Math 635: Chapter 5 Notes

David F. Anderson*

*anderson@math.wisc.edu

Department of Mathematics

University of Wisconsin - Madison

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Section 5.3: Reflection principle

Strong Markov Property of Brownian Motion:

Let τ be a stopping time with respect to Brownian filtration:

1. First time hitting closed set.
2. First time hitting open set.

Then,

$$Y_s \stackrel{def}{=} B_{s+\tau} - B_\tau, \quad s \geq 0$$

is a Brownian motion which is independent of $\mathcal{F}_\tau$.

So, intuitively, Brownian motion “restarts” at stopping times.

Section 5.3: Reflection principle

The following is not surprising then, but will have a surprising corollary!

Theorem

If τ is stopping time with respect to the filtration of B_t then, given $\tau < \infty$

$$\tilde{B}_t = \begin{cases} B_t & \text{if } t < \tau \\ B_\tau - (B_t - B_\tau) & \text{if } \tau \leq t \end{cases}$$

*is a standard BM (**picture!**).*

Further, the joint distribution of

$$B_t^* = \max\{B_s,\ 0 \leq s \leq t\},$$

and B_t satisfies

$$P(B_t^* \geq x, B_t \leq x - y) = P(B_t \geq x + y). \qquad x, y \geq 0$$

Section 5.3: Reflection principle

Why does second piece hold?

$$P(B_t^* \geq x, B_t \leq x - y) = P(B_t \geq x + y), \qquad \forall \quad x, y \geq 0$$

Let τ_x be hitting time of $x > 0$,

$$\begin{aligned}
P(B_t^* \geq x, B_t \leq x - y) &= P(\tau_x \leq t, B_t \leq x - y) \\
&= P(\tau_x \leq t, B_t - B_{\tau_x} \leq -y) \\
&= P(\tau_x \leq t, \tilde{B}_t - \tilde{B}_{\tau_x} \geq y) \\
&= P(\tau_x \leq t, \tilde{B}_t \geq x + y) \\
&= P(B_t^* \geq x, \tilde{B}_t \geq x + y) \\
&= P(\tilde{B}_t \geq x + y).
\end{aligned}$$

Any interesting things to be inferred by this?

Distribution of B_t^*

We have

$$P(B_t^* \geq x, B_t \leq x - y) = P(B_t \geq x + y)$$

Taking y to be 0 in the above yields

$$P(B_t^* \geq x, B_t \leq x) = P(B_t \geq x).$$

Similarly (and straightforward as $\{B_t^* \geq x\} \subset \{B_t \geq x\}$),

$$P(B_t^* \geq x, B_t \geq x) = P(B_t \geq x).$$

Summing yields

$$P(B_t^* \geq x) = 2P(B_t \geq x) = P(B_t \geq x) + P(B_t \leq -x) = P(|B_t| \geq x).$$

$\implies B_t^* = \sup\{B_s, 0 \leq s \leq t\}$ **and** $|B_t|$ **have the same distribution!**
This is amazing!

Distribution of B_t^*

$$P(B_t^* \geq x) = P(|B_t| \geq x).$$

We can get the density of B_t^*:

$$f_{B^*}(u) = \frac{d}{dx}P(|B_t| \leq x) = \frac{d}{dx}\frac{1}{\sqrt{2\pi t}}\int_{-x}^{x} e^{-s^2/2t}ds = \sqrt{\frac{2}{\pi t}}e^{-x^2/2t}.$$

For example, we can get the density of hitting time τ_a:

$$P(\tau_a \leq t) = P(B_t^* \geq a) = 1 - P(B_t^* < a) = 1 - \int_0^a f_{B_t^*}(u)du.$$

Differentiation (in t) and then integrating in u gives

$$f_{\tau_a}(t) = -\frac{d}{dt}\int_0^a \sqrt{\frac{2}{\pi t}}e^{-u^2/2t}du = \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/2t}.$$

Can answer any question you want about hitting times now...

Distribution of B_t^*

One easy corollary: we can get sharp tail probabilities for τ_a.

We will only use that density of standard normal is bounded:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2} \leq \frac{1}{\sqrt{2\pi}} = 0.3989422 < 1/2.$$

$$P(\tau_a \geq t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} ds \leq \int_t^\infty \frac{a}{2s^{3/2}} ds = \frac{a}{\sqrt{t}}.$$

Looked crude, but not so bad: for large t, the integrand concentrates on $s = t$. For example, when $t = 1000$, $a = 5$,

$$\frac{2}{\sqrt{2\pi}} \frac{a}{\sqrt{t}} = 0.1261566, \quad \text{and} \quad \int_t^\infty f_{\tau_a}(s) ds = 0.12563293.$$

Coding up: my answer (100,000 trials): 0.1282

Section 5.4: The invariance principle and Donsker's Theorem

Let X_n be i.i.d. with mean μ and variance $\sigma^2 < \infty$. Then, the CLT says

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}} \Rightarrow N(0, \sigma^2),$$

where "$\Rightarrow$" means convergence in distribution in usual sense:

$$P\left(n^{-1/2}(S(n) - n\mu) \leq x\right) \to \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-x^2/2\sigma^2} dx, \text{ as } n \to \infty.$$

- This is an "invariance" principle, because resulting limit is invariant to the details of the X_i (except for σ).
- Invariance has huge implications in getting confidence intervals: can do so without knowledge of underlying distribution.
- Can this be generalized?

Section 5.4: The invariance principle and Donsker's Theorem

Let X_i be i.i.d. sequence of mean zero, variance one random variables. Let

$$S_n = \sum_{i=1}^{n} X_i,$$

and define interpolated process:

$$S(t) = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) X_{\lfloor t \rfloor + 1}.$$

Scale it by $\sqrt{n}$, and define

$$B_t^{(n)} = \frac{S(nt)}{\sqrt{n}}.$$

Why? We have $\mathbb{E} B_t^{(n)} = 0$ and

$$\begin{aligned} Var(B_t^{(n)}) &= \frac{1}{n} Var\left(\sum_{i=1}^{\lfloor nt \rfloor} X_i + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1} \right) \\ &= n^{-1} \lfloor nt \rfloor + n^{-1} (nt - \lfloor nt \rfloor) \\ &= t. \end{aligned}$$

Central limit theorem says

$$\lim_{n \to \infty} P(B_t^{(n)} \le x) = P(B_t \le x)$$

for a fixed t.

Section 5.4: The invariance principle and Donsker's Theorem

Can even show for any finite sequence $x_1 \leq x_2 \leq \cdots \leq x_d$:

$$\lim_{n\to\infty} P(B^{(n)}_{t_1} \leq x_1, B^{(n)}_{t_2} \leq x_2, \ldots, B^{(n)}_{t_d} \leq x_d) = P(B_{t_1} \leq x_1, \ldots, B_{t_d} \leq x_d).$$

and many more such results.

Note:

1. results invariant to distribution of X_i.
2. What is most general result?
3. Feels like distribution of any path property will converge:
   - $\int B^{(n)}_s ds \implies \int B_s ds.$
   - $\max_{0\leq u\leq 1} B^{(n)}_u \implies \max_{0\leq u\leq 1} B_u$
4. Would be crazy to try to prove every such theorem individually.

Section 5.4: The invariance principle and Donsker's Theorem

- Let $C[0,1]$ be space of continuous functions on $[0,1]$.
- We have a norm on this space: $\|f\|_\infty = \sup_{0 \le t \le 1} |f(t)|$.
- This induces a metric:

$$d(f,g) = \|f - g\|_\infty.$$

- What does it mean for $H : C[0,1] \to \mathbb{R}$ to be continuous?
  - * If $f_n \to f$ in $C[0,1]$, then $H(f_n) \to H(f)$ in $\mathbb{R}$.
- Examples:
  1. $H(f) = f(1)$.
  2. $H(f) = \max_{0 \le x \le 1} f(t)$.

Section 5.4: The invariance principle and Donsker's Theorem

Theorem (Donsker's Invariance Principle- Functional Central Limit Theorem)

For any continuous function $H : C[0,1] \to \mathbb{R}$, the interpolated and scaled random walk $\{B_t^{(n)} : 0 \le t \le 1\}$ satisfies

$$\lim_{n\to\infty} P[H(B_{(\cdot)}^{(n)}) \le x] = P[H(B_{(\cdot)}) \le x].$$

So, $H(B_{(\cdot)}^{(n)})$ converges in distribution to $H(B_{(\cdot)})$. We write $B^{(n)} \Rightarrow B$.

Examples:

1. $H(f) = f(t)$, for some fixed t, gives usual CLT.
2. $H(f) = \sup_{0\le u\le 1} f(u)$ implies

$$P(\max_{0\le t\le 1} B_t^{(n)} \le x) \to P(\max_{0\le t\le 1} B_t \le x) \quad (= P(|B_1| \le x))$$

3. $H(f) = \int_0^1 f(s)ds$ says distribution of integral converges...

Section 5.4: The invariance principle and Donsker's Theorem

Question: if $B^{(n)} \Rightarrow B$, in sense of Donsker, when can I conclude that

$$g \circ B^{(n)} \Rightarrow g \circ B?$$

Just need that for all continuous $H : C[0,1] \to \mathbb{R}$,

$$H(g \circ B^{(n)}) \Rightarrow H(g \circ B).$$

Exercise: Suppose that g is globally Lipschitz on $[0,1]$ (Holder continuous with $\alpha = 1$). Suppose that $H : C[0,1] \to \mathbb{R}$ is continuous. Then the function $H \circ g : C[0,1] \to \mathbb{R}$, defined via

$$(H \circ g)(f) \overset{def}{=} H(g \circ f)$$

is continuous.

So, if $B^{(n)} \Rightarrow B$ in sense of Donsker's theorem, then for all continuous H,

$$H(g \circ B^{(n)}) = (H \circ g)(B^{(n)}) \Rightarrow (H \circ g)(B) = H(g \circ B),$$

Next slide has example with $g(x) = e^x$.

Section 5.4: The invariance principle and Donsker's Theorem

Consider the following family, indexed by n, of simple models for the price of a stock:

1. Let ξ_i be i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$.
2. We discretize $[0, 1]$ into n pieces and define

$$X^{(n)}(\frac{k+1}{n}) = (1 + \frac{\sigma}{\sqrt{n}}\xi_{k+1})X^{(n)}(\frac{k}{n}) \implies X^{(n)}(t) = \prod_{i=1}^{nt}(1 + \frac{\sigma}{\sqrt{n}}\xi_i).$$

3. Then, for $t = k/n$,

$$\ln(X^{(n)}(t)) = \sum_{i=1}^{nt} \ln(1 + \frac{\sigma}{\sqrt{n}}\xi_i)$$

   By Taylor's formula:

$$\ln(X^{(n)}(t)) = \sum_{i=1}^{nt} \left[\frac{\sigma}{\sqrt{n}}\xi_i - \frac{1}{2}\frac{\sigma^2}{n}\xi_i^2 + O(n^{-3/2})\right].$$

4. Hence, taking exponentials and applying theorem, we have

$$\sigma^{-1}\ln(X^{(n)}(t)) + \sigma^{-1}\frac{1}{2}\frac{\sigma^2}{n} \Rightarrow B_t$$

$$X^{(n)}(\cdot) \Rightarrow e^{\sigma B_{(\cdot)} - \sigma^2 t/2}$$

   in sense of Donsker's theorem (applied previous with $g(x) = \exp\{\sigma x\}$).