Math 831 – Spring 2012

Homework 1

Due: Thursday, September 13th, 2012.

1. Suppose f is a measurable mapping from one measurable space S to another measurable space U. If A is a measurable subset of S, does it follow that the image $f(A)$ is a measurable subset of U?

2. Exercise 1.1.1 from text.

   (i) If $\mathcal{F}_i$, $i \in I$ are σ-fields, then $\cap_{i\in I}\mathcal{F}_i$ is also. (I is an arbitrary index set– possibly uncountable.)

   (ii) Use (i) to show that if we are given a set Ω and a collection $\mathcal{A}$ of subsets of Ω, then there is a smallest $\sigma-$field containing $\mathcal{A}$. This is called the σ-field generated by $\mathcal{A}$ and we denote it by $\sigma(\mathcal{A})$.

3. Exercise 1.1.2 from text. Let $\Omega = \mathbb{R}$, and let $\mathcal{F}$ be all subsets of $\mathbb{R}$ such that either A or A^c is countable. Define $\mathbf{P}(A) = 0$ if A is countable and $\mathbf{P}(A) = 1$ if uncountable. Show that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space.

4. Exercise 1.2.1 from text. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

5. Exercise 1.2.5 from text. Suppose X has a continuous density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density

$$\frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$$

   for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = (y - b)/a$, so the answer is $(1/a)f((y - b)/a)$.

6. Exercise 1.3.1. Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.