Lecture on Poisson processes: November 20th

Math 831 – Fall 2012

University of Wisconsin at Madison

Instructor: David F. Anderson

Recall Theorem

Theorem (3.6.1-page 147)

For each n, let $X_{n,m}$, with $1 \le m \le n$, be independent (Bernoulli) RVs with

$$P(X_{n,m} = 1) = p_{n,m}, \quad P(X_{n,m} = 0) = 1 - p_{n,m}.$$

Suppose

(i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$, *and*

(ii) $\max_{\{1 \le m \le n\}} p_{n,m} \to 0$.

If we let

$$S_n = X_{n,1} + \cdots + X_{n,n},$$

then $S_n \Rightarrow Z$ where Z is Poisson(λ).

i.e.

$$P(Z = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

and

$$\varphi_Z(t) = \mathbb{E} e^{itZ} = \sum_k e^{itk} e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= \exp\{-\lambda + \lambda e^{it}\} = \exp\{\lambda(e^{it} - 1)\}.$$

proof

$$\varphi_{n,m}(t) = \mathbb{E}(\exp(itX_{n,m})) = (1 - p_{n,m}) + p_{n,m}e^{it} = 1 + p_{n,m}(e^{it} - 1).$$

Then, using that $S_n = X_{n,1} + \cdots + X_{n,m}$,

$$\varphi_{S_n}(t) = \prod_{m=1}^{n} \varphi_{n,m}(t) = \prod_{m=1}^{n} (1 + p_{n,m}(e^{it} - 1))$$

What do we want? We want:

$$\left| \prod_{m=1}^{n} (1 + p_{n,m}(e^{it} - 1)) - \exp(\lambda(e^{it} - 1)) \right| \quad \text{use } \sum_{m=1}^{n} p_{n,m} \approx \lambda$$

$$\approx \left| \prod_{m=1}^{n} (1 + p_{n,m}(e^{it} - 1)) - \exp\left(\sum_{m=1}^{n} p_{n,m}(e^{it} - 1) \right) \right|$$

$$= \left| \prod_{m=1}^{n} (1 + p_{n,m}(e^{it} - 1)) - \prod_{m=1}^{n} \exp\left(p_{n,m}(e^{it} - 1) \right) \right| \quad \text{use } \left|\prod_i z_i - \prod_i w_i\right| \leq \sum_i |z_i - w_i|$$

$$\leq \sum_{m=1}^{n} \left| (1 + p_{n,m}(e^{it} - 1)) - \exp\left(p_{n,m}(e^{it} - 1) \right) \right| \quad \text{now use } |e^b - (1 + b)| \leq |b^2|$$

$$\leq \sum_{m=1}^{n} |p_{n,m}|^2 |e^{it} - 1|^2 \leq 4 \left[\max_{1 \leq m \leq n} p_{n,m} \right] \sum_{m=1}^{n} p_{n,m} \to 0.$$

Slight variant

Theorem (3.6.6-page 154)

Let $X_{n,m}$, with $1 \le m \le n$, be independent, non-negative integer valued RVs with

$$P(X_{n,m} = 1) = p_{n,m}, \quad P(X_{n,m} \ge 2) = \epsilon_{n,m}.$$

Suppose

(i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$ *(same), and*

(ii) $\max_{\{1 \le m \le n\}} p_{n,m} \to 0$ *(same), and*

(iii) $\sum_{m=1}^{n} \epsilon_{n,m} \to 0$ *(very unlikely to see 2 or more events - same as before where it was zero).*

If we let

$$S_n = X_{n,1} + \cdots + X_{n,n},$$

then $S_n \Rightarrow Z$ where Z is Poisson(λ).

i.e. same conclusion.

Think of $\epsilon_{n,m} = o(1/n)$. Maybe $\epsilon_{n,m} = c/n^2$.

proof

Let $X'_{n,m}$ almost be $X_{n,m}$. That is, we let

$$X'_{n,m} = \begin{cases} 1 & \text{if } X_{n,m} = 1 \\ 0 & \text{if } X_{n,m} \neq 1. \end{cases}$$

Let $S'_n = X'_{n,1} + \cdots + X'_{n,n}$.

1. By previous theorem, $S'_n \Rightarrow Z$ (which is Poisson(λ)).
2. By $P(X_{n,m} \geq 2) = \epsilon_{n,m}$, and $\sum_{m=1}^{n} \epsilon_{n,m} \to 0$

$$P(S_n \neq S'_n) \leq P(X_{n,m} \geq 2, \text{ for some m}) \leq \sum_{m=1}^{n} \epsilon_{n,m} \to 0.$$

We have

(i) $S'_n \Rightarrow Z$, and

(ii) $S_n - S'_n \xrightarrow{p} 0 \implies S'_n - S_n \Rightarrow 0$,

and so by your HW exercise (3.2.13), $S_n \Rightarrow Z$.

Poisson process-a first pass

We want to think of $N(s,t)$ as the number of events (customers arriving, etc) happening in (s,t). We make the following modeling choices.

(i) the # of arrivals in disjoint intervals are independent.

(ii) the distribution of $N(s,t)$ only depends upon $t-s$ (weaken later)

(iii) $P(N(0,h)=1)=P(N(t,t+h)=1)=\lambda h+o(h)$, and

(iv) $P(N(0,h)\geq 2)=P(N(t,t+h)\geq 2)=o(h)$.

Theorem

If four conditions above hold, then $N(0,t)$ has a Poisson distribution with mean/parameter λt.

Proof.

Simply let

$$X_{n,m}=N\left(\frac{mt}{n}-\frac{t}{n},\frac{mt}{n}\right),\quad \text{for } 1\leq m\leq n,$$

apply previous theorem: $P(X_{n,m}=1)=p_{n,m}=\lambda(t/n)+o(1/n)$.

(i) $\sum_{m=1}^{n} p_{n,m}\to\lambda t$, and

(ii) $\max_{\{1\leq m\leq n\}} p_{n,m}\to 0$, and

(iii) $\sum_{m=1}^{n}\epsilon_{n,m}\to 0$.

□

Poisson process-a first pass

Definition

A family of RVs N_t, $t \geq 0$ satisfying

1. Independent increments: $t_0 < t_1 < \cdots < t_n$ implies

$$\{N(t_k) - N(t_{k-1})\}, \quad 1 \leq k \leq n,$$

   are independent.

2. $N(t) - N(s)$ is Poisson$(\lambda(t-s))$

is called a Poisson process with rate λ.

Note that if $N(t)$ is Poisson then it satisfies other conditions since:

$$P(N(h) - N(0) = 0) = e^{-\lambda h} = 1 - \lambda h + l(h)$$

$$P(N(h) - N(0) = 1) = e^{-\lambda h}\lambda h \approx \lambda h + o(h)$$

$$P(N(h) - N(0) \geq 2) = o(h).$$

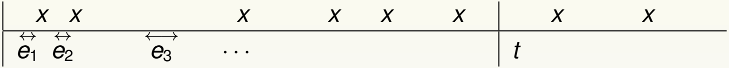
We now have two equivalent definitions of a Poisson process. We are now going to build it in a different fashion.

Constructing a Poisson process

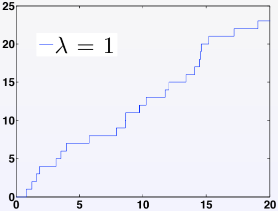
Will can also view a Poisson process, $N(t) = N([0, t])$, through the lens of an underlying point process. (More on that later)

(a) Let $\{e_i\}$ be i.i.d. exponential random variables with parameter λ: $f_{e_1}(x) = \lambda e^{-\lambda x}$.

(b) Now, put points down on a line with spacing equal to the e_i:



- Let $N_\lambda(t)$ denote the number of points hit by time t.
- In the figure above, $N_1(t) = 6$.

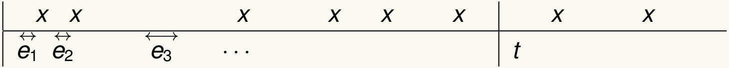


Constructing a Poisson process

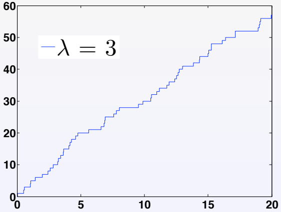
Will can also view a Poisson process, $N(t) = N([0, t])$, through the lens of an underlying point process. (More on that later)

(a) Let $\{e_i\}$ be i.i.d. exponential random variables with parameter λ: $f_{e_1}(x) = \lambda e^{-\lambda x}$.

(b) Now, put points down on a line with spacing equal to the e_i:



- Let $N_\lambda(t)$ denote the number of points hit by time t.
- In the figure above, $N_1(t) = 6$.



Constructing a Poisson process

Formally, we let e_i be independent exponential parameter λ, and let

$$T_n = e_1 + \cdots + e_n,$$

and

$$N(t) = \sup\{n : T_n \leq t\}.$$

Why is this a Poisson process? Essentially it's the loss of memory property of exponential random variables. See page 155/156 for details.

Transformations of poisson processes

First question: can I get a Poisson process with rate $\lambda > 0$ from one with a rate of one?

Answer: Yes. Just move hand quicker in previous picture!
Let

$$N_\lambda(t) \stackrel{\text{def}}{=} N_1(\lambda t).$$

Check conditions:

1. Let $t_0 < t_1 < \cdots < t_n$, then

   $$\{N_\lambda(t_k) - N_\lambda(t_{k-1})\} = \{N_1(\lambda t_k) - N_1(\lambda t_{k-1})\},$$

   are independent since $\lambda t_0 < \lambda t_1 < \cdots < \lambda t_n$.
2. For $k \geq 0$,

   $$\begin{aligned} N_\lambda(t) - N_\lambda(s) &= N_1(\lambda t) - N_1(\lambda s) \sim \text{Poisson}(1 \cdot (\lambda t - \lambda s)) \\ &= \text{Poisson}(\lambda(t - s)). \end{aligned}$$

Also note:

$$P(N_\lambda(t+h) - N_\lambda(t) = 1) = P(N_1(\lambda t + \lambda h) - N_1(\lambda t) = 1) = \lambda h + o(h).$$

This is fun! Are there other Poisson processes we can define?

Sure! Move hand at varying (non-negative speeds)!

Let's define a non-homogeneous Poisson process with intensity function $\lambda(t)$, to be a process $N(t)$ such that:

1. Independent increments: $t_0 < t_1 < \cdots < t_n$ implies
$$\{N(t_k) - N(t_{k-1})\}, \quad 1 \leq k \leq n,$$
are independent.
2.
$$N(t) - N(s) \sim \text{Poisson}\left(\int_0^t \lambda(s)ds\right).$$

Note:

$$P(N(t+h) - N(t) = 1) = e^{\int_t^{t+h} \lambda(s)ds} \int_t^{t+h} \lambda(s)ds \approx \lambda(t)h + o(h).$$

Transformations of poisson processes

Can I construct a non-homogenous Poisson process with intensity $\lambda(t) \geq 0$ from a homogeneous, rate one Poisson process?

Let

$$N_\lambda(t) \stackrel{\text{def}}{=} N_1\left(\int_0^t \lambda(s)ds\right).$$

Check conditions:

1. Let $t_0 < t_1 < \cdots < t_n$, then

$$\{N_\lambda(t_k) - N_\lambda(t_{k-1})\} = \left\{N_1\left(\int_0^{t_k} \lambda(s)ds\right) - N_1\left(\int_0^{t_{k-1}} \lambda(s)ds\right)\right\},$$

   are independent since

$$\int_0^{t_0} \lambda(s)ds \leq \int_0^{t_1} \lambda(s)ds \leq \cdots \leq \int_0^{t_n} \lambda(s)ds.$$

2. For $k \geq 0$,

$$\begin{aligned} N_\lambda(t) - N_\lambda(s) &= N_1\left(\int_0^t \lambda(s)ds\right) - N_1\left(\int_0^s \lambda(s)ds\right) \\ &= \text{Poisson}\left(\int_0^t \lambda(r)dr - \int_0^s \lambda(r)dr\right) \end{aligned}$$

The magic of unit rate Poisson processes

So, we learned that we can construct many Poisson processes with only a unit-rate Poisson process as a building block. Idea was to transform time.

Question: Are there other ways to transform processes that we haven't considered yet?

Great question! The answer is yes, but we need a more inclusive definition of a Poisson "process". Will look more generally into Point processes.

Point processes

Idea: Want to be able to model a random distribution of points in a space, usually a subset of Euclidean space

- $\mathbb{R}$
- $[0, \infty)$
- $\mathbb{R}^d$, $d \geq 1$.

Example

Renewal processes distribute points on $[0, \infty)$ so that gaps between points are iid random variables.

Example

The Poisson process is a renewal process which distributes points so gaps are iid exponential RVs.

Modeling examples:

1. The breakdown times of a machine.
2. Position of proteins on a cell membrane (Ankit's Ph.D. thesis).
3. the positions and times of earthquakes in the next 50 years.
4. Positions of lightning strikes.

Point processes basics: see Resnick, Adventures in Stochastic Processes, 1992

We suppose that E is a subset of Euclidian space, $\mathbb{R}^d$ (or $[0, \infty), \mathbb{R}^2$, etc). We

- Suppose that $\{X_n, n \geq 0\}$ are random elements of E, which represent points in the state space E.
- Define the discrete (random) measure

$$\epsilon_{X_n}(A) = \begin{cases} 1, & \text{if } X_n \in A, \\ 0, & \text{if } X_n \notin A, \end{cases}$$

  Note: $\epsilon_{X_n}(\cdot)$ takes $A \subset E$ as an input and asks whether or not $X_n \in A$.
- By summing over n, we get the total number of random points X_n which fall in A.
- We then define the counting measure, N, by

$$N(\cdot) = \sum_n \epsilon_{X_n}(\cdot),$$

  so that

$$N(A) = \sum_n \epsilon_{X_n}(A),$$

  is the random number of points that fall in the set A.

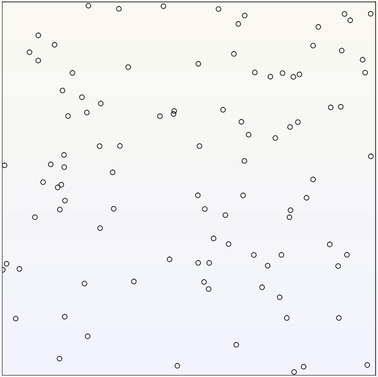
Example: Binomial Process

1. Place $n = 100$ points at random locations inside a bounded region $W \subset \mathbb{R}^2$.
2. Let $X_1, \ldots, X_{100}$ be i.i.d. points uniformly distributed in W.
3. That is, the density of each X_i is

$$f(x) = \begin{cases} 1/\lambda_2(W) & \text{if } x \in W \\ 0 & \text{else} \end{cases}$$

where $\lambda_2(W)$ is the area of W.

A realization could be:



Example: Binomial Process

For any bounded set B in $\mathbb{R}^2$, we have

$$P(X_i \in B) = \int_B f(x)dx = \int_{B \cap W} 1/\lambda_2(W)ds = \frac{\lambda_2(B \cap W)}{\lambda_2(W)},$$

And

$$N(B) = \sum_{i=1}^{n} 1(X_i \in B).$$

Note that $N(B)$ has a binomial distribution with parameters $n = 100$ and

$$p = \frac{\lambda_2(B \cap W)}{\lambda_2(W)},$$

which is why process is called binomial process.

Also note that if B_1 and B_2 are disjoint, then **not** in general independent. For example if $B_1 \cup B_2 = W$ and disjoint, we know

$$N(B_1) = 100 - N(B_2).$$

Point process basics

Definition

N is called a **point process** and $\{X_n\}$ are called the **points**.

Note: The counting measure N depends explicitly on the points. In this sense, it is a *random measure*.

An important statistic for a point process is the mean measure, or intensity. It is

$$\mu(A) \stackrel{\text{def}}{=} \mathbb{E}N(A).$$

The expected number of points in the region A.

- If N is a poisson process with parameter λ, then $\mu([a, b]) = \mathbb{E}(N(b) - N(a)) = \lambda(b - a)$.
- For binomial process

$$\mathbb{E}N(A) = n \cdot \frac{\lambda_2(A \cap W)}{\lambda_2(W)}.$$

Poisson random measure

Definition

Let E be a subset of $\mathbb{R}^d$, and let μ be a measure on E which is finite on every compact set. The Poisson process N on E with intensity measure μ is a point process on E such that

1. For compact $A \subset E$, the count $N(A)$ has a Poisson distribution with mean $\mu(A)$.
2. If $A_1, \ldots, A_k$ are disjoint compact subsets of E, then $N(A_1), \ldots, N(A_k)$ are independent.

N is also called a Poisson process with mean measure μ or a Poisson random measure, PRM(μ), if

See example 3.6.8 page 158 for formal construction when $\mu(E) < \infty$: Let X_n be i.i.d. with measure $\nu(\cdot) = \mu(\cdot)/\mu(E)$, let Y be independent Poisson($\mu(E)$). Then let $N(A) = |\{j \leq N : X_j \in A\}|$. (thinning)

If $\mu(E) = \infty$, build on disjoint pieces and tie together.

Poisson random measure

When the mean measure is a multiple of Lebesgue, i.e.

- **length** when $E = \mathbb{R}$, **area** when $E = \mathbb{R}^2$, **volume** when $E = \mathbb{R}^3$,

we call the process *homogeneous*.

- In homogeneous case, there is an $\alpha > 0$ such that for any $A \in \mathcal{E}$ we have that $N(A)$ is Poisson with mean

$$\mathbb{E}N(A) = \alpha|A|$$

  (where $|A|$ is Lebesgue measure of A).

- When $E = \mathbb{R}$, the parameter α is called the rate or intensity of the (homogeneous) Poisson process.

Non-homogenous Poisson process: part 2

Suppose that for open intervals $(a, b) \subset \mathbb{R}$, the mean measure μ for a Poisson process is

$$\mu((a, b)) = G(b) - G(a),$$

for some non-decreasing, absolutely continuous function G.

For example: $G(t) = t^2$

If G has density g, i.e. $g(t) = 2t$, then

$$\mu((a, b)) = G(b) - G(a) = \int_a^b g(s)ds, \quad \text{or more generally } \mu(A) = \int_A g(s)ds.$$

So long as $G(t) \neq ct$ for some t, i.e. $g(t) \neq c$, then this is non-homogeneous. We have that

$$P(N(a, b) = k) = e^{-(G(b)-G(a))} \frac{(G(b) - G(a))^k}{k!} = e^{-(\int_a^b g(s)ds)} \frac{(\int_a^b g(s)ds)^k}{k!}.$$

Example

Say restaurants are distributed relative to your restaurant as a spatial Poisson process with rate $\alpha = 3$ per square mile. What is expected distance to nearest competitor?

Let R be the distance of the nearest competitor and let $d(r)$ be a disc of radius r centered at your rest. Then,

$$P(R > r) = P[N(d(r)) = 0] = e^{-3|d(r)|}.$$

Obviously we have that

$$|d(r)| = \pi r^2.$$

Therefore,

$$P[R > r] = e^{-3\pi r^2}.$$

The expected distance is then

$$\mathbb{E}(R) = \int_0^\infty P[R > r]dr = \int_0^\infty e^{-3\pi r^2} dr \qquad \text{(use the subs. } u/\sqrt{2} = \sqrt{3\pi}r)$$
$$= \frac{1}{\sqrt{2}\sqrt{3\pi}} \int_0^\infty e^{-u^2/2} du = \frac{1}{\sqrt{2}\sqrt{3\pi}} \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-u^2} du$$
$$= \frac{1}{2} \frac{1}{\sqrt{3}} \approx .2887\text{miles}.$$

Back to original good question: are other transformations possible?

We suppose that

1. $N(\cdot) = \sum_n \epsilon_{X_n}(\cdot)$ is a Poisson process with state space E.
2. The mean measure is μ. So, $P(N(A) = k) = e^{-\mu(A)}\mu(A)^k/k!$.
3. T is some 1-1 transformation (function) with domain E and range E' (both Euclidean spaces):
$$T : E \to E'.$$

- Note that the function T^{-1} defines a set mapping from subsets of E' to subsets of E:
$$\text{for } A' \subset E' \quad \text{we have} \quad T^{-1}(A') = \{e \in E \ : \ T(e) \in A'\}.$$
That is: $T^{-1}(A')$ is the pre-image of A' under T. **Draw picture.**

Transformation

Theorem

Suppose that $T : E \to E'$ is a one-to-one (bijective) mapping between Euclidean spaces such that if $B' \subset E'$ is bounded, then so is $T^{-1}B' \subset E$. If N is PRM(μ) on E with points $\{X_n\}$, then $N' := N \circ T^{-1}$ is PRM(μ') on E' with points $\{T(X_n)\}$ and where $\mu' = \mu \circ T^{-1}$.

Thus: if you shift points of a Poisson process around you still have a Poisson process!

Proof.

We need to show Poisson distribution and independence property.
First, if B' bounded, then $T^{-1}(B')$ has finite number of points and

$$P(N'(B') = k) = P(N(T^{-1}(B')) = k) = e^{-\mu(T^{-1}(B'))}\frac{\mu(T^{-1}(B'))^k}{k!}.$$

Thus, $N'(\cdot)$ is Poisson distributed with mean measure $\mu' = \mu \circ T^{-1}$.

Next, if $B_1', \ldots, B_m'$ are disjoint, then so are $T^{-1}B_1', \ldots, T^{-1}B_m'$. Therefore,

$$\left(N'(B_1'), \cdots, N'(B_m')\right) = \Big(N(T^{-1}(B_1')), \cdots, N(T^{-1}(B_m'))\Big).$$

are independent.

□

Examples

Let

$$N = \sum_{n=1}^{\infty} \epsilon_{X_n}$$

be a homoegeneous Poisson process with rate $\alpha = 1$ on $E = [0, \infty)$. The mean measure is

$$\mu(A) = |A|,$$

and $\mu([0, t]) = t$.

Now we suppose that $Tx = x^2$. Then,

$$\sum_n \epsilon_{X_n^2}$$

is PRM and the mean measure $\mu'$is given by

$$\mu'([0, t]) = \mu(T^{-1}[0, t]) = \mu\{x \ : \ Tx \leq t\} = \mu\{x \ : \ x^2 \leq t\} = \mu([0, \sqrt{t}]) = \sqrt{t}.$$

Note, that this is a PRM with local intensity, or density, satisfying

$$\int_0^t \lambda(s)ds = \sqrt{t} \implies \lambda(t) = \frac{d}{dt}\sqrt{t} = \frac{1}{2}t^{-1/2}.$$

Examples

If $Tx = (x, x^2)$, then

$$\sum_n \epsilon_{TX_n} = \sum_n \epsilon_{(X_n, X_n^2)}$$

is Poisson on $\mathbb{R} \times \mathbb{R}^2$ with a mean measure that is zero off the graph (x, x^2).

Marking and thinning

We are going to "**mark**" each point of the Poisson process with a random variable $\{J_n\}$, which are independent RVs that are also independent from the Poisson process. Then, (X_n, J_n) will also be a Poisson process on an enlarged state space.

Idea: Think of J_n as marking toys coming off assembly line. Marks could be 1 for functional and 0 for dysfunctional. (we will see that this is thinning).

Marking and thinning

Theorem

Suppose that $\{X_n\}$ are random elements of state space E_1 such that

$$\sum_n \epsilon_{X_n}(\cdot)$$

is PRM(μ). Suppose that J_n are iid RVs of state space E_2 (usually subset of $\mathbb{R}$) with common distribution function F. We also suppose sequence $\{J_n\}$ and process are independent. Then, the point process,

$$\sum_n \epsilon_{(X_n, J_n)}(\cdot, \cdot)$$

on $E_1 \times E_2$ is PRM with mean measure $\mu \times F$, meaning that if $A_1 \subset E_1$ and $A_2 \subset E_2$, then

$$\mu \times F(A_1 \times A_2) = \mu(A_1)F(A_2) = \mu(A_1)P[J_1 \in A_2].$$

Marking and thinning

We won't prove, (see Resnick 1992), but note that mean measure is correct in sense

$$\begin{aligned}
\mathbb{E}\sum_n \epsilon_{(X_n,J_n)}(A_1 \times A_2) &= \sum_n P[(X_n, J_n) \in A_1 \times A_2] \\
&= \sum_n P[X_n \in A_1]P[J_n \in A_2] \quad \text{(by independence)} \\
&= \sum_n P[X_n \in A_1]P[J_1 \in A_2] \quad \text{(by iid of } J_n\text{)} \\
&= \left(E\sum_n \epsilon_{X_n}(A_1)\right) P[J_1 \in A_2] \\
&= \mu(A_1)P[J_1 \in A_2].
\end{aligned}$$

Special case: thinning

Suppose

1. that $N = \sum_n \epsilon_{X_n}$ is a Poisson process on the state space E with mean measure μ.

2. we keep each point with a probability of $p \in (0, 1)$. We delete with probability $q = 1 - p$.

3. Let N_r be the point process of retained points and N_d the point process of deleted points.

Will show: $N_r(\cdot)$ and $N_d(\cdot)$ are independent Poisson processes with mean measures $p\mu(\cdot)$ and $q\mu(\cdot)$, respectively.

Thinning

Let $\{B_i\}$ be iid Bernoulli RVs independent of the points of the process $\{X_n\}$ so that

$$P(B_1 = 1) = p, \quad P(B_1 = -1) = 1 - p.$$

We know that

$$\sum_n \epsilon_{(X_n, B_n)}$$

is a Poisson process on $E \times \{-1, 1\}$ with mean measure $\mu \times P(B_1 = \cdot)$.

- Think of $\{X_n \ : \ B_n = 1\}$ as the retained points;
- Think of $\{X_n \ : \ B_n = -1\}$ as the deleted points.

Then,

$$N_r(\cdot) = \sum_n \epsilon_{(X_n, B_n)}((\cdot) \times \{1\}) = \sum_{n: B_n = 1} \epsilon_{X_n}(\cdot)$$

$$N_d(\cdot) = \sum_n \epsilon_{(X_n, B_n)}((\cdot) \times \{-1\}) = \sum_{n: B_n = -1} \epsilon_{X_n}(\cdot)$$

are independent processes. Also, for $A \subset E$ we have

$$N_r(A) = \sum_n \epsilon_{(X_n, B_n)}((A) \times \{1\}) \sim \text{Poisson}\Big(P\{B_n = 1\} \cdot \mu(A)\Big) = \text{Poisson}(p \cdot \mu(A))$$

$$N_d(A) = \sum_n \epsilon_{(X_n, B_n)}((A) \times \{-1\}) = \text{Poisson}\Big(P\{B_n = -1\} \cdot \mu(A)\Big) = \text{Poisson}(q \cdot \mu(A)).$$

Thinning

Proof for thinning. We need:

1. N_r and N_d are *each* Poisson.
2. Independent from each other.

First note: if $A_1, \ldots, A_n$ are disjoint, then $N_r(A_i)$ depends only upon $N(A_i)$ and independent Bernoulli's. Hence $N_r(A_1), \ldots, N_r(A_n)$ are independent. Same for N_d.

Also get $N_r(A_i)$ and $N_d(A_j)$ for $j \neq i$ are independent.

Still need:

(i) N_r, N_d have Poisson distributions and

(ii) independent from each other on a given set A.

Handle together.

Proof continued

Let $Z_n =$ number of $B_1, \ldots, B_n$ that gave one $\sim$ binomial(n, p).

$$\begin{aligned} P(N_r(A) = k, N_d(A) = j) &= P(N(A) = j + k, Z_{j+k} = k) \\ &= e^{-\mu(A)} \frac{(\mu(A))^{k+j}}{(k+j)!} \cdot \frac{(j+k)!}{k!j!} p^k (1-p)^k \\ &= e^{-p\mu(A)} \frac{(p\mu(A))^k}{k!} e^{-q\mu(A)} \frac{(q\mu(A))^j}{j!} \end{aligned}$$

So we have Poisson and independent.

Thinning into M strips is no harder.

Thinning

This is amazing

Example

People arrive according to rate $\lambda = 5$. 50% are women. Knowing there have been 100 women, does not tell you how many men there have been!