In these notes, we will prove the following

Theorem 1. *Let $f \in C_b^3(\mathbb{R})$ (the bounded continuous functions with three bounded continuous derivatives. If B_t is a standard Brownian motion with respect to $\{F_t\}$, then*

$$f(B_t) - f(0) - \int_0^t \frac{1}{2} f''(B_s) ds$$

is a $\{\mathcal{F}_t\}$-martingale.

Proof. Let $r < t$. We consider an arbitrary (fine) discretization of $[r, t]$ and have by Taylor's theorem

$$\begin{aligned}
\mathbb{E}[f(B_t) - f(B_r)|\mathcal{F}_r] &= \sum_i \mathbb{E}[f(B_{t_{i+1}}) - f(B_{t_i})|\mathcal{F}_r] \\
&= \sum_i \mathbb{E}[f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_r] + \sum_i \mathbb{E}[\frac{1}{2} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2|\mathcal{F}_r] \\
&\quad + \sum_i \mathbb{E}\frac{1}{6}[f'''(\xi_i)(B_{t_{i+1}} - B_{t_i})^3|\mathcal{F}_r],
\end{aligned}$$

where $\xi_i \in [B_{t_i}, B_{t_{i+1}}]$. Since $\mathcal{F}_r \subset \mathcal{F}_{t_i}$, we have that

$$\begin{aligned}
\mathbb{E}[f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_r] &= \mathbb{E}[\mathbb{E}[f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i}]|\mathcal{F}_r] \\
&= \mathbb{E}[f'(B_{t_i})\mathbb{E}[(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i}]|\mathcal{F}_r] \\
&= 0.
\end{aligned}$$

Further,

$$\begin{aligned}
\mathbb{E}[\frac{1}{2} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2|\mathcal{F}_r] &= \mathbb{E}[\mathbb{E}[\frac{1}{2} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2|\mathcal{F}_{t_i}]|\mathcal{F}_r] \\
&= \mathbb{E}[\frac{1}{2} f''(B_{t_i})\mathbb{E}[(B_{t_{i+1}} - B_{t_i})^2|\mathcal{F}_{t_i}]|\mathcal{F}_r] \\
&= \mathbb{E}[\frac{1}{2} f''(B_{t_i})(t_{i+1} - t_i)|\mathcal{F}_r].
\end{aligned}$$

Hence, collecting we have

$$\mathbb{E}[f(B_t) - f(B_r)|\mathcal{F}_r] = \mathbb{E}\sum_i \frac{1}{2} f''(B_{t_i})(t_{i+1} - t_i)|\mathcal{F}_r] + \mathbb{E}\frac{1}{6}[\sum_i f'''(\xi_i)(B_{t_{i+1}} - B_{t_i})^3|\mathcal{F}_r].$$

The final term goes to zero almost surely as the mesh gets smaller since:

$$\begin{aligned}
|\mathbb{E}[\sum_i f'''(\xi_i)(B_{t_{i+1}} - B_{t_i})^3|\mathcal{F}_r]| &\leq \mathbb{E}[\sum_i |f'''(\xi_i)| \cdot |(B_{t_{i+1}} - B_{t_i})|^3] \\
&\leq \|f'''\|_\infty C' \sum_i (t_{i+1} - t_i)^{3/2} \to 0,
\end{aligned}$$

as the mesh size goes to zero. Finally, by calculus we have that with a probability of one (since B_t is continuous)

$$\sum_i \frac{1}{2} f''(B_{t_i})(t_{i+1} - t_i) \to \int_r^t f''(B_s) ds.$$

Also,

$$|\sum_i \frac{1}{2} f''(B_{t_i})(t_{i+1} - t_i) \leq \frac{1}{2} \|f''\|_\infty t.$$

So by the conditional version of the dominated convergence theorem (page 279),

$$\mathbb{E}[\sum_i \frac{1}{2} f''(B_{t_i})(t_{i+1} - t_i) | \mathcal{F}_r] \to \mathbb{E}[\int_r^t \frac{1}{2} f''(B_s) ds | \mathcal{F}_r]$$

almost surely. Hence, we see that,

$$\mathbb{E}[f(B_t) - f(B_r) - \int_r^t \frac{1}{2} f''(B_s) ds | \mathcal{F}_r] = 0,$$

which after rearranging terms gives the result. □