values are uniformly bounded, and hence the process is necessarily non-explosive (Norris 1998).

We recall that the graph of a Markov chain is defined in the following manner:

1. the vertices of the graph are given by the state space;
2. the directed edges of the graph, denoted as either (x, y) or $x \rightarrow y$, for $x, y \in \mathcal{T}$, are determined by the transitions of the chain;
3. the labels on the edges are determined by the transition rates (in the case of a continuous-time Markov chain) and by the transition probabilities (in the case of a discrete-time Markov chain).

Note that the process we are considering is a continuous-time Markov chain whose graph is a tree (with root o). (For more on Markov chains on trees, we refer to Woess (2009).) We will denote the graph of the process by T . For example, in the case of 2 monomers, M_1 and M_2 , the process can be visualized via the graph in Figure 1, with growth progressing downward and detachment corresponding to upward edges. Each vertex represents a polymer (i.e., a finite sequence of monomers), and directed edges correspond to monomer attachment or detachment at the end of the polymer chain. Arrows labeled with k_1^+ or k_2^+ indicate the rate of appending the monomers M_1 and M_2 , respectively, while arrows labeled with k_1^- or k_2^- represent the rate at which the ending monomer detaches.

Returning to the general case of d monomers, we write $|x|$ for the length of a polymer $x \in \mathcal{T}$, i.e., the number of monomers in the polymer. When $|x| \geq 1$, we define the predecessor x^- of x to be the unique neighbor of x that is closer to the root, so that $|x^-| = |x| - 1$. For example, if $x = M_1M_2M_3$, then $x^- = M_1M_2$.

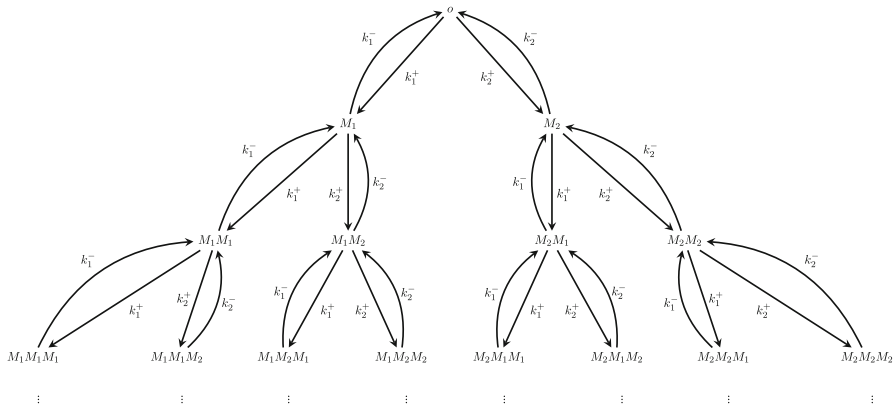


Fig. 1 Reaction graph, T , of the copolymerization process involving two monomer types. Each vertex corresponds to a polymer, and edges represent possible transitions due to monomer attachment and detachment. Note the tree-like structure of the graph

We denote the embedded discrete-time Markov chain (DTMC) for the process X via Z . Specifically, if we denote τ_n as the n th jump time of the process X , with τ_0 taken to be zero, then $Z_n = X(\tau_n)$ (Norris 1998). In this case, the transition probabilities, $\{p(x, y)\}_{x, y \in \mathcal{T}}$ of Z satisfy the following:

- for $x \in \mathcal{T}$, $j \in \{1, \dots, d\}$, we have

$$p(xM_j, xM_jM_i) = \frac{k_i^+}{k_j^- + \sum_{r=1}^d k_r^+}, \quad \text{for all } i = 1, \dots, d, \quad (2.1)$$

$$p(xM_j, x) = \frac{k_j^-}{k_j^- + \sum_{r=1}^d k_r^+};$$

- for the root o , we have

$$p(o, M_i) = \frac{k_i^+}{\sum_{r=1}^d k_r^+}, \quad \text{for all } i = 1, \dots, d.$$

After setting up the model, we can now clearly state the main questions we study in this paper.

Question 1. What are the criteria on the parameters $\{k_i^+, k_i^-\}_{i=1}^d$ for when the process X is transient, null recurrent, or positive recurrent?

Question 2. When the process X is transient, what is the limiting proportion of the d different monomer types, as functions of the parameters $\{k_i^+, k_i^-\}_{i=1}^d$? Specifically, if at time t we denote the length of the polymer by $|X(t)|$, and the number of monomers of type M_i by $N_i^X(t)$, then we want to know if there are values $\bar{\sigma}_i \in [0, 1]$ for which

$$\lim_{t \rightarrow \infty} \frac{N_i^X(t)}{|X(t)|} = \bar{\sigma}_i, \quad \text{for all } i = 1, \dots, d,$$

almost surely. Moreover, we want to calculate the values $\bar{\sigma}_i$.

Question 3. When the process X is transient, what is the limiting velocity of the process? Specifically, we would like to know if there is a value $v \in (0, \infty)$ for which

$$\lim_{t \rightarrow \infty} \frac{|X(t)|}{t} = v,$$

almost surely. Moreover, we want to calculate the value v .

Before proceeding to the technical proofs that address Questions 1–3, we briefly comment on the relationship between natural heuristic arguments and the more probabilistic approach developed in this paper. A common first instinct in growth models of this type is to attempt to describe the long-time behavior by writing down formal evolution equations for suitable averaged quantities, with the goal of extracting limiting velocities or asymptotic compositions from such relations.

However, even at this heuristic level, such arguments inevitably involve the distribution of the terminal (tip) monomer at time t , since attachment and detachment events depend only on the current end of the polymer. This tip distribution is not directly determined by the global empirical composition of the polymer, which instead reflects its bulk structure. As a result, these heuristic equations do not form a closed system in terms of the most natural macroscopic observables.

To rigorously relate these two levels of description, we introduce in Section 4 an auxiliary *boundary process*, which records the terminal monomer types along successive growth events of the polymer. This process allows us to establish almost-sure limits for the empirical composition. In Section 5, we then use these structural results, together with a careful analysis of the continuous-time dynamics between growth events, to obtain a rigorous characterization of the asymptotic growth velocity. A key technical difficulty is that these growth events do not occur at stopping times for the original process, so their cumulative effect cannot be analyzed by standard renewal or martingale arguments without additional work.

3 Criterion for Positive Recurrence, Null Recurrence, and Transience

Let $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-}$. In this section, we prove that α determines the recurrence properties of the CTMC X . Specifically, we will prove the following theorem.

Theorem 3.1 *If $\alpha < 1$, then the process X is positive recurrent. If $\alpha = 1$, then X is null recurrent. If $\alpha > 1$, then X is transient.*

To prove the theorem, we first analyze the criteria for recurrence and transience, postponing the distinction between null and positive recurrence until the end. For determining recurrence or transience of the CTMC $\{X(t)\}_{t \geq 0}$, it is sufficient to study the corresponding criteria for the embedded DTMC $\{Z_n\}_{n \in \mathbb{N}}$ (Norris 1998, Theorem 3.4.1). This first portion of the proof is essentially an application of the material in (Woess 2009, Chapter 9) (though the second portion, distinguishing between null and positive recurrence, is not).

The plan for the proof of Theorem 3.1 is to leverage a particular symmetry of the process. Specifically, for each monomer, M_i , the states of the form xM_i , for any $x \in \mathcal{T}$, are similar in a sense that will be made precise below. This will allow us to define d different classes, termed “cone types,” one for each monomer type $M_i, i \in \{1, \dots, d\}$. We will then define a matrix A associated with the various cone types, and the spectral radius of A will determine whether the process is recurrent or transient.

We begin with two key definitions:

Definition 3.2 For $x \in \mathcal{T} \setminus \{o\}$, we define

$$\mathcal{T}_x = \{z \in \mathcal{T} : \text{the first } |x| \text{ monomers of } z \text{ is exactly } x\},$$

and $\overline{\mathcal{T}}_x = \mathcal{T}_x \cup \{x^-\}$. We define T_x to be the graph with vertices $\overline{\mathcal{T}}_x$ and directed edges

$$\{y \rightarrow z : y \rightarrow z \text{ is an edge of } T \text{ and } y \in \mathcal{T}_x\},$$

which are precisely the edges with “starting” polymer contained within $\mathcal{T}_x$ (so that $x \rightarrow x^-$ is included but $x^- \rightarrow x$ is not). Finally, the labels for the edges of the graph T_x are inherited from T . That is, the label for $y \rightarrow z$ in T_x is the same as the label for $y \rightarrow z$ in T .

Note that T_x can be viewed as a subtree rooted at x , containing all extensions of x , such as $xM_i, xM_iM_j, xM_iM_jM_k$, and so on. For technical reasons, it also includes the precursor state x^- and the transition from x to x^- .

Definition 3.3 The two subtrees T_x and T_y are isomorphic if there is a root-preserving bijection between their underlying graphs that preserves edges and labels. We will term the isomorphism classes *cone types* and for $x \neq o$ denote the cone type of T_x as $C(x)$.

Based on Definitions 3.2 and 3.3, for each monomer type $M_i \in \mathcal{M}$, the associated subtrees rooted at xM_i share the same cone type, which we denote by C_i . Thus, the number of cone types is exactly d , one for each monomer type. Note that C is a function from $\mathcal{T}$ to $I = \{C_1, \dots, C_d\}$ defined via $C(xM_i) = C_i$ for all $x \in \mathcal{T}$. For technical reasons later, we will want C to be defined on the root as well and so we define $C(o) = C_0$, but we do not call C_0 a cone type. Finally, when referring to the “cone type of x ”, we always mean the cone type of the associated subtree T_x .

For a visual example, we again return to the case of two monomers. See Figure 2 for a version of Figure 1, but with two representative subtrees of cone type C_1 colored blue and two representative subtrees of cone type C_2 colored red. There are, in fact, infinitely many subtrees of each cone type, and each such subtree is infinite (for example, the subtree T_{M_1} is also of cone type C_1), but only a small number are shown in the figure for visual clarity.

We are in position to prove the main theorem of this section.

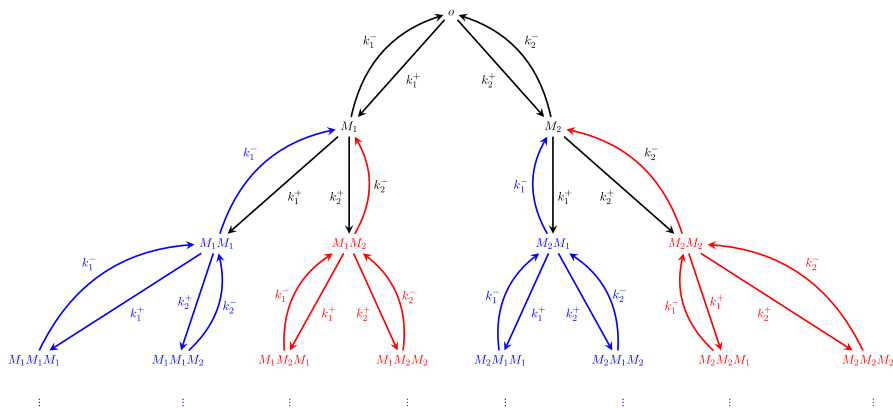


Fig. 2 Reaction graph T of the copolymerization process with two monomer types M_1 and M_2 . Vertices and edges in blue correspond to the subtrees $T_{M_1M_1}$ and $T_{M_2M_1}$, both having cone type C_1 , while those in red correspond to the subtrees $T_{M_1M_2}$ and $T_{M_2M_2}$, both having cone type C_2

Proof of Theorem 3.1 We define A , a $d \times d$ matrix, whose spectral radius will determine whether the process is transient or recurrent. For $i, j \in \{1, \dots, d\}$, we set

$$a(C_i, C_j) = \frac{k_j^+}{k_i^- + \sum_{r=1}^d k_r^+}, \quad \text{and} \quad a(C_i^-) = \frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+}, \quad (3.1)$$

where $a(C_i, C_j)$ is the transition probability from a state xM_i to xM_iM_j , and $a(C_i^-)$ is the probability of moving from xM_i to x (compare with (2.1)). Then, for $i, j \in \{1, \dots, d\}$, we define (see formula (9.77) in Woess (2009))

$$A_{ij} = \frac{a(C_i, C_j)}{a(C_i^-)} = \frac{k_j^+}{k_i^-}. \quad (3.2)$$

Therefore, the matrix A takes the form:

$$A = \left(\frac{k_j^+}{k_i^-} \right)_{1 \leq i, j \leq d} = \begin{pmatrix} \frac{k_1^+}{k_1^-} & \frac{k_2^+}{k_1^-} & \dots & \frac{k_d^+}{k_1^-} \\ \frac{k_1^+}{k_2^-} & \frac{k_2^+}{k_2^-} & \dots & \frac{k_d^+}{k_2^-} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{k_1^+}{k_d^-} & \frac{k_2^+}{k_d^-} & \dots & \frac{k_d^+}{k_d^-} \end{pmatrix}.$$

Observe that A is a rank-one matrix of the form $A = \mathbf{u}\mathbf{v}^T$, where

$$\mathbf{u} = \left(\frac{1}{k_1^-}, \frac{1}{k_2^-}, \dots, \frac{1}{k_d^-} \right)^T, \quad \mathbf{v} = (k_1^+, k_2^+, \dots, k_d^+)^T.$$

Such a matrix has one nonzero eigenvalue equal to the inner product $\mathbf{v}^T \mathbf{u} = \sum_{i=1}^d \frac{k_i^+}{k_i^-} > 0$, and the remaining $d - 1$ eigenvalues are all zero. Hence, the spectral radius is

$$\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-}.$$

Therefore, according to Theorem 9.78 in Woess (2009) and Theorem 3.4.1 in Norris (1998), we may conclude the following.

- If $\sum_{i=1}^d \frac{k_i^+}{k_i^-} \leq 1$, then the DTMC Z , and hence the CTMC X , is recurrent;
- If $\sum_{i=1}^d \frac{k_i^+}{k_i^-} > 1$, then the DTMC Z , and hence the CTMC X , is transient.

We now distinguish between positive and null recurrence for X . The process X is positive recurrent if and only if there exists a unique probability distribution $\{\mu(x) : x \in \mathcal{T}\}$ satisfying the global balance equations

$$q(x)\mu(x) = \sum_{y \neq x} \mu(y)q(y, x), \quad (3.3)$$

where $q(x) = \sum_{y \neq x} q(x, y)$ is the total exit rate from state x (see, for example, (Norris 1998, Theorem 3.5.3)). This characterization assumes that the process is non-explosive, which holds in our setting because the total jump rate from any state, namely $q(x)$, is uniformly bounded above (see Norris (1998)).

We now define a measure $\mu : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$ by

$$\mu(x) := \mu(o) \prod_{i=1}^d \left(\frac{k_i^+}{k_i^-} \right)^{\beta_i(x)}, \quad (3.4)$$

where $\beta_i(x)$ denotes the number of monomers of type M_i in polymer x , and $\mu(o)$ is a normalizing constant.

Proposition 3.4 *The measure defined in (3.4) satisfies the balance equations (3.3).*

Proof We simply check that the balance equation (3.3) holds for each state x . We make use of the fact that for any $x \in \mathcal{T}$ and $M_i \in \mathcal{M}$,

$$\mu(xM_i) = \mu(x) \left(\frac{k_i^+}{k_i^-} \right). \quad (3.5)$$

- We begin by verifying (3.3) for the root, $x = o$. Since $q(o) = \sum_{i=1}^d k_i^+$, we have $q(o)\mu(o) = \left(\sum_{i=1}^d k_i^+\right)\mu(o)$. Moreover,

$$\begin{aligned}\sum_{y \neq o} \mu(y)q(y, o) &= \sum_{i=1}^d \mu(M_i)q(M_i, o) \\ &= \sum_{i=1}^d \mu(o) \left(\frac{k_i^+}{k_i^-} \right) \cdot k_i^- \quad (\text{using (3.5) and that } q(M_i, 0) = k_i^-) \\ &= \left(\sum_{i=1}^d k_i^+ \right) \mu(o).\end{aligned}$$

Hence, $q(o)\mu(o) = \sum_{y \neq o} \mu(y)q(y, o)$.

- We now consider states of the form xM_j , with $x \in \mathcal{T}$ and $M_j \in \mathcal{M}$. We first note that (3.5) yields

$$q(xM_j)\mu(xM_j) = \left(k_j^- + \sum_{i=1}^d k_i^+ \right) \mu(x) \left(\frac{k_j^+}{k_j^-} \right).$$

Next, we have

$$\begin{aligned}\sum_{y \neq xM_j} \mu(y)q(y, xM_j) &= \mu(x)q(x, xM_j) + \sum_{i=1}^d \mu(xM_jM_i)q(xM_jM_i, xM_j) \\ &= \mu(x)k_j^+ + \sum_{i=1}^d \mu(x) \left(\frac{k_j^+}{k_j^-} \right) \left(\frac{k_i^+}{k_i^-} \right) \cdot k_i^- = \mu(x)k_j^+ + \mu(x) \left(\frac{k_j^+}{k_j^-} \right) \sum_{i=1}^d k_i^+ \\ &\quad (\text{using (3.5) twice}) \\ &= \left(k_j^- + \sum_{i=1}^d k_i^+ \right) \mu(x) \left(\frac{k_j^+}{k_j^-} \right).\end{aligned}$$

Therefore, $q(xM_j)\mu(xM_j) = \sum_{y \neq xM_j} \mu(y)q(y, xM_j)$. Since both sides match, the proposition has been proved. $\square$

$\square$

Now we need to give the condition under which $\{\mu_x : x \in \mathcal{T}\}$ forms a probability distribution. This requires the measure sums to 1. Note that the number of polymers of length ℓ that consist of β_i monomers of type M_i (so that $\beta_1 + \cdots + \beta_d = \ell$) is precisely the multinomial coefficient $\binom{\ell}{\beta_1, \dots, \beta_d}$. This accounts for the number of

distinct sequences (i.e., orderings) of monomers with those multiplicities. Therefore,

$$\begin{aligned}\sum_{x \in \mathcal{T}} \mu(x) &= \mu(o) + \sum_{\ell=1}^{\infty} \sum_{\substack{\beta_1 + \dots + \beta_d = \ell \\ \beta_i \geq 0}} \mu(o) \binom{\ell}{\beta_1, \dots, \beta_d} \prod_{i=1}^d \left(\frac{k_i^+}{k_i^-} \right)^{\beta_i} \\ &= \mu(o) \left(1 + \sum_{\ell=1}^{\infty} \left(\sum_{i=1}^d \frac{k_i^+}{k_i^-} \right)^{\ell} \right),\end{aligned}$$

where the final equality follows from the multinomial theorem.

Hence, $\mu(0)$ can be chosen for $\sum_{x \in \mathcal{T}} \mu(x)$ to equal one if and only if $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} < 1$.

Hence, if $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} < 1$, then the process X is positive recurrent.

Now consider the case where $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} = 1$. The system still admits a unique stationary measure $\{\mu(x) : x \in \mathcal{T}\}$ given by (3.4) because stationary measures for irreducible recurrent continuous-time Markov chains are unique up to scalar multiples (see (Norris 1998, Theorem 3.5.2)). However, in the case $\alpha = 1$ this measure cannot be normalized to a probability distribution. Hence, in this case, the process X cannot be positive recurrent, and so must be null recurrent, concluding the proof. $\square$

4 Limiting Proportion of Each Monomer Type

We now turn to our second question. Throughout this section we assume that $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} > 1$, so that the process X is transient and $\lim_{t \rightarrow \infty} |X_t| = \infty$, almost surely (Woess 2009, Theorem 9.18). For each $i \in \{1, \dots, d\}$ and any $t \geq 0$, we denote the number of occurrences of the monomer M_i in the polymer $X(t)$ by $N_i^X(t)$. Using the notation of the last section (in (3.4)), we note $N_i^X(t) = \beta_i(X(t))$. The proportion of monomer M_i at time t is then

$$\sigma_i(t) := \frac{N_i^X(t)}{\sum_{j=1}^d N_j^X(t)} = \frac{N_i^X(t)}{|X(t)|}, \quad (4.1)$$

with each $\sigma_i(t)$ taken to be zero when $X(t) = o$.

In this section, we prove the following.

Theorem 4.1 *In the transient regime, i.e. when $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} > 1$, for each $i \in \{1, \dots, d\}$ we have $\lim_{t \rightarrow \infty} \sigma_i(t) = \bar{\sigma}_i$, almost surely, where*

$$\bar{\sigma}_i = \frac{k_i^+}{m + k_i^-},$$

with m being the unique value satisfying $\sum_{r=1}^d \frac{k_r^+}{m+k_r^-} = 1$.

Remark 4.2 Although we do not use this interpretation in the analysis, the constant m admits a natural probabilistic meaning as an effective *escape rate* from the root. In particular, one can show that

$$m = \left(\sum_{r=1}^d k_r^+ \right) P_o (X \text{ never returns to } o).$$

Thus, m may be viewed as the total rate at which growth attempts are initiated at the root, multiplied by the probability that such an attempt leads to a trajectory that never returns. Equivalently, m may be interpreted as the effective drift of the process away from the root in the transient regime.

To prove the theorem, it suffices to study the embedded discrete-time Markov chain $\{Z_n\}_{n \geq 0}$. Moreover, and without loss of generality, we will assume throughout this section that the process has an initial state given by the root; that is, $Z_0 = o$ with probability one.

We begin by defining some of the key objects for the next two sections. First, we define the k th *level* of the graph T to be the subset of the state space $\mathcal{T}$ consisting of polymers with length k . For example, when $d = 2$, the second level is the set $\{M_1 M_1, M_1 M_2, M_2 M_1, M_2 M_2\}$. Next, the random times $\{e_k\}_{k \geq 0}$ are defined to be the *last* time the process Z visits level k . That is,

$$e_k = \sup \{n \geq 0 : |Z_n| = k\}.$$

Note that because the process is assumed to be transient, we have $e_k < \infty$ for each k , with probability one, and that for any $k \geq 0$, we have $|Z_{e_k}| = k$ and $|Z_n| > k$ for all $n > e_k$. Note also that the e_k are *not* stopping times.

We now construct the process $\{W_k\}_{k \geq 0}$, sometimes referred to as the *boundary process* (Woess 2009). For each $k \geq 0$, we set

$$W_k = Z_{e_k}, \tag{4.2}$$

which records the state visited at the last time the process Z is at level k .

It follows that W_k is the polymer of length k that forms the first k monomers of the limiting infinite polymer. In particular, note that $(W_0, W_1, W_2, \dots, W_k)$ converges, as $k \rightarrow \infty$, to an infinite length polymer, and that the fractional representation of each monomer in the W process is the object of our interest in this section. For readers who wish to see a concrete example immediately, a visualization of the boundary process in the two-monomer case appears in Section 6.

The plan is the following. According to Theorem 4.3 below, the process $\{W_k\}_{k \geq 0}$ is itself a Markov chain. Define the associated cone type of W_k to be U_k . That is,

$$U_k = C(W_k). \tag{4.3}$$

The process U_k is then a Markov chain on the finite state space $\{C_1, \dots, C_d\}$. We will prove below that U_k is irreducible. Hence, it has a unique limiting stationary distribution. Moreover, this distribution yields the desired limiting proportion of each monomer type. Thus, our remaining goal is to characterize the limiting (stationary) distribution of the process U_k .

Our first order of business is to characterize the transition probabilities for $\{W_k\}_{k \geq 0}$. To that end, for any $x, y \in \mathcal{T}$, define $f^{(n)}(x, y) = 0$ and for $n \geq 1$,

$$f^{(n)}(x, y) = P_x(Z_n = y, Z_m \neq y \text{ for } 0 < m < n),$$

which is the probability that the first time the process Z enters state y is at time n (after n jumps), given that the process Z starts at state x . We then define

$$F(x, y) := \sum_{n=0}^{\infty} f^{(n)}(x, y) = P_x(\text{the process enters state } y \text{ in finite time}). \quad (4.4)$$

It is intuitively clear that for any monomer type $M_i \in \mathcal{M}$ and any $x \in \mathcal{T}$, the value $F(xM_i, x)$ only depends on the cone type $C(xM_i) = C_i$. (For a reference to this fact, see Chapter 9, page 276 in Woess (2009).) Hence, for each $M_i \in \mathcal{M}$ and any $x \in \mathcal{T}$, we denote

$$F_i := F(xM_i, x). \quad (4.5)$$

Note that each F_i is strictly greater than zero (and, in fact, lower bounded by $\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+}$) and is also strictly less than one (Woess 2009, Lemma 9.98).

We now state the following key result from Woess (2009).

Theorem 4.3 *In the transient case, i.e., $\alpha > 1$, the process $\{W_k\}_{k \geq 1}$ is a Markov chain. For $x \in \mathcal{T}$ with $|x| = k$ and any $y \in \mathcal{T}$ for which $x = y^-$, we have the following transition probabilities*

$$P(W_k = y \mid W_{k-1} = x) = F(x, x^-) \cdot \frac{1 - F(y, x)}{1 - F(x, x^-)} \cdot \frac{p(x, y)}{p(x, x^-)},$$

where the p are given in and around (2.1).

From this, we can immediately calculate the transition probabilities for W_k in terms of the F_i in (4.5). In particular, for the polymers xM_i and xM_iM_j , with $x \in \mathcal{T}$ and $|x| = k - 1$,

$$\begin{aligned} P(W_{k+1} = xM_iM_j \mid W_k = xM_i) &= F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{p(xM_i, xM_iM_j)}{p(xM_i, x)} \\ &= F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{\frac{k_j^+}{k_i^- + \sum_{r=1}^d k_r^+}}{\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+}} \end{aligned}$$

$$= F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{k_j^+}{k_i^-}. \quad (4.6)$$

Note that each term is well defined because $F_i < 1$ and that each term is also strictly positive. We then immediately conclude that the process $\{U_k\}_{k \geq 1}$ is irreducible and has the following transition probabilities

$$P(U_{k+1} = C_j \mid U_k = C_i) = F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{k_j^+}{k_i^-}.$$

We can now give a $d \times d$ transition matrix $V = (V_{ij})_{1 \leq i, j \leq d}$ for the Markov chain $\{U_k\}_{k \geq 1}$:

$$V_{ij} := P(U_{k+1} = C_j \mid U_k = C_i) = F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{k_j^+}{k_i^-} > 0. \quad (4.7)$$

This matrix V indicates $\{U_k\}_{k \geq 1}$ is irreducible and positive recurrent (under our transient assumption). Let $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_d)$ denote the stationary distribution of this Markov chain $\{U_k\}_{k \geq 1}$. Then $\bar{\sigma}$ satisfies:

$$\bar{\sigma} V = \bar{\sigma}, \quad \sum_{i=1}^d \bar{\sigma}_i = 1. \quad (4.8)$$

Thus, all that remains is to calculate the F_i of (4.5) and derive the stationary distribution for the process.

Before that, we give the following propositions for preparation.

Proposition 4.4 *The F_i of (4.5) satisfy the following system of d equations,*

$$\frac{k_i^-}{\left(k_i^- + \sum_{r=1}^d k_r^+\right) - \sum_{r=1}^d k_r^+ F_r} = F_i, \quad \text{for each } i \in \{1, \dots, d\}. \quad (4.9)$$

The proof of the above proposition can be found in and around (Woess 2009, Equation 9.76).

Proposition 4.5 *The solution for (4.9) exists and is equal to*

$$F_i = \frac{k_i^-}{m + k_i^-}, \quad i \in \{1, \dots, d\}$$

where m is the unique solution satisfying

$$\sum_{r=1}^d \frac{k_r^+}{m + k_r^-} = 1.$$

Remark 4.6 When the number of monomer types satisfies $d < 5$, this equation can be solved analytically by reducing it to a polynomial of degree at most four. However, for $d \geq 5$, the equation is not in general solvable in radicals due to the Abel-Ruffini theorem. However, in that case the value of m can be computed numerically.

Proof of Proposition 4.5 Manipulating (4.9) shows that for any $i \in \{1, \dots, d\}$,

$$\frac{k_i^-}{F_i} - k_i^- = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^+ F_r. \quad (4.10)$$

Denote

$$m = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^+ F_r = \sum_{r=1}^d k_r^+ (1 - F_r), \quad (4.11)$$

and note that $m > 0$ because each $F_r \in (0, 1)$. Combining (4.10) and (4.11) yields

$$\frac{k_i^-}{F_i} - k_i^- = m \implies F_i = \frac{k_i^-}{m + k_i^-}. \quad (4.12)$$

Plugging this back into (4.11) yields

$$\begin{aligned} m &= \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^+ F_r = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^+ \left(\frac{k_r^-}{m + k_r^-} \right) \\ &= \sum_{r=1}^d k_r^+ \frac{m}{m + k_r^-} = m \sum_{r=1}^d k_r^+ \frac{1}{m + k_r^-}. \end{aligned}$$

Since $m > 0$, we may divide by m and conclude

$$\sum_{r=1}^d \frac{k_r^+}{m + k_r^-} = 1.$$

Note that the value of m satisfying the above is unique because the function $g(m) = \sum_{r=1}^d \frac{k_r^+}{m + k_r^-}$ is monotonically decreasing in m , $g(0) = \sum_{r=1}^d \frac{k_r^+}{k_r^-} > 1$, and approaches zero as $m \rightarrow \infty$. $\square$

We can now give the stationary distribution for the process U .

Proposition 4.7 The stationary distribution for the process $\{U_k\}_{k \geq 1}$ is

$$\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_d) = \left(\frac{k_1^+}{m + k_1^-}, \frac{k_2^+}{m + k_2^-}, \dots, \frac{k_d^+}{m + k_d^-} \right), \quad (4.13)$$

where m is the unique solution to

$$\sum_{r=1}^d \frac{k_r^+}{m + k_r^-} = 1.$$

Proof From (4.10),

$$\frac{k_i^-(1 - F_i)}{F_i} = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^+ F_r.$$

Note that the right-hand side of the above does not depend upon i . Thus, for any $i, j \in \{1, \dots, d\}$, we have $\frac{k_i^-(1 - F_i)}{F_i} = \frac{k_j^-(1 - F_j)}{F_j}$. Hence,

$$\frac{1 - F_j}{1 - F_i} = \frac{k_i^- F_j}{k_j^- F_i}.$$

Plugging this into (4.7) yields

$$V_{ij} := F_i \cdot \frac{1 - F_j}{1 - F_i} \cdot \frac{k_j^+}{k_i^-} = F_i \cdot \frac{k_i^- F_j}{k_j^- F_i} \cdot \frac{k_j^+}{k_i^-} = F_j \frac{k_j^+}{k_j^-}.$$

Since V is a transition matrix,

$$\sum_{j=1}^d F_j \frac{k_j^+}{k_j^-} = 1, \quad (4.14)$$

and so for $i \in \{1, 2, \dots, d\}$,

$$F_i \frac{k_i^+}{k_i^-} = F_i \frac{k_i^+}{k_i^-} \left(\sum_{j=1}^d F_j \frac{k_j^+}{k_j^-} \right) = \sum_{j=1}^d F_i \frac{k_i^+}{k_i^-} F_j \frac{k_j^+}{k_j^-} = \sum_{j=1}^d V_{ji} \left(F_j \frac{k_j^+}{k_j^-} \right). \quad (4.15)$$

Hence, according to (4.14) and (4.15) and Proposition 4.5, the stationary distribution is

$$\bar{\sigma}_i = F_i \frac{k_i^+}{k_i^-} = \frac{k_i^+}{m + k_i^-}, \quad i \in \{1, 2, \dots, d\}, \quad (4.16)$$

where m is the unique solution given by $\sum_{r=1}^d \frac{k_r^+}{m + k_r^-} = 1$. $\square$

At this point, we have determined the stationary distribution of the chain $\{U_k\}_{k \geq 1}$, which describes the limiting frequency of cone types along the boundary process W_k .

Intuitively, this already suggests that the limiting proportion of each monomer type in the polymer should be given by (4.13). However, the connection is not yet completely rigorous: the limiting frequencies of cone types in the boundary process must be related back to the original proportion $\sigma_i(t)$ of (4.1) for the process X . Specifically, if we denote the number of occurrences of the monomer M_i in the polymer W_k by $N_i^W(k)$, we now know that from ergodic theorem, almost surely,

$$\lim_{k \rightarrow \infty} \frac{N_i^W(k)}{|W_k|} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{C_i}(C(W_k)) = \bar{\sigma}_i,$$

all that remains is to show

$$\lim_{t \rightarrow \infty} \sigma_i(t) = \lim_{k \rightarrow \infty} \frac{N_i^W(k)}{|W_k|},$$

almost surely. This requires carefully embedding the continuous-time process into the discrete W process. The remainder of the proof is devoted to establishing this connection.

Proof of Theorem 4.1 Define

$$J_t = \max\{n \in \mathbb{N} \mid \tau_n \leq t\},$$

to be the number of jumps of X up to time t . Since $X_t = Z_{J_t}$,

$$\lim_{t \rightarrow \infty} \sigma_i(t) = \lim_{t \rightarrow \infty} \frac{N_i^X(t)}{|X(t)|} = \lim_{t \rightarrow \infty} \frac{N_i^Z(J_t)}{|Z_{J_t}|},$$

where the number of occurrences of the monomer M_i in the polymer Z_n by $N_i^Z(n)$.

Because $J_t \rightarrow \infty$ a.s. Norris (1998), it follows that

$$\lim_{t \rightarrow \infty} \sigma_i(t) = \lim_{n \rightarrow \infty} \frac{N_i^Z(n)}{|Z_n|}, \quad \text{with probability 1.} \quad (4.17)$$

We need to embed this limit onto W . As above, let e_k be the last time Z visits level k and define

$$\hat{k}(n) = \max\{k : e_k \leq n\},$$

giving the length of the boundary process at time n . Then $\hat{k}(n) \rightarrow \infty$ as $n \rightarrow \infty$ a.s. (Woess 2009, Page 295), and so

$$\lim_{n \rightarrow \infty} \frac{N_i^W(\hat{k}(n))}{|W_{\hat{k}(n)}|} = \lim_{k \rightarrow \infty} \frac{N_i^W(k)}{|W_k|}, \quad \text{with probability 1.} \quad (4.18)$$

Combining (4.17) and (4.18), we now simply need to show the following holds almost surely,

$$\lim_{n \rightarrow \infty} \frac{N_i^W(\hat{\mathbf{k}}(n))}{|W_{\hat{\mathbf{k}}(n)}|} = \lim_{n \rightarrow \infty} \frac{N_i^Z(n)}{|Z_n|}.$$

To that end, we decompose the right hand side (where we assume n is large enough so that none of the denominators are zero),

$$\frac{N_i^Z(n)}{|Z_n|} = \frac{N_i^Z(n)}{N_i^W(\hat{\mathbf{k}}(n))} \cdot \frac{N_i^W(\hat{\mathbf{k}}(n))}{|W_{\hat{\mathbf{k}}(n)}|} \cdot \frac{|W_{\hat{\mathbf{k}}(n)}|}{|Z_n|}. \quad (4.19)$$

We will show that the first and third ratios limit to 1, almost surely, in which case we are done.

We tackle the first ratio. First note that $Z_n \in \mathcal{T}_{W_{\hat{\mathbf{k}}(n)}}$ (i.e., the first $\hat{k}(n)$ monomers of Z_n coincide with $W_{\hat{\mathbf{k}}(n)}$). Hence, for each choice of i and n ,

$$1 \leq \frac{N_i^Z(n)}{N_i^W(\hat{\mathbf{k}}(n))}.$$

We also have an upper bound,

$$\begin{aligned} \frac{N_i^Z(n)}{N_i^W(\hat{\mathbf{k}}(n))} &= \frac{N_i^Z(n) - N_i^W(\hat{\mathbf{k}}(n)) + N_i^W(\hat{\mathbf{k}}(n))}{N_i^W(\hat{\mathbf{k}}(n))} \\ &\leq \frac{|Z_n| - |W_{\hat{\mathbf{k}}(n)}| + N_i^W(\hat{\mathbf{k}}(n))}{N_i^W(\hat{\mathbf{k}}(n))} \quad (\text{since } Z_n \in \mathcal{T}_{W_{\hat{\mathbf{k}}(n)}}) \\ &\leq \frac{e_{\hat{\mathbf{k}}(n)+1} - e_{\hat{\mathbf{k}}(n)} + N_i^W(\hat{\mathbf{k}}(n))}{N_i^W(\hat{\mathbf{k}}(n))} \quad (\text{since } e_{\hat{\mathbf{k}}(n)} \leq n \leq e_{\hat{\mathbf{k}}(n)+1}) \\ &= 1 + \frac{e_{\hat{\mathbf{k}}(n)+1} - e_{\hat{\mathbf{k}}(n)}}{N_i^W(\hat{\mathbf{k}}(n))}. \end{aligned}$$

By (Woess 2009, Page 295),

$$\lim_{n \rightarrow \infty} \frac{e_{\hat{\mathbf{k}}(n)+1} - e_{\hat{\mathbf{k}}(n)}}{N_i^W(\hat{\mathbf{k}}(n))} = 0,$$

almost surely, and so the first ratio is handled.

Turning to the third ratio of (4.19), by similar arguments we have

$$1 \leq \frac{|Z_n|}{|W_{\hat{\mathbf{k}}(n)}|} \leq 1 + \frac{e_{\hat{\mathbf{k}}(n)+1} - e_{\hat{\mathbf{k}}(n)}}{|W_{\hat{\mathbf{k}}(n)}|},$$

and by (Woess 2009, Page 295), the second term $\frac{e^{\hat{k}(n)+1} - e^{\hat{k}(n)}}{|W_{\hat{k}(n)}|}$ tends to 0 almost surely. This completes the proof. $\square$

5 Asymptotic Growth Rate

In this section, we are interested in the asymptotic growth rate of the polymer in the transient regime. Specifically, we ask whether there exists a constant $v \in (0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{|X(t)|}{t} = v, \quad \text{almost surely,}$$

and we want to characterize the value of v . We will prove the following.

Theorem 5.1 *Let $\bar{\sigma}_r$ denote the limiting proportion of monomers of type M_r , as given in Theorem 4.1. Then, in the transient regime, i.e. when $\alpha = \sum_{i=1}^d \frac{k_i^+}{k_i^-} > 1$, the process admits a deterministic asymptotic growth velocity $v \in (0, \infty)$ given by*

$$v := \lim_{t \rightarrow \infty} \frac{|X(t)|}{t} = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^- \bar{\sigma}_r, \quad \text{almost surely,}$$

where $\{k_r^+, k_r^-\}_{r=1}^d$ are the attachment and detachment rates of the respective monomer types.

Intuitively, the polymer's growth velocity should reflect the net rate of monomer addition, weighted by how often the process occupies states ending in each monomer type. If we let

$$A_r(t) = \frac{1}{t} \int_0^t \mathbf{1}_{\{\text{terminal monomer of } X(s) \text{ is } M_r\}} ds,$$

then a natural heuristic is

$$v = \sum_{r=1}^d k_r^+ - \sum_{r=1}^d k_r^- \lim_{t \rightarrow \infty} A_r(t),$$

interpreting $\lim_{t \rightarrow \infty} A_r(t)$ as the long-time fraction of time that X spends at polymers whose terminal monomer is M_r .

The cone-type and boundary-process analysis in Section 4 identifies $\bar{\sigma}_r$ as the limiting fraction of *growth steps* at which the terminal monomer is M_r . It is therefore tempting to set $\lim_{t \rightarrow \infty} A_r(t) = \bar{\sigma}_r$ and conclude the formula for v directly. However, this identification is not immediate: the boundary process records only the sequence of terminal monomers at successive growth events and ignores the random excursions of the continuous-time process X between these events. In particular, the growth events

do not occur at stopping times for X , so standard ergodic or renewal arguments cannot be applied without further work.

This section is devoted to bridging this gap. Rather than directly analyzing the time averages $A_r(t)$, we instead work directly with the process $|X(t)|$ and establish a strong law of large numbers for $\frac{|X(t)|}{t}$ in the transient regime. The resulting expression for the velocity coincides with the above heuristic formula, thereby providing a rigorous justification for interpreting $\bar{\sigma}_r$ as the effective contribution of each monomer type to the long-time growth rate.

Before proving Theorem 5.1, we require some preliminary results. It is most convenient to shift our analysis, as much as possible, to the DTMC Z . Recall that τ_n is the time of the n -th jump of X , with $\tau_0 = 0$, and $Z_n = X(\tau_n)$ is the embedded DTMC. Define

$$J_t := \max \{n \in \mathbb{N} \mid \tau_n \leq t\},$$

which represents the number of jumps of X that have occurred at or before time t . Since $X_t = Z_{J_t}$, it follows that

$$\lim_{t \rightarrow \infty} \frac{t}{|X_t|} = \lim_{t \rightarrow \infty} \frac{t}{|Z_{J_t}|}, \quad (5.1)$$

if the limits exist.

We begin by getting useful upper and lower bounds on the numerator t . The process X is a CTMC and so its holding times are exponentially distributed. Since the j th state visited by the chain is Z_j , we may denote these holding times via H_{Z_j} . We then note that $\tau_{n+1} = \sum_{j=0}^n H_{Z_j}$, and so define $H^n := \sum_{j=0}^n H_{Z_j}$ and

$$\begin{aligned} (H^n)_i &:= \sum_{j=0}^n H_{Z_j} \cdot \mathbf{1}_{C_i}(C(Z_j)), \quad i = 1, \dots, d, \\ (H^n)_o &:= \sum_{j=0}^n H_{Z_j} \cdot \mathbf{1}_o(Z_j). \end{aligned}$$

The above give (i) the total amount of time the process has spent in states with various cone types up to time τ_{n+1} and (ii) the total amount of time the process has spent in the root. Clearly,

$$H^n = (H^n)_o + \sum_{i=1}^d (H^n)_i. \quad (5.2)$$

Moreover, $H^{J_t-1} \leq t \leq H^{J_t}$, and hence

$$\frac{H^{J_t-1}}{|Z_{J_t}|} \leq \frac{t}{|Z_{J_t}|} \leq \frac{H^{J_t}}{|Z_{J_t}|}.$$

Applying the squeeze theorem, it therefore suffices to determine

$$\lim_{t \rightarrow \infty} \frac{H^{J_t-1}}{|Z_{J_t}|} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{H^{J_t}}{|Z_{J_t}|}.$$

Note that because the process is transient, we have $\lim_{t \rightarrow \infty} \frac{(H^{J_t-1})_o}{|Z_{J_t}|} = 0$ and $\lim_{t \rightarrow \infty} \frac{(H^{J_t})_o}{|Z_{J_t}|} = 0$ almost surely. Hence, in view of (5.2), our analysis reduces to analyzing

$$\lim_{t \rightarrow \infty} \frac{(H^{J_t-1})_i}{|Z_{J_t}|} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{(H^{J_t})_i}{|Z_{J_t}|} \quad (5.3)$$

for each $i \in \{1, \dots, d\}$. The arguments for the two limits in (5.3) are essentially the same and so we only focus on the second.

We require one more bit of notation. For each $i \in \{1, \dots, d\}$, we let

$$\chi_i(n) := \sum_{j=0}^n \mathbf{1}_{C_i}(C(Z_j)), \quad (5.4)$$

be the number of visits to polymers with cone type C_i in the first $n+1$ states of the process Z , and let $\chi_o(n)$ be the number of visits to the root. Since $J_t \rightarrow \infty$ almost surely as $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} \frac{(H^{J_t})_i}{|Z_{J_t}|} = \lim_{n \rightarrow \infty} \frac{(H^n)_i}{|Z_n|} = \lim_{n \rightarrow \infty} \left(\frac{(H^n)_i}{\chi_i(n)} \cdot \frac{\chi_i(n)}{n} \cdot \frac{n}{|Z_n|} \right), \quad (5.5)$$

with probability one, so long as the limits exist. Hence, it is sufficient to calculate the following three limits:

$$\lim_{n \rightarrow \infty} \frac{(H^n)_i}{\chi_i(n)}, \quad \lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{n}{|Z_n|}.$$

The first of the above limits is straightforward. From the previous section, we know that $\bar{\sigma}_i > 0$, and so $\chi_i(n) \rightarrow \infty$, as $n \rightarrow \infty$. Hence, from the law of large numbers,

$$\lim_{n \rightarrow \infty} \frac{(H^n)_i}{\chi_i(n)} = \frac{1}{k_i^- + \sum_{r=1}^d k_r^+}, \quad \text{almost surely.} \quad (5.6)$$

Moreover, the third limit is known, and we simply cite a result (see (Woess 2009, Theorem 9.100, Exercise 9.101)).

Lemma 5.2 *The following limit holds with probability one,*

$$\lim_{n \rightarrow \infty} \frac{|Z_n|}{n} = \bar{v}, \quad \text{where} \quad \bar{v} = \left(\sum_{i=1}^d \bar{\sigma}_i \cdot \frac{F_i}{\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+} (1 - F_i)} \right)^{-1}.$$

For the middle term, $\lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n}$, we have the following lemma.

Lemma 5.3 *The following limit holds with probability one,*

$$\lim_{k \rightarrow \infty} \frac{\chi_i(n)}{n} = \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}. \quad (5.7)$$

The proof of Lemma 5.3 is somewhat lengthy, so we postpone it until later. For now, we rely on it to establish Theorem 5.1, the main result of this section.

Proof of Theorem 5.1 The proof essentially consists of plugging in the three pieces detailed above. Noting that $\lim_{t \rightarrow \infty} \frac{(H^{J_t})_o}{|Z_{J_t}|} = 0$ almost surely, together with the three limits needed for (5.5) above, we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{H^{J_t}}{|Z_{J_t}|} &= \lim_{t \rightarrow \infty} \sum_{i=1}^d \frac{(H^{J_t})_i}{|Z_{J_t}|} + \lim_{t \rightarrow \infty} \frac{(H^{J_t})_o}{|Z_{J_t}|} \\ &= \sum_{i=1}^d \frac{1}{k_i^- + \sum_{r=1}^d k_r^+} \cdot \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)} \cdot \frac{1}{\bar{v}}, \end{aligned}$$

with probability one, where according to Lemma 5.2,

$$\frac{1}{\bar{v}} = \sum_{a=1}^d \bar{\sigma}_a \cdot \frac{F_a}{\frac{k_a^-}{k_a^- + \sum_{r=1}^d k_r^+} (1 - F_a)}.$$

After some algebra, we have the following almost sure limit

$$\lim_{t \rightarrow \infty} \frac{H^{J_t}}{|Z_{J_t}|} = \frac{1}{\sum_{r=1}^d k_r^+ - \sum_{r=1}^d \bar{\sigma}_r k_r^-}.$$

By the same token, with probability one we also have

$$\lim_{t \rightarrow \infty} \frac{H^{J_t-1}}{|Z_{J_t}|} = \sum_{i=1}^d \frac{1}{k_i^- + \sum_{r=1}^d k_r^+} \cdot \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)} \cdot \frac{1}{\bar{v}}$$

$$= \frac{1}{\sum_{r=1}^d k_r^+ - \sum_{r=1}^d \bar{\sigma}_r k_r^-}.$$

Recalling $\frac{H_{J_t}^{J_t-1}}{|Z_{J_t}|} \leq \frac{t}{|Z_{J_t}|} \leq \frac{H_{J_t}^{J_t}}{|Z_{J_t}|}$, an application of the squeeze theorem completes the proof. $\square$

With our main result in hand, the remainder of this section, and the appendix, is dedicated to proving Lemma 5.3.

5.1 Proof of Lemma 5.3

The proof of Lemma 5.3 consists of two main steps.

1. We first establish the limit

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n} = \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}. \quad (5.8)$$

2. We then prove that the limit $\lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n}$ exists almost surely.

Assuming the above are established, the proof is straightforward.

Proof of Lemma 5.3 Since $\left| \frac{\chi_i(n)}{n} \right| \leq 1$, according to the Bounded Convergence Theorem, we have

$$\lim_{k \rightarrow \infty} \frac{\chi_i(n)}{n} = \lim_{k \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n} = \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}. \quad (5.9)$$

where the last equality comes from (5.8). $\square$

We break the analysis of the two points above into two separate subsections.

5.1.1 Proof of the Limit (5.8)

For any two states $x, y \in \mathcal{T}$, we define the function

$$G(x, y) := \sum_{n=0}^{\infty} p^{(n)}(x, y), \quad (5.10)$$

where $p^{(n)}(x, y) = P_x(Z_n = y)$. Note that $G(x, y)$ gives the total expected number of visits to state y given an initial condition of x . From (Nagnibeda and Woess 2002, Section 2), we know $G(x, y) < \infty$ for any $x, y \in \mathcal{T}$, and we also know the following lemma holds.

Lemma 5.4 (Nagnibeda and Woess 2002, Lemma 2.1) *Let F be the function defined in (4.4). Then the following relations hold: for any $x \in \mathcal{T}$ and any monomer type $M_i \in \mathcal{M}$,*

$$\begin{aligned} F(xM_i, o) &= F(xM_i, x) \cdot F(x, o) \quad \text{if } x \neq o, \\ G(x, x) &= \frac{1}{1 - F(x, x)}, \\ G(x, y) &= F(x, y) \cdot G(y, y) \quad \text{if } x \neq y. \end{aligned} \quad (5.11)$$

We define recursively the following (reversible) measure,

$$\mu(o) = 1, \quad \text{and} \quad \mu(x) = \mu(x^-) \cdot \frac{p(x^-, x)}{p(x, x^-)} \quad \text{for } x \neq o. \quad (5.12)$$

Corollary 5.5 *For any $y \in \mathcal{T}$,*

$$G(o, y) \cdot \mu(o) = G(y, o) \cdot \mu(y).$$

Proof Let $\Gamma_{x,y}^n$ denote the set of all admissible paths of length n from x to y , that is,

$$\Gamma_{x,y}^n := \left\{ (z_0, z_1, \dots, z_n) \in \mathcal{T}^{n+1} \mid z_0 = x, z_n = y, p(z_i, z_{i+1}) > 0, \forall i \in \{0, \dots, n-1\} \right\}.$$

For any $y \in \mathcal{T}$, the n -step transition probability $p^{(n)}(o, y)$ can then be expressed as the sum over all such paths:

$$p^{(n)}(o, y) = \sum_{(z_0, z_1, \dots, z_n) \in \Gamma_{o,y}^n} p(z_0, z_1) p(z_1, z_2) \cdots p(z_{n-1}, z_n).$$

Using the reversibility condition in (5.12), which relates μ and p for each adjacent pair (z_i, z_{i+1}) , we have for $(z_0, z_1, \dots, z_n) \in \Gamma_{o,y}^n$:

$$\begin{aligned} \mu(o)p(o, z_1) &= \mu(z_1)p(z_1, o), \\ \mu(z_1)p(z_1, z_2) &= \mu(z_2)p(z_2, z_1), \\ &\vdots \\ \mu(z_{n-1})p(z_{n-1}, z_n) &= \mu(z_n)p(z_n, z_{n-1}). \end{aligned}$$

Applying this relation repeatedly yields

$$\begin{aligned} &\mu(o)p(o, z_1)p(z_1, z_2) \cdots p(z_{n-1}, y) \\ &= \mu(z_1)p(z_1, o)p(z_1, z_2) \cdots p(z_{n-1}, y) \\ &= \mu(z_2)p(z_2, z_1)p(z_1, o)p(z_2, z_3) \cdots p(z_{n-1}, y) \\ &\vdots \\ &= \mu(y)p(y, z_{n-1})p(z_{n-1}, z_{n-2}) \cdots p(z_1, o). \end{aligned} \quad (5.13)$$

Summing over all such paths yields

$$\begin{aligned}\mu(o)p^{(n)}(o, y) &= \mu(o) \sum_{(z_0, \dots, z_n) \in \Gamma_{o,y}^n} p(z_0, z_1) \cdots p(z_{n-1}, z_n) \\ &= \mu(y) \sum_{(z_0, \dots, z_n) \in \Gamma_{y,o}^n} p(z_0, z_1) \cdots p(z_{n-1}, z_n) \\ &= \mu(y)p^{(n)}(y, o),\end{aligned}$$

where the second equality uses (5.13), the third reindexes the reversed paths. Finally, summing over $n \geq 0$, we obtain

$$G(o, y) \cdot \mu(o) = \sum_{n=0}^{\infty} p^{(n)}(o, y) \mu(o) = \sum_{n=0}^{\infty} p^{(n)}(y, o) \mu(y) = G(y, o) \cdot \mu(y),$$

and the result is shown. $\square$

For each $i, j \in \{1, \dots, d\}$, we now compute the ratio $\frac{G(o, xM_j)}{G(o, xM_i)}$, which will play an important role later.

Proposition 5.6 *For any $x \in \mathcal{T}$ and any monomer types $M_i, M_j \in \mathcal{M}$,*

$$R_{ij} := \frac{G(o, xM_j)}{G(o, xM_i)} = \frac{\bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}. \quad (5.14)$$

Proof By Corollary 5.5, for $x = o$ we have

$$\frac{G(o, M_i)}{G(o, o)} = \frac{\mu(o)G(o, M_i)}{\mu(o)G(o, o)} = \frac{\mu(M_i)G(M_i, o)}{\mu(o)G(o, o)}.$$

From (5.12) and Lemma 5.4, we have

$$\frac{\mu(M_i)}{\mu(o)} = \frac{p(o, M_i)}{p(M_i, o)}, \quad \text{and} \quad G(M_i, o) = F(M_i, o)G(o, o),$$

respectively. Hence,

$$\begin{aligned}\frac{G(o, M_i)}{G(o, o)} &= \frac{p(o, M_i) F(M_i, o)G(o, o)}{p(M_i, o)G(o, o)} \\ &= \frac{p(o, M_i) F_i}{\left(\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+} \right)}.\end{aligned}$$

By the same argument,

$$\frac{G(o, M_j)}{G(o, o)} = \frac{p(o, M_j) F_j}{\left(\frac{k_j^-}{k_j^- + \sum_{r=1}^d k_r^+} \right)}.$$

Moreover, for each $x \in \mathcal{T}$, $x \neq o$ and $M_i, M_j \in \mathcal{M}$, we have

$$\frac{G(o, xM_i)}{G(o, x)} = \frac{\mu(o)G(o, xM_i)}{\mu(o)G(o, x)} = \frac{\mu(xM_i)G(xM_i, o)}{\mu(x)G(x, o)},$$

by Corollary 5.5. From (5.12) and Lemma 5.4, respectively, we have

$$\begin{aligned} \frac{\mu(xM_i)}{\mu(x)} &= \frac{p(x, xM_i)}{p(xM_i, x)}, \\ G(xM_i, o) &= F(xM_i, o) G(o, o) = F(xM_i, x) F(x, o) G(o, o), \\ G(x, o) &= F(x, o) G(o, o). \end{aligned}$$

Combining the above yields,

$$\begin{aligned} \frac{G(o, xM_i)}{G(o, x)} &= \frac{p(x, xM_i) F(xM_i, x) F(x, o) G(o, o)}{p(xM_i, x) F(x, o) G(o, o)} = \frac{p(x, xM_i) F(xM_i, x)}{p(xM_i, x)} \\ &= \frac{p(x, xM_i) F_i}{\left(\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+} \right)}. \end{aligned}$$

By the same argument,

$$\frac{G(o, xM_j)}{G(o, x)} = \frac{p(x, xM_j) F_j}{\left(\frac{k_j^-}{k_j^- + \sum_{r=1}^d k_r^+} \right)}.$$

Finally, because

$$\frac{p(x, xM_i)}{p(x, xM_j)} = \frac{k_i^+}{k_j^+},$$

for all $x \in \mathcal{T}$, we obtain

$$\frac{G(o, xM_j)}{G(o, xM_i)} = \frac{p(x, xM_j) F_j}{\frac{k_j^-}{k_j^- + \sum_{r=1}^d k_r^+}} \cdot \frac{\frac{k_i^-}{k_i^- + \sum_{r=1}^d k_r^+}}{p(x, xM_i) F_i} = \frac{k_j^+ k_i^- F_j (k_j^- + \sum_{r=1}^d k_r^+)}{k_i^+ k_j^- F_i (k_i^- + \sum_{r=1}^d k_r^+)}$$

$$= \frac{\bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)},$$

where the last equality follows from (4.16). $\square$

We are now in a position to give the following proposition to compute the limit

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n}.$$

Proposition 5.7 *For each $i \in \{1, 2, \dots, d\}$, the following limit holds:*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{\sum_{j=1}^d \mathbb{E}_o[\chi_j(n)]} = \frac{1}{\sum_{j=1}^d R_{ij}} = \frac{\bar{\sigma}_i \left(k_i^- + \sum_{r=1}^d k_r^+ \right)}{\sum_{j=1}^d \bar{\sigma}_j \left(k_j^- + \sum_{r=1}^d k_r^+ \right)}, \quad (5.15)$$

where $\bar{\sigma}_i$ is the limiting proportion of monomer M_i as given in Theorem 4.1 and R_{ij} is the ratio defined in (5.14).

Proof From the definition of $\{\chi_j(n)\}_{j \in \{1, 2, \dots, d\}}$ in (5.4), we have

$$\sum_{j=1}^d \chi_j(n) + \chi_o(n) = n + 1.$$

From (Nagnibeda and Woess 2002, Section 2),

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_o(n)]}{n} = 0.$$

Consequently,

$$\lim_{n \rightarrow \infty} \sum_{j=1}^d \frac{\mathbb{E}_o[\chi_j(n)]}{n} = \lim_{n \rightarrow \infty} \frac{n + 1 - \mathbb{E}_o[\chi_o(n)]}{n} = 1,$$

and hence

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{\sum_{j=1}^d \mathbb{E}_o[\chi_j(n)]}.$$

This establishes the first equality in (5.15) and the last equality comes from (5.14). It remains to prove the second equality.

According to (5.14), we obtain for each $i, j \in \{1, \dots, d\}$,

$$\begin{aligned} R_{ij} &= \frac{\sum_{x \in \mathcal{T}} \frac{G(o, xM_j)}{G(o, xM_i)} \cdot G(o, xM_i)}{\sum_{x \in \mathcal{T}} G(o, xM_i)} = \frac{\sum_{x \in \mathcal{T}} G(o, xM_j)}{\sum_{x \in \mathcal{T}} G(o, xM_i)} \\ &= \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_j)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_i)}. \end{aligned}$$

From (5.4), we have for any $i, j \in \{1, \dots, d\}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_j(n)]}{\mathbb{E}_o[\chi_i(n)]} &= \lim_{n \rightarrow \infty} \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_j)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_i)} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_j)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_j)} \cdot \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_j)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_i)} \\ &\quad \cdot \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_i)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_i)}. \end{aligned}$$

For the first term and last term, we have the following from the monotone convergence theorem:

$$\lim_{n \rightarrow \infty} \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_j)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_j)} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^{\infty} p^{(\ell)}(o, xM_i)}{\sum_{x \in \mathcal{T}} \sum_{\ell=0}^n p^{(\ell)}(o, xM_i)} = 1.$$

Recognizing that the middle term is simply R_{ij} , we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_j(n)]}{\mathbb{E}_o[\chi_i(n)]} = R_{ij}.$$

Finally,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{\sum_{j=1}^d \mathbb{E}_o[\chi_j(n)]} = \lim_{n \rightarrow \infty} \frac{1}{\sum_{j=1}^d \frac{\mathbb{E}_o[\chi_j(n)]}{\mathbb{E}_o[\chi_i(n)]}} = \frac{1}{\sum_{j=1}^d R_{ij}},$$

which completes the proof. $\square$

5.1.2 Proof that $\lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n}$ Exists Almost Surely

With the value $\lim_{n \rightarrow \infty} \frac{\mathbb{E}_o[\chi_i(n)]}{n} = \frac{1}{\sum_{j=1}^d R_{ij}}$ in hand, it remains to show that $\lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n}$ exists for each $i \in \{1, \dots, d\}$. To that end, we now introduce the

following notation for $x, y \in \mathcal{T}$, $x \neq y$:

$$\begin{aligned} S_i^{(n)}(x, y) &= \sum_{s=n}^{\infty} P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right), \\ S_i(x, y) &= \sum_{n=0}^{\infty} S_i^{(n)}(x, y), \\ S_i'(x, y) &= \sum_{n=0}^{\infty} n S_i^{(n)}(x, y). \end{aligned} \quad (5.16)$$

Remark 5.8 $S_i^{(n)}(x, y)$ denotes the probability that the process Z visits y for the *first* time after at least n steps, having visited polymers ending with M_i exactly n times before arriving at y .

Similarly, we define

$$\begin{aligned} T_i^{(n)}(x, y) &= \sum_{s=n}^{\infty} P_x \left(Z_s = y, \sum_{r=1}^s \mathbf{1}_{(C(Z_r)=C_i)} = n \right), \\ T_i(x, y) &= \sum_{n=0}^{\infty} T_i^{(n)}(x, y). \end{aligned} \quad (5.17)$$

Remark 5.9 $T_i^{(n)}(x, y)$ is the probability that the process Z reaches y after at least n steps, having visited polymers ending with M_i exactly n times *before or at* $Z_s = y$ (excluding the starting state x).

With these definitions, we can now observe the following relationships:

Proposition 5.10 For any $x, y \in \mathcal{T}$, $x \neq y$, $S_i(x, y) = F(x, y) < 1$ for all $i \in \{1, \dots, d\}$.

Proof Let $x, y \in \mathcal{T}$ with $x \neq y$. Since Z is transient, and because our state space is a tree, we have $F(x, y) < 1$. Moreover, a straightforward calculation yields

$$\begin{aligned} F(x, y) &= P_x(Z \text{ eventually hits } y \text{ in finite time}) = \sum_{s=0}^{\infty} P_x(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s) \\ &= \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{n=0}^{\infty} S_i^{(n)}(x, y) = S_i(x, y).
\end{aligned}$$

□

Proposition 5.11 For any $x, y \in \mathcal{T}$, $x \neq y$, $T_i(x, y) = G(x, y) < \infty$ for all $i \in \{1, \dots, d\}$.

Proof Since Z is transient, $G(x, y) < \infty$ for any $x, y \in \mathcal{T}$. By a similar argument as in the proof of Proposition 5.10 above, for any $x, y \in \mathcal{T}$, $i \in \{1, \dots, d\}$,

$$\begin{aligned}
G(x, y) &= \sum_{s=0}^{\infty} p^{(s)}(x, y) = \sum_{s=0}^{\infty} P_x(Z_s = y) = \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} P_x \left(Z_s = y, \sum_{r=1}^s \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{n=0}^{\infty} T_i^{(n)}(x, y) = T_i(x, y).
\end{aligned}$$

□

Proposition 5.12 For $x, y \in \mathcal{T}$, with $x \neq y$, let $F'(x, y) = \sum_{n=0}^{\infty} n f^{(n)}(x, y)$. Then, $S'_i(x, y) \leq F'(x, y)$ for all $i \in \{1, \dots, d\}$.

Proof Let $x, y \in \mathcal{T}$, with $x \neq y$. For $i \in \{1, \dots, d\}$,

$$\begin{aligned}
S'_i(x, y) &= \sum_{n=0}^{\infty} n S_i^{(n)}(x, y) = \sum_{n=0}^{\infty} n \sum_{s=n}^{\infty} P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} n P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{s=0}^{\infty} \sum_{n=0}^s n P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&\leq \sum_{s=0}^{\infty} \sum_{n=0}^s s P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{s=0}^{\infty} s \sum_{n=0}^s P_x \left(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n \right) \\
&= \sum_{s=0}^{\infty} s P_x(Z_s = y, Z_l \neq y \text{ for } 0 \leq l < s) = \sum_{s=0}^{\infty} s f^{(s)}(x, y) = F'(x, y).
\end{aligned}$$

□

Remark 5.13 For any $x \in \mathcal{T} \setminus \{o\}$, the quantity $S_i^{(n)}(x, x^-)$ only depends upon the path of the process up to and including the first hitting time of x^- when starting from x .

By the tree structure of our process, all transitions occur along the edges inside the subtree T_x . Once the cone type of T_x is known, the transition probabilities along these edges are fully determined, and hence the probability $S_i^{(n)}(x, x^-)$ is also determined. From this, we see that for each $j \in \{1, \dots, d\}$, all values $S_i^{(n)}(xM_j, x)$ are identical. Therefore, $S_i^{(n)}(x, x^-)$ depends only on n, i and cone type of T_x . In particular, for all $x \in \mathcal{T}$ and for each $i, j \in \{1, \dots, d\}$, we have

$$\begin{aligned} 0 < S_i^j &:= S_i(xM_j, x) = F(xM_j, x) = F_j < 1, \\ 0 < (S_i^j)' &:= S_i'(xM_j, x) \leq F_j' := F'(xM_j, x) < \infty, \end{aligned} \quad (5.18)$$

where the bound $F'(xM_j, x) < \infty$ follows from (Woess 2009, Lemma 9.98).

With the quantities $S_i^{(n)}(x, y)$ defined in (5.16), we can now describe the transition mechanism of the process $(W_k, \chi_i(e_k))_{k \geq 1}$. This is one of our main results. We note that this result is similar to Proposition 9.55 in Woess (2009) where it was shown that (W_k, e_k) is a Markov chain. Here we are studying $(W_k, \chi_i(e_k))$. This difference is subtle but critical.

Proposition 5.14 *The process $(W_k, \chi_i(e_k))_{k \geq 1}$ is a Markov chain for each $i \in \{1, \dots, d\}$. In particular, for $x, y \in \mathcal{T}$ with $|x| = k \geq 1$ and $y^- = x$ (so $|y| = k + 1$), and $m, n \in \mathbb{Z}_{n \geq 0}$ with $n \geq m$, the transition probability is*

$$\begin{aligned} P_o(W_{k+1} = y, \chi_i(e_{k+1}) = n \mid W_k = x, \chi_i(e_k) = m) \\ = \frac{p(x, y)}{p(x, x^-)} \left(\frac{F(x, x^-)}{1 - F(x, x^-)} \right) \left(\frac{1 - F(y, x)}{F(y, x)} \right) S_i^{(n-m)}(y, x), \end{aligned}$$

where $S_i^{(n-m)}(y, x)$ is defined at (5.16):

$$S_i^{(n-m)}(y, x) = \sum_{s=n-m}^{\infty} P_y \left(Z_s = x, Z_l \neq x \text{ for } 0 \leq l < s, \sum_{r=0}^{s-1} \mathbf{1}_{(C(Z_r)=C_i)} = n - m \right).$$

We relegate the proof of Proposition 5.14 to Appendix A. Continuing, we denote the increment

$$(\Delta_k)_i = \chi_i(e_k) - \chi_i(e_{k-1}).$$

We have the following proposition.

Proposition 5.15 *The process $(W_k, (\Delta_k)_i)_{k \geq 1}$ is a Markov chain. In particular, for $x, y \in \mathcal{T}$ with $|x| = k \geq 1$ and $y^- = x$ (so $|y| = k + 1$), and $m, n \in \mathbb{Z}_{\geq 0}$, its transition probability is*

$$P_o(W_{k+1} = y, (\Delta_{k+1})_i = n \mid W_k = x, (\Delta_k)_i = m)$$

$$= \frac{p(x, y)}{p(x, x^-)} \left(\frac{F(x, x^-)}{1 - F(x, x^-)} \right) \left(\frac{1 - F(y, x)}{F(y, x)} \right) S_i^{(n)}(y, x).$$

Proof Let $x, y \in \mathcal{T}$ with $|x| = k \geq 1$ and $y^- = x$. For all $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} P_o(W_{k+1} = y, (\Delta_{k+1})_i = n \mid W_k = x, (\Delta_k)_i = m) \\ = \frac{P_o(W_{k+1} = y, (\Delta_{k+1})_i = n, W_k = x, (\Delta_k)_i = m)}{P_o(W_k = x, (\Delta_k)_i = m)}. \end{aligned}$$

We treat the numerator and denominator separately. We first handle the denominator $P_o(W_k = x, (\Delta_k)_i = m)$. Since $(\Delta_k)_i = \chi_i(e_k) - \chi_i(e_{k-1})$, partitioning on $\chi_i(e_{k-1})$ yields

$$\begin{aligned} P_o(W_k = x, (\Delta_k)_i = m) &= \sum_{l=0}^{\infty} \\ &P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m). \end{aligned}$$

Now consider the numerator

$$P_o(W_{k+1} = y, (\Delta_{k+1})_i = n, W_k = x, (\Delta_k)_i = m).$$

Using $(\Delta_{k+1})_i = \chi_i(e_{k+1}) - \chi_i(e_k)$ and partitioning on $\chi_i(e_{k-1})$ yields

$$\begin{aligned} P_o(W_{k+1} = y, (\Delta_{k+1})_i = n, W_k = x, (\Delta_k)_i = m) \\ = P_o(W_{k+1} = y, \chi_i(e_{k+1}) - \chi_i(e_k) = n, W_k = x, \chi_i(e_k) - \chi_i(e_{k-1}) = m) \\ = \sum_{l=0}^{\infty} P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m, \\ W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n). \end{aligned} \tag{5.19}$$

Then we want to calculate the inner probability in (5.19). First, we want to condition on the event

$$\{W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m\}$$

and using Markov Property:

$$\begin{aligned} P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m, W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n) \\ = P_o(W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n \mid W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m) \\ \quad \times P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m) \\ = P_o(W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n \mid W_k = x, \chi_i(e_k) = l + m) \\ \quad \times P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m). \end{aligned}$$

Turning back to (5.19), we have

$$\begin{aligned}
 & P_o(W_{k+1} = y, (\Delta_{k+1})_i = n, W_k = x, (\Delta_k)_i = m) \\
 &= \sum_{l=0}^{\infty} P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m, \\
 & W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n) \\
 &= \sum_{l=0}^{\infty} P_o(W_{k+1} = y, \chi_i(e_{k+1}) = l + m + n \mid W_k = x, \chi_i(e_k) = l + m) \\
 & \quad \times P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m) \\
 &= \frac{p(x, y)}{p(x, x^-)} \left(\frac{F(x, x^-)}{1 - F(x, x^-)} \right) \left(\frac{1 - F(y, x)}{F(y, x)} \right) S_i^{(n)}(y, x) \\
 & \quad \times \sum_{l=0}^{\infty} P_o(W_{k-1} = x^-, \chi_i(e_{k-1}) = l, W_k = x, \chi_i(e_k) = l + m),
 \end{aligned}$$

where the last equality follows from (A.8).

Collecting the above, we obtain

$$\begin{aligned}
 & P_o(W_{k+1} = y, (\Delta_{k+1})_i = n \mid W_k = x, (\Delta_k)_i = m) \\
 &= \frac{P_o(W_{k+1} = y, (\Delta_{k+1})_i = n, W_k = x, (\Delta_k)_i = m)}{P_o(W_k = x, (\Delta_k)_i = m)} \\
 &= \frac{p(x, y)}{p(x, x^-)} \left(\frac{F(x, x^-)}{1 - F(x, x^-)} \right) \left(\frac{1 - F(y, x)}{F(y, x)} \right) S_i^{(n)}(y, x).
 \end{aligned} \tag{5.20}$$

□

For the above transition probability (5.20), according to (4.5), $\frac{F(x, x^-)}{1 - F(x, x^-)}$ and $\frac{1 - F(y, x)}{F(y, x)}$ depend on the cone types of T_x and T_y , respectively. Moreover, $S_i^{(n)}(y, x)$ depends only on n, i , and the cone type of T_y from (5.18). Given all of the above, we conclude that the transition probability

$$P_o(W_{k+1} = y, (\Delta_{k+1})_i = n \mid W_k = x, (\Delta_k)_i = m)$$

depends only on n, i , and the cone types of T_x and T_y . Therefore, we can factorize with respect to the cone types, which implies that

$$(U_k, (\Delta_k)_i)_{k \geq 1}$$

forms a Markov chain on $I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)$. The transition probability for $(U_k, (\Delta_k)_i)_{k \geq 1}$ is, for any $a, b \in \{1, 2, \dots, d\}$,

$$\tilde{q}((C_a, m), (C_b, n)) := P_o(U_{k+1} = C_b, (\Delta_{k+1})_i = n \mid U_k = C_a, (\Delta_k)_i = m)$$

$$\begin{aligned}
&= \frac{k_b^+}{k_a^-} \cdot \frac{F_a}{1-F_a} \cdot \left(\frac{1-F_b}{F_b} \right) \cdot S_i^{b,n} \\
&= \frac{k_b^+}{k_a^-} \cdot \frac{F_a}{1-F_a} \cdot (1-F_b) \cdot \frac{S_i^{b,n}}{S_i^b} \\
&= V_{ab} \cdot \frac{S_i^{b,n}}{S_i^b} > 0,
\end{aligned}$$

where we define $S_i^{b,n} := S_i^{(n)}(xM_b, x)$, $x \in \mathcal{T}$ and use $S_i^b = F_b$ from (5.18) in the last equality and

$$\{V_{ab}\}_{a,b \in \{1,2,\dots,d\}}$$

are the transition probabilities of the Markov chain $(U_k)_{k \geq 1}$, as defined in (4.7).

With these probabilities in hand, we obtain the following proposition.

Proposition 5.16 Fix $i \in \{1, \dots, d\}$. The bi-variate process $(U_k, (\Delta_k)_i)_{k \geq 1}$ is a positive recurrent Markov chain on $I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)$. Its stationary probability measure $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(C_a, n) = \bar{\sigma}_a \frac{S_i^{a,n}}{S_i^a}, \quad \text{for all } (C_a, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0),$$

where $\bar{\sigma}_a$ denotes the limiting proportion of cone type C_a (equivalently, the limiting fraction of monomer M_a , as characterized in Theorem 4.1).

Proof We see that $S_i^{b,n} > 0$, $V_{ab} > 0$ for any $(C_a, m), (C_b, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)$, then $\tilde{q}((C_a, m), (C_b, n)) > 0$ for any $(C_a, m), (C_b, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)$, so $(U_k, (\Delta_k)_i)_{k \geq 1}$ is irreducible. Also, it's straightforward that $\tilde{\sigma}$ is a stationary probability measure. Since $S_i^{i,0} = 0$,

- $\tilde{\sigma}$ is a probability measure:

$$\sum_{(C_a, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)} \tilde{\sigma}(C_a, n) = \sum_{a=1}^d \sum_{n \in \mathbb{Z}_{\geq 0}} \bar{\sigma}_a \frac{S_i^{a,n}}{S_i^a} = \sum_{a=1}^d \bar{\sigma}_a = 1.$$

- $\tilde{\sigma}$ is a stationary probability measure: for any $(C_b, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)$,

$$\begin{aligned}
\sum_{(C_a, m) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)} \tilde{\sigma}(C_a, m) \tilde{q}((C_a, m), (C_b, n)) &= \sum_{a=1}^d \sum_{m \in \mathbb{Z}_{\geq 0}} \bar{\sigma}_a \frac{S_i^{a,m}}{S_i^a} \cdot V_{ab} \cdot \frac{S_i^{b,n}}{S_i^b} \\
&= \sum_{a=1}^d \bar{\sigma}_a V_{ab} \frac{S_i^{b,n}}{S_i^b} \sum_{m \in \mathbb{Z}_{\geq 0}} \frac{S_i^{a,m}}{S_i^a} \\
&= \bar{\sigma}_b \frac{S_i^{b,n}}{S_i^b} = \tilde{\sigma}(C_b, n) > 0,
\end{aligned}$$

where we used that $\bar{\sigma}$ satisfies the stationary equation for the base cone-type Markov chain $(U_k)_{k \geq 1}$, i.e.,

$$\bar{\sigma}_b = \sum_{a=1}^d \bar{\sigma}_a V_{ab}.$$

Since there exists a positive stationary probability measure for $(U_k, (\Delta_k)_i)_{k \geq 1}$, $(U_k, (\Delta_k)_i)_{k \geq 1}$ is positive recurrent and the proof of Proposition 5.16 is complete. $\square$

Finally, we need the expectation of $(\Delta_k)_i$ under the stationary distribution just computed. For that purpose, consider the projection $g_i : I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0) \rightarrow \mathbb{Z}_{\geq 0}$, $(a, n) \mapsto n$ for $(U_k, (\Delta_k)_i)_{k \geq 1}$. We have

$$\begin{aligned} E_i &:= \int_{I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)} g_i d\bar{\sigma} = \sum_{(C_a, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)} n \bar{\sigma}(C_a, n) \\ &= \sum_{(C_a, n) \in I \times \mathbb{Z}_{\geq 0} \setminus (C_i, 0)} n \bar{\sigma}_a \frac{S_i^{a, n}}{S_i^a} = \sum_{a=1}^d \sum_{n=0}^{\infty} n \bar{\sigma}_a \frac{S_i^{a, n}}{S_i^a} \\ &= \sum_{a=1}^d \bar{\sigma}_a \frac{(S_i^a)'}{S_i^a} \\ &\leq \sum_{a=1}^d \bar{\sigma}_a \frac{F_a'}{F_a} \quad (\text{by (5.18)}) \\ &< \infty. \end{aligned}$$

By the ergodic theorem for positive recurrent Markov chains, almost surely, for each $i \in \{1, \dots, d\}$,

$$\lim_{k \rightarrow \infty} \frac{\chi_i(e_k) - \chi_i(e_0)}{k} = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{m=1}^k g_i(U_m, (\Delta_m)_i) = E_i < \infty.$$

Moreover, as noted in Woess (2009),

$$\lim_{k \rightarrow \infty} \frac{\chi_i(e_0)}{k} = 0.$$

Combining the previous two points yields the almost sure limit

$$\lim_{k \rightarrow \infty} \frac{\chi_i(e_k)}{k} = E_i < \infty. \quad (5.21)$$

Note that this result pertains to the boundary process. Hence, we must shift it to the nominal process X .

We recall the following integer-valued random variables

$$\hat{k}(n) = \max \{k : e_k \leq n\}.$$

We have the following almost sure inequalities from Woess (2009),

$$\hat{k}(n) \rightarrow \infty, \quad \frac{e_{\hat{k}(n)+1}}{e_{\hat{k}(n)}} \rightarrow 1, \quad \frac{\chi_i(e_{\hat{k}(n)+1})}{\chi_i(e_{\hat{k}(n)})} \rightarrow 1,$$

so that

$$\begin{aligned} 0 &\leq \frac{n - e_{\hat{k}(n)}}{n} \leq \frac{e_{\hat{k}(n)+1} - e_{\hat{k}(n)}}{n} \leq \frac{e_{\hat{k}(n)+1} - e_{\hat{k}(n)}}{e_{\hat{k}(n)}} \rightarrow 0, \\ 0 &\leq \frac{\chi_i(n) - \chi_i(e_{\hat{k}(n)})}{\chi_i(n)} \leq \frac{\chi_i(e_{\hat{k}(n)+1}) - \chi_i(e_{\hat{k}(n)})}{\chi_i(n)} \leq \frac{\chi_i(e_{\hat{k}(n)+1}) - \chi_i(e_{\hat{k}(n)})}{\chi_i(e_{\hat{k}(n)})} \rightarrow 0. \end{aligned}$$

Then for each $i = \{1, 2, \dots, d\}$, we have as $n \rightarrow \infty$, almost surely,

$$\frac{e_{\hat{k}(n)}}{n} \rightarrow 1, \quad \frac{\chi_i(e_{\hat{k}(n)})}{\chi_i(n)} \rightarrow 1,$$

so that for each $i = \{1, \dots, d\}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\chi_i(n)}{n} &= \lim_{n \rightarrow \infty} \frac{\chi_i(n)}{\chi_i(e_{\hat{k}(n)})} \cdot \frac{\chi_i(e_{\hat{k}(n)})}{e_{\hat{k}(n)}} \cdot \frac{e_{\hat{k}(n)}}{n} = \lim_{n \rightarrow \infty} \frac{\chi_i(e_{\hat{k}(n)})}{e_{\hat{k}(n)}} \\ &= \lim_{k \rightarrow \infty} \frac{\chi_i(e_k)}{e_k} = \lim_{k \rightarrow \infty} \frac{\chi_i(e_k)}{k} \cdot \frac{k}{e_k} = E_i \cdot \bar{v} < \infty, \end{aligned}$$

where the final equality follows from both (5.21) and (Woess 2009, Theorem 9.100). Hence, the existence of the limit has been verified.

6 Copolymerization Process Involving Two Monomer Types

To illustrate our general results, we now specialize to the case of two monomer types. Thus, in this section we consider a copolymerization process X involving the monomers M_1 and M_2 . The constants k_i^+ and k_i^- represent the attachment and detachment rates of monomer M_i , for $i \in \{1, 2\}$. The process can be visualized as in Figure 1. According to Theorem 3.1, the recurrence/transience criterion for the copolymer-

ization process X is given by the parameter

$$\alpha = \frac{k_1^+}{k_1^-} + \frac{k_2^+}{k_2^-}.$$

Specifically, X is positive recurrent if $\alpha < 1$, null recurrent if $\alpha = 1$, and transient if $\alpha > 1$.

We begin by providing closed form solutions in this two-monomer case. Note that the results given here are consistent with those presented in Section III of the paper “Extracting chemical energy by growing disorder: Efficiency at maximum power” (Esposito et al. 2010).

In the transient regime, the limiting proportion of each monomer type is given by Theorem 4.1 (see Section 4). For the two-monomer case, when X is transient, we obtain explicit formulas for the limiting proportions $\bar{\sigma}_1$ and $\bar{\sigma}_2$ of M_1 and M_2 , respectively. We first consider the special case $k_1^- = k_2^-$. In this scenario, the almost-sure limiting proportions of M_1 and M_2 are

$$\bar{\sigma}_1 = \lim_{t \rightarrow \infty} \sigma_1(t) = \frac{k_1^+}{k_1^+ + k_2^+}, \quad \bar{\sigma}_2 = \lim_{t \rightarrow \infty} \sigma_2(t) = \frac{k_2^+}{k_1^+ + k_2^+}. \quad (6.1)$$

In the case of $k_1^- \neq k_2^-$, we obtain the almost-sure limiting proportions of M_1 and M_2 as

$$\begin{aligned} \bar{\sigma}_1 &= \lim_{t \rightarrow \infty} \sigma_1(t) = \frac{k_1^+ + k_2^+ + k_1^- - k_2^- - \sqrt{(k_1^+ + k_2^+ + k_1^- - k_2^-)^2 + 4k_1^+ k_2^- - 4k_1^+ k_1^-}}{2(k_1^- - k_2^-)}, \\ \bar{\sigma}_2 &= \lim_{t \rightarrow \infty} \sigma_2(t) = \frac{k_1^+ + k_2^+ + k_2^- - k_1^- - \sqrt{(k_1^+ + k_2^+ + k_2^- - k_1^-)^2 + 4k_2^+ k_1^- - 4k_2^+ k_2^-}}{2(k_2^- - k_1^-)}. \end{aligned} \quad (6.2)$$

By Theorem 5.1, the asymptotic growth velocity for the two-monomer case is given by

$$v = \lim_{t \rightarrow \infty} \frac{|X_t|}{t} = k_1^+ + k_2^+ - \bar{\sigma}_1 k_1^- - \bar{\sigma}_2 k_2^-,$$

where $\bar{\sigma}_1$ and $\bar{\sigma}_2$ given at (6.1) and (6.2).

Next, we provide simulated results of the two-monomer process with the following parameters,

$$k_1^+ = 1, \quad k_1^- = 1.8, \quad k_2^+ = 1.2, \quad \text{and} \quad k_2^- = 2.592. \quad (6.3)$$

Note that for these parameters, we have $k_1^+ < k_1^-$ and $k_2^+ < k_2^-$, but $\alpha = \frac{1}{1.8} + \frac{1.2}{2.592} \approx 1.0185 > 1$, which puts us in the transient regime even though the detachment rate for each monomer is higher than its attachment rate. Specifically, we will visualize the

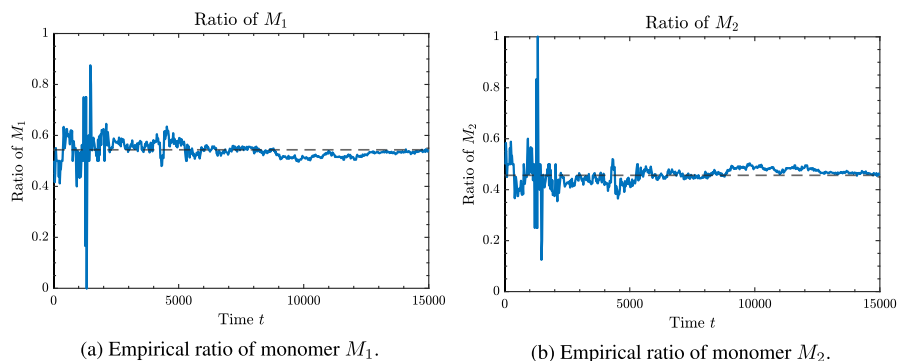


Fig. 3 Evolution of the empirical proportions of monomers M_1 and M_2 in the polymer with parameters given in (6.3). The blue curves represent simulation results, while the gray dashed lines indicate the theoretical limiting values: $\bar{\sigma}_1 \approx 0.5436$ (left) and $\bar{\sigma}_2 \approx 0.4564$ (right).

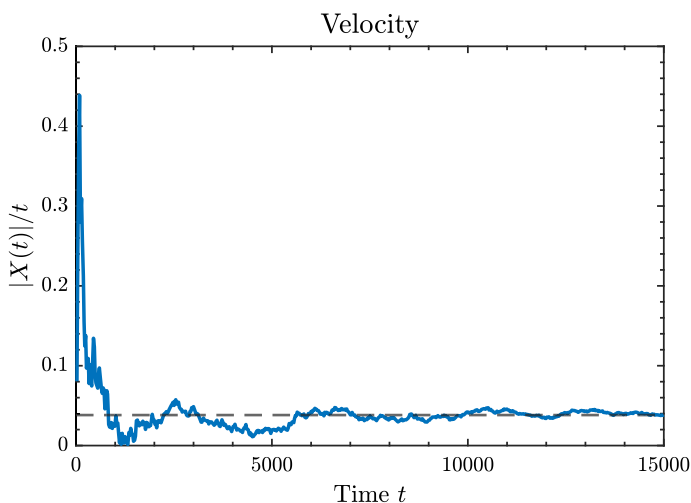


Fig. 4 Empirical polymer growth velocity $\frac{|X_t|}{t}$ for the process with parameters (6.3). Note that the blue curve approaches the theoretical value $v \approx 0.0382$ (gray dashed line).

convergence to the limiting proportion of each monomer type (Theorem 4.1) and the limiting velocity of the growth of the process (Theorem 5.1). Then, we will provide a visualization of the boundary process, which we used analytically throughout this paper.

For the parameters given in (6.3), we have

$$\bar{\sigma}_1 \approx 0.5436 \quad \text{and} \quad \bar{\sigma}_2 \approx 0.4564.$$

See Figure 3 for our simulated results demonstrating the convergence of the proportions to these values.

In the simulation of the velocity $\frac{|X_t|}{t}$ in Figure 4 we see that the empirical velocity also stabilizes around the theoretical benchmark.

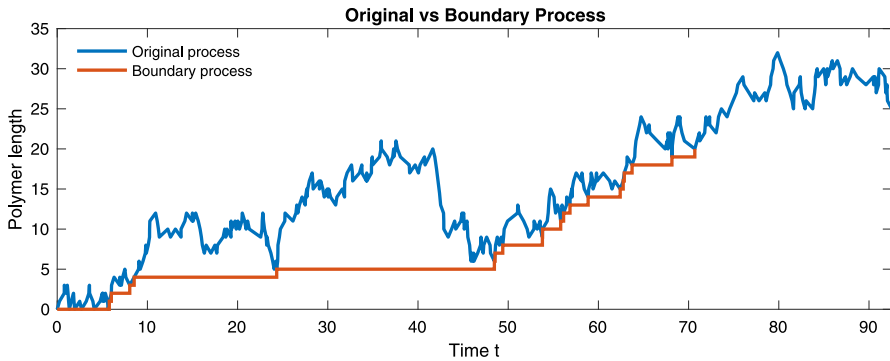


Fig. 5 Comparison between the actual process and the boundary process in terms of polymer length over time period $[0, 70]$.

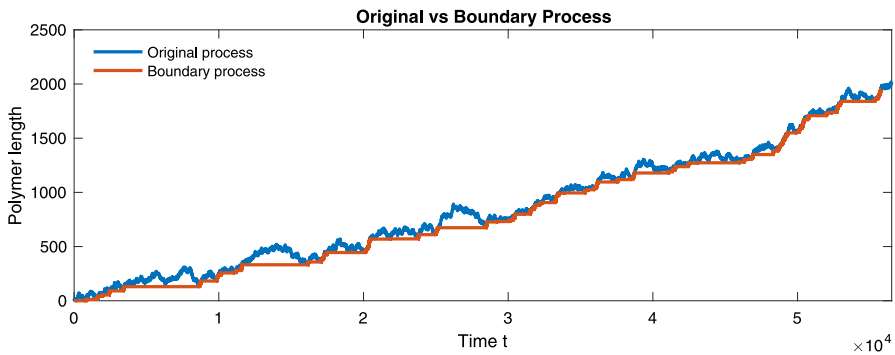


Fig. 6 Comparison between the actual process and the boundary process in terms of polymer length over time period $[0, 50,000]$.

We turn to a visualization of the boundary process. We remind the reader that this process played a critical role in the analysis carried out in sections 4 and 5. In particular, we used that the boundary process remained “close” to the original process for all time (in a very specific manner), and we hope to demonstrate that visually here. We recall that the boundary process was defined in order to keep track of the “exit state” at each level. That is, W_k was the particular state of our tree from level k (i.e., was a polymer with k monomers) that appears in the limiting “infinite length” polymer. That is, W_k is the unique prefix of the limiting polymer.

Note that the issue with simulating the boundary process is that the “last exit time” from a level is not a stopping time. Since “simulating to time infinity” is not an option, we instead chose to simulate to a very large time, $T = 200,000$, and then restrict our visualization to a much smaller time-frame. As before, we used the parameters (6.3). See Figures 5 and 6, where close agreement between the actual process and the boundary process can be observed, especially as the time-frame increases.

7 Discussion

Motivated by models from the Origins-of-Life literature, in this paper we studied a stochastic model of polymer growth. Earlier treatments focused on two monomer types and relied on heuristic arguments. We provided rigorous analysis and, using this framework, extended the analysis seamlessly to the case of d monomer types. The main contributions are as follows.

- We formulated the copolymerization process with finitely many monomer types as a continuous-time Markov chain (CTMC) on an infinite, tree-like state space. By considering the embedded discrete-time Markov chain (DTMC), we characterized the positive recurrent, null recurrent, and transient conditions using spectral theory for random walks on trees with finitely many cone types.
- In the transient regime, we provide explicit formulas for the limiting monomer proportions $\{\bar{\sigma}_i\}$. These limits are characterized by the stationary distribution of an associated cone-type Markov chain obtained from the boundary process.
- We derived an explicit formula for the asymptotic velocity of polymer growth in the transient case. The expression involves the limiting monomer proportions $\bar{\sigma}_i$, and the transition rates, and relies on reversibility arguments.

Together, these results provide, to the best of our knowledge, the first mathematically rigorous treatment of this class of copolymerization models, generalizing and justifying earlier physics-based work. The methods introduced here—particularly the spectral criterion for transience, the cone-type Markov chain formalism, and the explicit construction of boundary processes—are broadly applicable to other stochastic models of assembly processes with hierarchical or rule-based structures.

Beyond the specific copolymerization setting, the analytic framework developed—particularly in Section 5 and particularly the use of boundary processes and the control of continuous-time excursions between successive growth levels—addresses a general class of problems for Markov chains on tree-like state spaces. In such models, natural macroscopic observables (such as growth rates or empirical frequencies) are often coupled to local transition mechanisms in a way that is not directly accessible through moment equations alone. The methods introduced here provide a systematic way to extract almost-sure asymptotic behavior in this setting, and we expect them to be useful in other stochastic growth processes with similar hierarchical structure.

To conclude, we mention possible avenues for future research. Most biochemically relevant processes can be modeled as rule-based systems (Danos et al. 2007), in which a finite set of rules gives rise to an infinite cascade of assemblies and functions. These systems can sometimes be formally described using the double-pushout approach from category theory (Ehrig et al. 1973; Andersen et al. 2016), and significant work in the computer science and mathematics communities has focused on formulating them as continuous-time Markov chains via generating function techniques (Behr et al. 2016; Behr 2021), as well as developing methods for their simulation (Boutillier et al. 2018). More recently, algebraic approaches based on the Fock space formalism have been introduced in the physics literature to study rule-based systems (Rousseau and Kinney 2025). The copolymerization process examined in this paper provides a simple yet illustrative example of a rule-based system. We hope that the rigorous treatment

of the copolymerization model presented here will serve as a common platform for unifying different mathematical approaches to rule-based systems and for inspiring their extension to biologically significant problems.

A Proof of Proposition 5.14

Proof of Proposition 5.14 Let $x, y \in \mathcal{T}$ with $|x| = k \geq 1$ and $y^- = x$. Then for all $m, n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, we have

$$\begin{aligned} P_o(W_{k+1} = y, \chi_i(e_{k+1}) = n \mid W_k = x, \chi_i(e_k) = m) \\ = \frac{P_o(W_{k+1} = y, \chi_i(e_{k+1}) = n, W_k = x, \chi_i(e_k) = m)}{P_o(W_k = x, \chi_i(e_k) = m)}. \end{aligned} \quad (\text{A.1})$$

We handle the numerator and denominator separately. We begin with the denominator $P_o(W_k = x, \chi_i(e_k) = m)$. Partitioning on e_k yields

$$P_o(W_k = x, \chi_i(e_k) = m) = \sum_{l=m}^{\infty} P_o(W_k = x, e_k = l, \chi_i(e_k) = m). \quad (\text{A.2})$$

For $l \geq m \geq 0, k \geq 1$, the conditional probability formula and the Markov property yield

$$\begin{aligned} P_o(W_k = x, e_k = l, \chi_i(e_k) = m) \\ = P_o\left(Z_l = x, Z_r \in \mathcal{T}_x \setminus \{x\} \forall r \geq l+1, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m\right) \\ = P_o\left(Z_r \in \mathcal{T}_x \setminus \{x\} \forall r \geq l+1 \mid Z_l = x, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m\right) \\ \quad \times P_o\left(Z_l = x, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m\right) \\ = P_x(Z_r \in \mathcal{T}_x \setminus \{x\} \forall r \geq 1) \times P_o\left(Z_l = x, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m\right). \end{aligned} \quad (\text{A.3})$$

From (Nagnibeda and Woess 2002, Equation 4.1) we have:

$$P_x(Z_r \in \mathcal{T}_x \setminus \{x\} \forall r \geq 1) = \sum_{y: y^- = x} p(x, y) (1 - F(y, x)) = p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right).$$

Substituting this into (A.3) gives

$$P_o(W_k = x, e_k = l, \chi_i(e_k) = m) = p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right) \\ P_o \left(Z_l = x, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m \right).$$

Returning to (A.2):

$$P_o(W_k = x, \chi_i(e_k) = m) = \sum_{l=m}^{\infty} P_o(W_k = x, e_k = l, \chi_i(e_k) = m) \\ = p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right) \sum_{l=m}^{\infty} P_o \left(Z_l = x, \sum_{r=0}^l \mathbf{1}_{(C(Z_r)=C_i)} = m \right) \\ = p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right) \sum_{l=m}^{\infty} P_o \left(Z_l = x, \sum_{r=1}^l \mathbf{1}_{(C(Z_r)=C_i)} = m \right) \\ = p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right) T_i^{(m)}(o, x),$$

where $T_i^{(m)}(o, x)$ is defined at (5.17).

Now we are ready to study the numerator of (A.1). We begin by partitioning the event over the possible values of e_k and $e_{k+1} - e_k$:

$$P_o(W_k = x, W_{k+1} = y, \chi_i(e_k) = m, \chi_i(e_{k+1}) = n) \\ = P_o(W_k = x, W_{k+1} = y, \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n - m) \\ = \sum_{l=m}^{\infty} \sum_{q=n-m}^{\infty} P_o(W_k = x, W_{k+1} = y, e_k = l, e_{k+1} - e_k = q, \\ \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n - m).$$

Now we study the inner probability. For $l \geq m \geq 0$, $q \geq n - m \geq 0$, and $k \geq 1$, we have:

$$P_o(W_k = x, W_{k+1} = y, e_k = l, e_{k+1} - e_k = q, \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n - m) \\ = P_o \left(\sum_{s=0}^l \mathbf{1}_{(C(Z_s)=C_i)} = m, \sum_{s=l+1}^{l+q} \mathbf{1}_{(C(Z_s)=C_i)} = n - m, \right. \\ \left. Z_l = x, Z_{l+r} \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_{l+q} = y, Z_{l+q+v} \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1 \right).$$

Conditioning on the event

$$\left\{ Z_l = x, \sum_{s=0}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right\}$$

and utilizing the Markov property yields

$$\begin{aligned} & P_o(W_k = x, W_{k+1} = y, e_k = l, e_{k+1} - e_k = q, \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n - m) \\ &= P_o \left(Z_{l+r} \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_{l+q} = y, Z_{l+q+v} \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1, \right. \\ &\quad \left. \sum_{s=l+1}^{l+q} \mathbf{1}_{(C(Z_s)=C_i)} = n - m \mid Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right) \\ &\quad \times P_o \left(Z_l = x, \sum_{s=0}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right) \\ &= P_x \left(\sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n - m \right. \\ &\quad \left. \times P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right) \right). \end{aligned}$$

Similarly, we can split out the event

$$\{Z_{q+v} \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1\}$$

using the conditional probability formula and the Markov property:

$$\begin{aligned} & P_o(W_k = x, W_{k+1} = y, e_k = l, e_{k+1} - e_k = q, \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n - m) \\ &= P_x \left(Z_{q+v} \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1 \mid Z_r \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_q = y, \right. \\ &\quad \left. \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n - m \right) \\ &\quad \times P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n - m \right) \\ &\quad \times P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right) \\ &= P_y(Z_v \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1) \\ &\quad \times P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n - m \right) \end{aligned}$$

$$\times P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right). \quad (\text{A.4})$$

From (Nagnibeda and Woess 2002, equation 4.1), we have for $W_k = x$ and $W_{k+1} = y$ (with $x = y^-$),

$$P_y (Z_v \in T_y \setminus \{y\} \forall v \geq 1) = \sum_{z: z^- = y} p(y, z)(1 - F(z, y)) = p(y, x) \left(\frac{1}{F(y, x)} - 1 \right). \quad (\text{A.5})$$

We will postpone the analysis of the term

$$P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right)$$

until later. For now, we focus on evaluating

$$P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right).$$

by making use of the reversibility property. From an argument analogous to that in (5.13), for any path of length q with $z_0 = x$, $z_q = y$, and $p(z_s, z_{s+1}) > 0$ for all $s \in \{0, \dots, q-1\}$, the reversibility condition gives

$$\mu(x) p(x, z_1) p(z_1, z_2) \cdots p(z_{q-1}, y) = \mu(y) p(y, z_{q-1}) \cdots p(z_2, z_1) p(z_1, x). \quad (\text{A.6})$$

Specifically, consider the set of paths

$$\bar{\Gamma}_{x,y}^{q,n-m} := \{(z_0, z_1, \dots, z_q) \in \mathcal{T}^{q+1}\}$$

of length q such that $z_0 = x$, $z_q = y$, $p(z_s, z_{s+1}) > 0$ for all $s \in \{0, \dots, q-1\}$, $z_s \in \mathcal{T}_y$ for all $s \in \{1, \dots, q-1\}$ and

$$\sum_{s=1}^q \mathbf{1}_{(C(z_s)=C_i)} = n-m.$$

Then we have

$$\begin{aligned} P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\ = \sum_{(z_0, z_1, \dots, z_q) \in \bar{\Gamma}_{x,y}^{q,n-m}} p(x, z_1) \cdots p(z_{q-1}, y) \end{aligned}$$

$$\begin{aligned}
&= \sum_{(z_0, z_1, \dots, z_q) \in \bar{\Gamma}_{x,y}^{q, n-m}} \frac{\mu(y)}{\mu(x)} p(y, z_{q-1}) \cdots p(z_1, x) && \text{(by (A.6))} \\
&= \sum_{(z_0, z_1, \dots, z_q) \in \bar{\Gamma}_{x,y}^{q, n-m}} \frac{p(x, y)}{p(y, x)} p(y, z_{q-1}) \cdots p(z_1, x) && \text{(by (5.12))} \\
&= \frac{p(x, y)}{p(y, x)} P_y \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_q = x, \sum_{s=0}^{q-1} \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right).
\end{aligned}$$

We now sum over the length q from $n-m$ to ∞ of the path $\bar{\Gamma}_{x,y}^{q, n-m}$,

$$\begin{aligned}
&\sum_{q=n-m}^{\infty} P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\
&= \frac{p(x, y)}{p(y, x)} \sum_{q=n-m}^{\infty} P_y \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, 2, \dots, q-1, Z_q = x, \sum_{s=0}^{q-1} \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\
&= \frac{p(x, y)}{p(y, x)} \sum_{q=n-m}^{\infty} P_y \left(Z_r \neq x \text{ for } r = 1, 2, \dots, q-1, Z_q = x, \sum_{s=0}^{q-1} \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\
&= \frac{p(x, y)}{p(y, x)} S_i^{(n-m)}(y, x).
\end{aligned} \tag{A.7}$$

The second equality holds because when $x = y^-$, the process Z starting from y must remain in $\mathcal{T}_y$ before entering x for the first time.

Now we come back to calculate $P_o(W_k = x, W_{k+1} = y, \chi_i(e_{k+1}) = n, \chi_i(e_k) = m)$:

$$\begin{aligned}
&P_o(W_k = x, W_{k+1} = y, \chi_i(e_{k+1}) = n, \chi_i(e_k) = m) \\
&= \sum_{l=m}^{\infty} \sum_{q=n-m}^{\infty} P_o(W_k = x, W_{k+1} = y, e_k = l, e_{k+1} \\
&\quad - e_k = q, \chi_i(e_k) = m, \chi_i(e_{k+1}) - \chi_i(e_k) = n-m) \\
&= \sum_{l=m}^{\infty} \sum_{q=n-m}^{\infty} P_y(Z_v \in \mathcal{T}_y \setminus \{y\} \forall v \geq 1) \\
&\quad \times P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\
&\quad \times P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right) && \text{(by (A.4))} \\
&= p(y, x) \left(\frac{1}{F(y, x)} - 1 \right) \times \sum_{l=m}^{\infty} P_o \left(Z_l = x, \sum_{s=1}^l \mathbf{1}_{(C(Z_s)=C_i)} = m \right)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{q=n-m}^{\infty} P_x \left(Z_r \in \mathcal{T}_y \text{ for } r = 1, \dots, q-1, Z_q = y, \sum_{s=1}^q \mathbf{1}_{(C(Z_s)=C_i)} = n-m \right) \\
& \hspace{15em} \text{(by (A.5))} \\
& = p(y, x) \left(\frac{1}{F(y, x)} - 1 \right) T_i^{(m)}(o, x) \frac{p(x, y)}{p(y, x)} S_i^{(n-m)}(y, x) \quad \text{(by (A.7) and (5.17))} \\
& = \left(\frac{1}{F(y, x)} - 1 \right) T_i^{(m)}(o, x) p(x, y) S_i^{(n-m)}(y, x).
\end{aligned}$$

Collecting the above we have:

$$\begin{aligned}
P_o(W_{k+1} = y, \chi_i(e_{k+1}) = n \mid W_k = x, \chi_i(e_k) = m) \\
& = \frac{P_o(W_{k+1} = y, \chi_i(e_{k+1}) = n, W_k = x, \chi_i(e_k) = m)}{P_o(W_k = x, \chi_i(e_k) = m)} \\
& = \frac{T_i^{(m)}(o, x) p(x, y) S_i^{(n-m)}(y, x) \left(\frac{1}{F(y, x)} - 1 \right)}{T_i^{(m)}(o, x) p(x, x^-) \left(\frac{1}{F(x, x^-)} - 1 \right)} \\
& = \frac{p(x, y)}{p(x, x^-)} \left(\frac{F(x, x^-)}{1 - F(x, x^-)} \right) \left(\frac{1 - F(y, x)}{F(y, x)} \right) S_i^{(n-m)}(y, x),
\end{aligned} \tag{A.8}$$

which completes the proof. $\square$

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