Math 635: Chapter 6 Notes

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Section 6.1: The Itô Integral: first steps

Aim: to define the Itô integral

$$I(f)(\omega) = \int_0^T f(\omega, t) dB_t.$$

Can we construct just using ideas of Riemann-Stieltjes integration:

$$\int_0^T f(s) dg(s)?$$

Recall that a function g has **bounded variation** on $[0, T]$ if

$$\sup_{P \in \text{ partitions}} \sum_i |g(x_{i+1}) - g(x_i)| < \infty.$$

Can define

$$\int_0^T f(s) dg(s) = \lim_{\|\Delta\| \to 0} \sum_i f(x_i)(g(x_{i+1}) - g(x_i)),$$

if g has bounded variation. **Does Brownian path have bounded variation?**

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Q: Does Brownian path have bounded variation? Answer: No.

Let's try to understand this a bit... first some definitions

1. Let $f : [a, b] \to \mathbb{R}$ be a real-valued function defined on $a \leq t \leq b$.
2. Let $\Delta_n = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ be a partition of $[a, b]$.
3. Define the **mesh** of the partition Δ_n by
$$\|\Delta_n\| = \max_{i \leq i \leq n}(t_i - t_{i-1}).$$
4. Then, for $p > 0$ define
$$Q_p(f; a, b, \Delta_n) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p.$$

Theorem

If $\{\Delta_n : n = 1, 2, 3, \dots\}$ is a sequence of partitions of $[a, b]$ such that $\|\Delta_n\| \to 0$, then

$$Q_2(B; a, b, \Delta_n) \to b - a \quad \text{in} \quad L^2(dP),$$

as $n \to \infty$.

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Theorem

If $\{\Delta_n : n = 1, 2, 3, \dots\}$ *is a sequence of partitions of* $[a, b]$ *such that* $\|\Delta_n\| \to 0$*, then*

$$Q_2(B; a, b, \Delta_n) \to b - a \quad \text{in} \quad L^2,$$

as $n \to \infty$*.*

Proof.

Define

$$X_i = (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}), \quad \text{and} \quad Y_n = \sum_{i=1}^{n} X_i.$$

Goal is to show that $Y_n \to 0$ in L^2: $\mathbb{E}[Y_n^2] \to 0$.

Squaring

$$Y_n^2 = \sum_{i=1}^{n} X_i^2 + 2 \sum_{i<j} X_i X_j.$$

Taking expectations yields

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2).$$

□

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Recall,

$$X_i = (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}), \quad \text{and} \quad Y_n = \sum_{i=1}^{n} X_i.$$

However,

$$\begin{aligned}\mathbb{E}(X_i)^2 &= \mathbb{E}(B_{t_i} - B_{t_{i-1}})^4 - 2(t_i - t_{i-1})\mathbb{E}(B_{t_i} - B_{t_{i-1}})^2 + (t_i - t_{i-1})^2 \\ &= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\ &= 2(t_i - t_{i-1})^2.\end{aligned}$$

Hence,

$$\mathbb{E}Y_n^2 = \sum_i \mathbb{E}X_i^2 = 2\sum_i (t_i - t_{i-1})^2 \leq 2\|\Delta_n\|(b-a) \to 0,$$

as $\|\Delta_n\| \to 0$, and $Y_n \to 0$ in L^2. Thus,

$$\|\sum_i |B_{t_i} - B_{t_{i-1}}|^2 - (b-a)\|_2 = \|Q_2(B; a, b, \Delta_n) - (b-a)\|_2 \to 0.$$

Since we have

$$\lim_{\|\Delta_n\| \to 0} \sum_i (B_{t_i} - B_{t_{i-1}})^2 \xrightarrow{L^2} (b-a),$$

we make the following definition:

Definition: The **quadratic variation** of Brownian motion B on the interval $[a, b]$ is defined to be

$$Q_2(B; a, b) = \lim_{\|\Delta_n\| \to 0} Q_2(B; a, b, \Delta_n) = (b-a),$$

where convergence is in L^2 (and hence in probability).

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Theorem

If $\|\Delta_n\| \to 0$*, with*

$$\sum_{n=1}^{\infty} \|\Delta_n\| < \infty,$$

then

$$Q_2(B; a, b, \Delta_n) \to b - a,$$

almost surely.

For example: $\Delta_n = \{k/2^n, k = 0, \ldots, 2^n\} \implies \|\Delta_n\| = 1/2^n$.

Proof.

Let $\epsilon > 0$. From Chebyshev:

$$\sum_{n=1}^{\infty} P(|Y_n| > \epsilon) \leq \frac{1}{\epsilon} \sum_n \mathbb{E} Y_n^2 \leq \frac{1}{\epsilon^2} 2(b-a) \sum_n \|\Delta_n\| < \infty.$$

Borel-Cantelli $\implies$ a.s. we have that

$$Y_n = \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 - (b-a)$$

greater to ϵ for only finite n. ϵ arbitrary, so $Y_n \to 0$ a.s. □

Corollary

If $\{\Delta_n, n = 1, 2, \dots\}$ *are partitions of* $[a, b]$ *then*

$$Q_1(B; a, b, \Delta_n) = \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| \to \infty,$$

almost surely.

Hence, Brownian motion **does not have finite variation**.

Proof.

Suppose it is bounded variation, with variation $V_1(B; a, b)$. Then,

$$\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2 \le \max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}| \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}.$$

By continuity of B, we know $\max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}| \to 0$. Thus, left side goes to zero unless

$$\max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}| \to \infty.$$

We know left hand side does not go to zero! □

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Point of all this: we need to be careful to define

$$I(f)(\omega) = \int_0^T f(\omega, t) dB_t.$$

We will assume that

1. $f(\cdot, t) \in \mathcal{F}_t$, and so is **adapted** to filtration $\mathcal{F}_t$.
2. $f(\cdot, \cdot) \in \mathcal{F}_t \times \mathcal{B}$ (where $\mathcal{B}$ is Borel of $[0, T]$). (so not pathological)

Note: every function you can think of that satisfies the first condition satisfies the second condition.

Examples:

1. $f(\omega, t) = tB_t$.
2. $f(\omega, t) = \exp\{B_t^4\} - t^2$.

This class is much too large for us right now and we will make further restrictions on f.

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Definition

We define $\mathcal{H}^2 = \mathcal{H}^2[0, T]$ to be all measurable, adapted functions f satisfying following integrability condition:

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] < \infty.$$

Note that

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] = \int_\Omega \int_0^T f^2(\omega, t)dt \, dP(\omega),$$

so condition says simply that $f \in L^2(dP \times dt)$

(in fact, is closed linear subspace of $L^2(dP \times dt)$: $f_n \in \mathcal{H}$, $f_n \xrightarrow{L^2} f \implies f \in \mathcal{H}$).

Fubini's theorem implies that for $f \in \mathcal{H}^2$

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] = \int_0^T \mathbb{E}[f^2(\omega, t)]dt.$$

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Start with simplest case possible:

$$f(\omega, t) = 1_{(a,b]} \quad (\text{ not random}).$$

We then **define**

$$I(f)(\omega) = \int_a^b dB_t = B_b - B_a.$$

What about if there is some randomness?

Example

$$f(\omega, t) = X1_{(a,b]},$$

with $X \in \mathcal{F}_a$. (X could be any function of B_s up to time a). We should have

$$I(f)(\omega) = \int_a^b X(\omega) dB_t = X(\omega) \int_a^b dB_t = X(B_b - B_a).$$

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More generally, consider linear combinations (simple functions):

$$f(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega) 1_{(t_i, t_{i+1}]}$$

with

1. $a_i \in \mathcal{F}_{t_i}$,
2. $\mathbb{E} a_i^2 < \infty$ and
3. $t_0 < t_1 < \cdots < t_n = T$.

The collection of these will be denoted by $\mathcal{H}_0^2$.

Insisting on linearity, the integral should be

$$I(f)(\omega) = \sum_{i=0}^{n-1} a_i(\omega)(B_{t_{i+1}} - B_{t_i}).$$

We would like to extend this to $\mathcal{H}^2$.

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Lemma (6.1 in text, Itô Isometry on $\mathcal{H}_0^2$)

For $f \in \mathcal{H}_0^2$ we have

$$||I(f)||_{L^2(dP)} = ||f||_{L^2(dP\times dt)}.$$

That is,

$$\mathbb{E}\left(\int_0^T f(\omega,t)dB_t\right)^2 = \mathbb{E}\int_0^T f^2(\omega,t)dt.$$

Proof.

Just compute out! Use independent increments...

$$\mathbb{E}\left(\int_0^T f(\omega,t)dB_t\right)^2 = \mathbb{E}\left(\sum_{i=0}^{n-1} a_i(\omega)(B_{t_{i+1}} - B_{t_i})\right)^2 = \sum_{i=0}^{n-1} \mathbb{E}a_i^2(B_{t_{i+1}} - B_{t_i})^2$$
$$= \sum_{i=0}^{n-1}(t_{i+1} - t_i)\mathbb{E}a_i^2$$

$$\mathbb{E}\int_0^T f^2(\omega,t)dt = \mathbb{E}\int_0^T \sum_{i=0}^{n-1} a_i^2 1_{(t_i,t_{i+1}]}dt = \mathbb{E}\sum_{i=0}^{n-1} a_i^2(t_{i+1} - t_i) = \sum_{i=0}^{n-1} \mathbb{E}a_i^2(t_{i+1} - t_i).$$

□

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1. Remember, we want the Itô integral on $\mathcal{H}^2[0, T]$ ($\subset L^2(dP \times dt)$) functions:

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] < \infty.$$

2. We will follow ideas from Riemann integration by approximating such general functions with simple functions $\mathcal{H}_0^2$ (sums of indicator functions)

The first necessary step is the following:

Lemma (6.2 in text)

$\mathcal{H}_0^2$ is dense in $\mathcal{H}^2$: for any $f \in \mathcal{H}^2$ there is a sequence $f_n \in \mathcal{H}_0^2$ such that

$$\|f - f_n\|_{L^2(dP\times dt)} = \mathbb{E}\int_0^T (f_n(t) - f(t))^2 dt \to 0, \text{ as } n \to \infty.$$

Corollary to main Theorem of Section 6.6:

Theorem

We define the approximation operator $A_n : \mathcal{H}^2 \to \mathcal{H}_0^2$ in the following way:

$$A_n(f) = \sum_{i=1}^{2^n-1} \frac{1}{t_i - t_{i-1}} \left[\int_{t_{i-1}}^{t_i} f(\omega, u)du \right] 1(t_i < t \leq t_{i+1}),$$

where $t_i = iT/2^n$ for $0 \leq i \leq 2^n$. This maps $\mathcal{H}^2[0, T]$ to $\mathcal{H}_0^2[0, T]$ and

$$||A_n(f)||_\infty \leq ||f||_\infty$$

$$||A_n(f)||_{L^2(dP\times dt)} \leq ||f||_{L^2(dP\times dt)}$$

$$\lim_{n\to\infty} ||A_n(f) - f||_{L^2(dP\times dt)} = 0$$

for all $f \in \mathcal{H}^2$.

Proof.

The first two statements are easy, the last one is a bit harder and we omit. □

Example: if $f(\omega, t) = B_t$, we average B_t over (t_{i-1}, t_i), so is in $\mathcal{F}_{t_i}$, and take that as value on $(t_i, t_{i+1}]$.

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Now we will be able to define $I(f)$ for any $f \in \mathcal{H}^2$!

1. Let $f_n \in \mathcal{H}_0^2$ be an approximating sequence (in $L^2(dP \times dt)$).
2. We know $I(f_n)$ are well-defined random variables in $L^2(dP)$ (since $f_n \in \mathcal{H}_0^2$).
3. Want to define $I(f) \in L^2(dP)$ via

$$I(f) = \lim_{n \to \infty} I(f_n),$$

   where convergence should be in $L^2(dP)$: Itô Isometry will let us go back and forth.

Question: is this well defined?

1. Does such a limit exist **for a given sequence** f_n?
2. Would a different approximating sequence give a different random variable?

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Lemma

This is well-defined.

Proof.

The limit exists.

1. We know that $f_n \to f$ in $L^2(dP \times dt)$ so the sequence is Cauchy:
   - i.e. for $\epsilon > 0$ there exists N such that $||f_k - f_l||_2 \leq \epsilon$ for any $k, l \geq N$.

2. By the Itô isometry

$$\mathbb{E} \int_0^T |f_n(t) - f_m(t)|^2 dt = \mathbb{E} \left(\int_0^T f_n(t) dB_t - \int_0^T f_m(t) dB_t \right)^2,$$

   $I(f_n)$ is Cauchy in $L^2(dP)$.

3. But $L^2(dP)$ is complete, i.e. any Cauchy sequence will be a converging sequence, so there will be a limiting $L^2(dP)$ random variable which we can denote by $I(f)$.

□

Lemma

This is well-defined.

Proof.

Different approximating sequences will give the same limit.

1. Suppose that f_n, f'_n are both approximating sequences for f.

2. Since they both converge to f, their difference will converge to 0 by triangle inequality:

$$\|f_n - f'_n\|_{L^2(dP\times dt)} \leq \|f_n - f'\|_{L^2(dP\times dt)} + \|f - f'_n\|_{L^2(dP\times dt)} \to 0.$$

3. By the Itô isometry we have

$$\begin{aligned} \|I(f_n) - I(f'_n)\|_{L^2(dP)} &= \mathbb{E}\left(\int_0^T f_n(t) - f'_n(t)dB_t\right)^2 \\ &= \mathbb{E}\int_0^T (f_n(t) - f'_n(t))^2 dt \to 0. \end{aligned}$$

which shows that the limit does not depend on the sequence.

□

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We can now extend the Itô Isometry to all of $\mathcal{H}^2[0, T]$!

Theorem (Itô Isometry on $\mathcal{H}^2$)

For any $f \in \mathcal{H}^2[0, T]$:

$$\|I(f)\|_{L^2(dP)} = \|I(f)\|_{L^2(dP \times dt)},$$

or

$$\mathbb{E}\left(\int_0^T f(\omega, t) dB_t\right)^2 = \mathbb{E}\int_0^T f^2(\omega, t) dt.$$

Proof.

1. Choose an approximating sequence f_n.
2. Then $||f - f_n||_{L^2(dP \times dt)} \to 0$ which gives (triangle inequality)

$$\|f_n\|_{L^2(dP \times dt)} \to \|f\|_{L^2(dP \times dt)}.$$

3. Similarly, we also have

$$\|I(f_n)\|_{L^2(dP)} \to \|I(f)\|_{L^2(dP)}$$

and since we know the Itô isometry in $\mathcal{H}_0^2$ the lemma follows.

That is

$$\mathbb{E}\int_0^T f_n^2(t)dt \to \mathbb{E}\int_0^T f^2(t)dt$$

and

$$\mathbb{E}\int_0^T f_n^2(t) = \mathbb{E}\left(\int_0^T f_n(t)dB_t\right)^2 \to \mathbb{E}\left(\int_0^T f(t)dB_t\right)^2.$$

So,

$$\mathbb{E}\int_0^T f^2(t)dt = \mathbb{E}\left(\int_0^T f(t)dB_t\right)^2.$$

□

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Linearity of the process.

1. If $f_n, g_n \in \mathcal{H}_0^2$, then by definition

$$\int_0^t f_n(s)dB_s + \int_0^t g_n(s)dB_s = \int_0^t (f_n(s) + g_n(s))dB_s.$$

2. Is it true for general $f, g \in \mathcal{H}^2$? Yes:
   - use that $f_n \to f$, $g_n \to g \implies f_n + g_n \to f + g$ (in $L^2(dP \times dt)$)
   - Then

$$\begin{aligned} I(f_n + g_n) &\to I(f+g) \quad (\text{ in } L^2(dP)) \\ I(f_n + g_n) = I(f_n) + I(g_n) &\to I(f) + I(g). \end{aligned}$$

   So,

$$I(f+g) = I(f) + I(g).$$

Homework: prove that $\int_0^t f(\omega, s)dB_s$ is mean zero.