Math 635: Chapter 4 Notes

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Definition (Precise definition of conditional expectation)

Let

- X be a random variable with $\mathbb{E}|X| < \infty$ on $(\Omega, \mathcal{F}, \mathrm{P})$ and
- $\mathcal{G} \subset \mathcal{F}$ be a σ-field (think of it as "generated" by Z, i.e. $\mathcal{G} = \sigma(Z)$).

We say that Y is the conditional expectation of X wrt $\mathcal{G}$ if Y is $\mathcal{G}$ measurable and

$$\mathbb{E}(X 1_A) = \mathbb{E}(Y 1_A) \qquad \text{for all } A \in \mathcal{G}$$

Notation: $Y = \mathbb{E}(X|\mathcal{G})$.

Conditional Expectation: Properties

Properties of Conditional Expectations:

1. $\mathbb{E}(X + Y|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Y|\mathcal{G})$

Proof.

Must show that for $A \in \mathcal{G}$:

$$\mathbb{E}\left[\mathbb{E}[X + Y|\mathcal{G}]1(A)\right] = \mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}]\right)1(A)\right].$$

Let $A \in \mathcal{G}$.

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}[X + Y|\mathcal{G}]1(A)\right] &= \mathbb{E}[(X + Y)1(A)] \quad \text{(by definition of Cond. Exp.)} \\
&= \mathbb{E}[X1(A)] + \mathbb{E}[Y1(A)] \quad \text{(by linearity of usual expec.)} \\
&= \mathbb{E}[\mathbb{E}[X|\mathcal{G}]1(A)] + \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]1(A)] \quad \text{(by def. of cond. Exp.)} \\
&= \mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}]\right)1(A)\right], \quad \text{(by linearity)}
\end{aligned}$$

and done by uniqueness.

□

Conditional Expectation: Properties

1. (Tower property) if $\mathcal{H} \subset \mathcal{G}$ then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H})$$

Special case: if $\mathcal{H} = \{\emptyset, \Omega\}$ trivial, only scalars $(Z(\omega) = c, \forall\omega)$ are in $\mathcal{H}$. Why?

$$\{\omega : Z(\omega) \leq x\} \in \mathcal{H}$$

for all x, means each set is either all or nothing! Only scalars.

Then, requiring Y to satisfy

$$\mathbb{E}X1(A) = \mathbb{E}Y1(A),$$

reduces (since trivial if $A = \emptyset$) to taking $A = \Omega$, in which case we simply require,

$$\mathbb{E}X = \mathbb{E}Y = \mathbb{E}(\mathbb{E}[X|H]) = \mathbb{E}[X|H],$$

since only scalars are measurable. Hence, in this case, the tower property reduces to

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}X.$$

Conditional Expectation: Properties

1. If X and XY are integrable (in L^1) and $Y \in \mathcal{G}$ then

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G})$$

2. Essentially all properties of expectations: i.e. $\mathbb{E}[aX|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}]$.

Group project: Prove the Tower Property: if $\mathcal{H} \subset \mathcal{G}$ then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H})$$

Section 4.3: Uniform Integrability

Note: important for proving that $M_{t\wedge\tau}$ is a martingale if τ is stopping time.

Definition

We say that a collection $\mathcal{C}$ of random variables is **uniformly integrable** if

$$\rho(x) = \sup_{Z\in\mathcal{C}} \mathbb{E}(|Z|1_{\{|Z|>x\}}), \qquad \text{satisfes } \rho(x) \to 0 \text{ as } x \to \infty.$$

Why? Recall that for integrable X (i.e. in L^1), we have

$$\mathbb{E}[|X|] = \mathbb{E}[|X|1_{\{|X|>x\}}] + \mathbb{E}[|X|1_{\{|X|\leq x\}}],$$

with first term going to zero as $x \to \infty$.

Hence, for **each** $X_i \in L^1$, there is a ρ_i such that

$$\rho_i(x) = \mathbb{E}(|X_i|1_{\{|X_i|>x\}}), \qquad \text{satisfes } \rho_i(x) \to 0 \text{ as } x \to \infty.$$

Uniformly integrable says there is only one ρ for *all* the RVs in $\mathcal{C}$.

Section 4.3: Uniform Integrability

Lemma

If $\mathcal{C} \subset L^1$ is finite then it is U.I.

Follows since for $Z \in \mathcal{C}$

$$\mathbb{E}(|Z|1_{\{|Z|>x\}}) \leq \max_{Z_i \in \mathcal{C}} \mathbb{E}(|Z_i|1_{\{|Z_i|>x\}}) = \max_i \rho_i(x) \stackrel{def}{=} \rho(x) \to 0, \quad \text{as } x \to \infty.$$

Lemma

If for $Z \in \mathcal{C}$ we have $|Z| \leq |X| \in L^1$ with a fixed X then $\mathcal{C}$ is U.I.

Section 4.3: Uniform Integrability

Lemma (4.1 in book, Uniform integrability and L^1 convergence)

If $Z_n \to Z$ a.s. and $\{Z_n\}$ is U.I. then $Z_n \to Z$ in L^1.

Proof.

By Fatou $Z \in L^1$ and $E|Z| \leq \rho(x-\epsilon) + x$ (for any x and $\epsilon > 0$) since

$$\mathbb{E}|Z| = \mathbb{E}|Z|1_{\{|Z|>x\}} + \mathbb{E}|Z|1_{\{|Z|\leq x\}} \leq \mathbb{E}|Z|1_{\{|Z|>x\}} + x$$

and

$$\begin{aligned}\mathbb{E}|Z|1_{\{|Z|>x\}} &\leq \mathbb{E}\limsup_{n\to\infty}|Z_n|1_{\{|Z_n|>x-\epsilon\}} \\ &\leq \liminf_{n\to\infty}\mathbb{E}|Z_n|1_{\{|Z_n|>x-\epsilon\}} \\ &\leq \rho(x-\epsilon).\end{aligned}$$

□

Lemma (4.1 in book, Uniform integrability and L^1 convergence)

If $Z_n \to Z$ a.s. and $\{Z_n\}$ is U.I. then $Z_n \to Z$ in L^1.

Proof.

We write

$$\begin{aligned} |Z_n - Z| &= |Z_n - Z| 1_{\{|Z_n \le x\}} + |Z_n - Z| 1_{\{|Z_n > x\}} \\ &\le |Z_n - Z| 1_{\{|Z_n| \le x\}} + |Z| 1_{\{|Z_n| > x\}} + |Z_n| 1_{\{|Z_n| > x\}} \end{aligned}$$

Must show that each term converges to zero:

1. First term: Dominated Convergence Thm (DCT) with $|Z| + x$.
2. Second term: DCT with $|Z|$ as the majorant: limit is $\rho(x)$.
3. Third term: at most $\rho(x)$.

So, we have that for any x

$$\lim_{n \to \infty} \mathbb{E}|Z_n - Z| \le 0 + \rho(x) + \rho(x).$$

By letting $x \to \infty$ we are done. □

Lemma

Conditional expectation is a contraction:

$$\mathbb{E}|\mathbb{E}(Z|\mathcal{G})| \leq \mathbb{E}|Z|$$

Proof.

Easy: consider $Z = Z_+ - Z_-$. Then,

$$\begin{aligned}
\mathbb{E}|\mathbb{E}[Z|\mathcal{G}]| &= \mathbb{E}|\mathbb{E}[Z_+|\mathcal{G}] - \mathbb{E}[Z_-|\mathcal{G}]| \\
&\leq \mathbb{E}|\mathbb{E}[Z_+|\mathcal{G}] + \mathbb{E}[Z_-|\mathcal{G}]| \\
&= \mathbb{E}[\mathbb{E}[|Z||\mathcal{G}]] \\
&= \mathbb{E}|Z|.
\end{aligned}$$

□

Question: L^p, for $p \geq 1$, contraction?

Section 4.3: Uniform Integrability

Lemma

If $Z_n \to Z$ a.s. and Z_n is U.I. then $E(Z_n|\mathcal{G}) \to E(Z|\mathcal{G})$ in L^1 and in probability.

Proof.

Previous lemmas.

1. We first get $Z_n \to Z$ in L^1 by Lemma 1.
2. then by the previous lemma $\mathbb{E}(Z_n|\mathcal{G}) \to \mathbb{E}(Z|\mathcal{G})$ in L^1

$$\mathbb{E}\big|\mathbb{E}[Z_n|\mathcal{G}] - \mathbb{E}[Z|\mathcal{G}]\big| \leq \mathbb{E}[\mathbb{E}[|Z_n - Z||\mathcal{G}]] = \mathbb{E}|Z_n - Z| \to 0.$$

L^1 convergence is stronger than convergence in prob, so done. □

Conditions for Uniform Integrability

How to check for uniform integrability?

Lemma

If $\phi(x)/x \to \infty$ as $x \to \infty$ and $\mathbb{E}\phi(|Z|) \leq B < \infty$ for $Z \in \mathcal{C}$, then $\mathcal{C}$ is U.I.

Proof.

1. Let $\Psi(x) = \dfrac{\phi(x)}{x} \implies x = \phi(x)/\Psi(x)$.
2. For any $Z \in \mathcal{C}$,

$$\begin{aligned}\mathbb{E}(|Z|1_{\{|Z|\geq x\}}) &= \mathbb{E}\left[\frac{\phi(|Z|)}{\Psi(|Z|)}1_{\{|Z|\geq x\}}\right] \\ &\leq \frac{1}{\min\{\Psi(y) : y \geq x\}}\mathbb{E}[\phi(|Z|)1_{\{|Z|\geq x\}}] \\ &\leq \frac{B}{\min\{\Psi(y) : y \geq x\}}\end{aligned}$$

But, $\Psi(x) \to \infty$ as $x \to \infty$. □

Example: $\phi(x) = x^2$. Says that if $\mathbb{E}|Z_n|^2 \leq B$ for all n, then *U.I.* (we already knew about convergence!)

More generally: if $\mathcal{C} \subset L^p$ with $p > 1$, then it is U.I.

Conditions for Uniform Integrability

Lemma

If Z is in L^1 then there exists convex ϕ with $\phi(x)/x \to \infty$ and $E(\phi(|Z|)) < \infty$.

Proof.

Omit. □

Lemma

If $\mathcal{C} = \{\mathbb{E}(Z|\mathcal{G}) : \mathcal{G} \subset \mathcal{F}\}$ then $\mathcal{C}$ is U.I.

Proof.

Use the previous lemma: $\mathbb{E}\phi(|Z|) < \infty$ and also by Jensen's inequality

$$\mathbb{E}\phi(|\mathbb{E}(Z|\mathcal{G})|) \leq \mathbb{E}(\mathbb{E}\phi(|Z|)|\mathcal{G})) = \mathbb{E}\phi(|Z|) \leq \infty.$$

This is enough for the U.I. by previous Lemma (using this specific ϕ). □

Section 4.4: Martingales in Continuous Time

Definition

If the collection

$$\{\mathcal{F}_t : 0 \leq t < \infty\}$$

of sub σ-fields of $\mathcal{F}$ (so $\mathcal{F}_t \subset \mathcal{F}$) satisfies

$$s \leq t \implies \mathcal{F}_s \subset \mathcal{F}_t,$$

then the collection is called a **filtration**.

Definition

If the process X_t is such that X_t is $\mathcal{F}_t$ measurable,

$$\{\omega : X_t(\omega) \leq x\} \in \mathcal{F}_t,$$

then we say that X_t is **adapted** to the filtration $\{\mathcal{F}_t\}$.

Section 4.4: Martingales in Continuous Time

Definition

We say that X_t is a **martingale** with respect to $\mathcal{F}_t$ if it is adapted to it, $\mathbb{E}|X_t| < \infty$ and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \quad \text{for } t > s,$$

and we say it is a **submartingale** if all assumptions hold with

$$\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s, \quad \text{for } t \geq s.$$

We will be interested in continuous martingales: i.e. there exists $\Omega_0 \subset \Omega$ such that X_t is continuous on Ω_0:

$$\omega \in \Omega_0 \implies t \to X_t(\omega) \text{ is continuous},$$

and $P(\Omega_0) = 1$.

Section 4.4: Martingales in Continuous Time

Important filtration: the one associated to the Brownian motion, B_t.

Natural choice: $\mathcal{F}_t = \sigma(B_s : s \leq t)$.

It turns out that this is not the nicest choice, so we also include all the probability zero events from $[0, T]$ and also any subsets of these (null sets). (This is denoted by $\mathcal{N}$.)

Then $\mathcal{F}_0 = \sigma(\mathcal{N})$ and

$$\mathcal{F}_t = \text{smallest } \sigma - \text{algebra containing } \mathcal{N} \text{ and } \sigma(B_s : s \leq t).$$

we have the nice property that

$$\mathcal{F}_t = \bigcap_{\{s: s > t\}} \mathcal{F}_s = \mathcal{F}_{t+} \qquad \text{right continuity property}$$

These

1. Having all sets of measure zero in filtration
2. Right continuity

are called the "usual conditions".

Stopping times: Same definition.

Definition

If $\{\mathcal{F}_t\}$ is a filtration, then $\tau : \Omega \to \mathbb{R} \cup \{\infty\}$ is a **stopping time** with respect to $\{\mathcal{F}_t\}$ if

$$\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t, \quad \text{for all } t \ge 0.$$

Also, as before, on the set $\{\omega : \tau(\omega) < \infty\}$, we can define the **stopped variable** X_τ via

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega).$$

Section 4.4: Martingales in Continuous Time

Main Theorem of chapter:

Theorem (Doob's Stopping time theorem:)

Assume that M_t is continuous martingale with respect to $\mathcal{F}_t$. If τ is a stopping time wrt $\{\mathcal{F}_t\}$, then

$$X_t = M_{\tau \wedge t}$$

is also a continuous martingale with respect to $\{\mathcal{F}_t\}$.

Proof: Note: continuity is inherited from continuity of M.

We need two things:

1. $\mathbb{E}|X_t| < \infty$ and
2. $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for $s \leq t$.

Idea: The proof is a bit harder than in the discrete case, but we can use the discrete result as an ingredient. Approximate with discrete processes and use previous results.

Section 4.4: Martingales in Continuous Time

Recall:

$$X_t = M_{\tau \wedge t}$$

First show: $\mathbb{E}|X_t| < \infty$.

Fix $s < t$ (for now take $s = 0$). For any $n \geq 1$, define random time τ_n to be smallest element of

$$S(n) = \left\{ s + (t-s)k\frac{1}{2^n} : 0 \leq k < \infty \right\}$$

such that

$$\tau \leq \tau_n.$$

and takes ∞ if $\tau(\omega) = \infty$.

We have that (i) $\tau_n(\omega) \to \tau(\omega)$ for all ω (mesh size gets finer and finer) and (ii) τ_n is a stopping time (you know when you hit it): for $x \in [u_i, u_{i+1})$ (each in $S(n)$)

$$\begin{aligned} \{\tau_n \leq x\} &= \{\min\{u \in S(n) : \tau \leq u\} \leq x\} \\ &= \{\tau \leq u_i\} \in \mathcal{F}_{u_i} \subset \mathcal{F}_x. \end{aligned}$$

We restrict $\{M, \mathcal{F}\}$ to the set $S(n)$:

$$\{M_u, \mathcal{F}_u\}_{S(n)}.$$

Then we get a discrete martingale $\{M_u, \mathcal{F}_u\}_{S(n)}$, and similarly $|M_u|$ is a discrete time submartingale.

Since $|M_u|$ is a (discrete) submartingale on $S(n)$, and $t, \tau_n \in S(n)$, we have

$$\mathbb{E}|M_{t\wedge\tau_n}| \leq \mathbb{E}|M_t| < \infty.$$

Letting $n \to \infty$ and using Fatou we get for all $t \geq 0$

$$\mathbb{E}|X_t| = \mathbb{E}|M_{t\wedge\tau}| \leq \liminf_{n\to\infty} \mathbb{E}|M_{t\wedge\tau_n}| \leq \mathbb{E}|M_t| < \infty,$$

which proves the integrability of X_t.

Section 4.4: Martingales in Continuous Time

To prove the martingale identity, we again use the fact that M_u is a discrete martingale on $S(n)$ to get

$$\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) = M_{s\wedge\tau_n}. \qquad (*)$$

where we used that $s, t, \tau_n \in S(n)$.

Now we need to show that as $n \to \infty$ both sides converge to the right thing.

By the a.s. continuity of $\{M_t\}$ and $\tau_n \to \tau$ we have

- $M_{t\wedge\tau_n} \to M_{t\wedge\tau} = X_t$ and
- $M_{s\wedge\tau_n} \to M_{s\wedge\tau} = X_s$

almost surely.

But we need convergence

$$\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \to \mathbb{E}(M_{t\wedge\tau}|\mathcal{F}_s),$$

which will follow if we prove that $M_{t\wedge\tau_n}$ is U.I.

For this we use the trick introduced at the end of the U.I. section: there exists a convex ϕ with $\phi(x)/x \to \infty$ s.t. $\mathbb{E}\phi(|M_t|) < \infty$ (t is fixed!).

By the convexity of ϕ (Jensen) and $\mathbb{E}\phi(|M_t|) < \infty$ we get that $\phi(|M_u|)$ is a discrete submartingale on $S(n)$.

So by the discrete version of the stopping time thm (used for submartingales) we get

$$\mathbb{E}\phi(|M_{t\wedge\tau_n}|) \leq \mathbb{E}\phi(|M_t|) < \infty$$

1. By lemma from last class (Lemma 4.4): we have the U.I. property for $M_{t\wedge\tau_n}$, which converges a.s. to $M_{t\wedge\tau}$.
2. SoLemma 4.3 gives the L^1 convergence

$$\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \xrightarrow{L^1} \mathbb{E}(M_{t\wedge\tau}|\mathcal{F}_s)$$

   and this is enough to prove the martingale identity.

- $\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \to \mathbb{E}(M_{t\wedge\tau}|\mathcal{F}_s)$ in L^1 and (if we look at the other side of the equation $(*)$) we have
- $\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \to M_{s\wedge\tau}$ a.s.

which means $\mathbb{E}(M_{t\wedge\tau}|\mathcal{F}_s) = M_{s\wedge\tau}$ a.s. (exercise 4.2 c).

Section 4.4: Martingales in Continuous Time

Theorem (Maximal inequality in cont. time)

If M_t is a cont. nonnegative submartingale and $\lambda > 0$, $p \geq 1$ then

$$\lambda^p P\left(\sup_{\{t:0\leq t\leq T\}} M_t > \lambda\right) \leq \mathbb{E}M_T^p$$

Also: if $\|M_T\|_p = \mathbb{E}|M_T^p| < \infty$, for $p > 1$, then

$$|| \sup_{\{t:0\leq t\leq T\}} M_t||_p \leq \frac{p}{p-1}||M_T||_p$$

Proof.

Restrict to $S(n,T) = \{t_i : t_i = tT/2^n, 0 \leq i \leq 2^n\}$ and use the discrete results with Fatou's lemma. Basic idea:

$$\sup_{t\in S(n,T)} M_t \approx \sup_{0\leq t\leq T} M_t$$

with equality in limit as $n \to \infty$. Specifically, we have (a.s.)

$$\lim_{n\to\infty} 1(\sup_{t\in S(n,T)} M_t > \lambda) = 1(\sup_{0\leq t\leq T} M_t > \lambda)$$

Now apply Fatou with discrete result. □

Section 4.4: Martingales in Continuous Time

Theorem (Martingale convergence theorems in continuous time)

If

1. $\{M_t\}$ *is a continuous martingale,*
2. $p > 1$ *and* $\mathbb{E}|M_t|^p \leq B < \infty$ *for all* t*,*

then $M_t \to M_\infty$ *a.s and in* L^p

$$\mathbb{E}|M_t - M_\infty|^p \to 0, \text{ as } t \to \infty,$$

and $\mathbb{E}|M_\infty|^p \leq B$*.*

If $\{M_t\}$ *is a cont martingale and* $\mathbb{E}|M_t| \leq B < \infty$ *for all* t *then* $M_t \to M_\infty$ *a.s and* $\mathbb{E}|M_\infty| \leq B$*.*

Proof: Use the discrete result to get that $M_n \to M_\infty$ $(n \in \{0, 1, 2, \dots\})$, then we only need to show that the fluctuations (in non-integer parts) are small.

Note that for any integer $m \leq t$, we have

$$|M_t - M_\infty| \leq |M_m - M_\infty| + \sup_{\{t : m \leq t < \infty\}} |M_t - M_\infty|.$$

First term is trivial as $m \to \infty$, it goes to zero with prob. 1. Need limit of second term.

Section 4.4: Martingales in Continuous Time

Need

$$\lim_{m\to\infty} \sup_{\{t:m\le t<\infty\}} |M_t - M_\infty| = 0$$

This can be done by the maximal inequality.

$$P(\sup_{\{t:m\le t\le n\}} |M_t - M_m| > \lambda) \le \lambda^{-p}\mathbb{E}(|M_n - M_m|^p).$$

which implies (since $M_n \to M_\infty$ in L^p),

$$P(\sup_{\{t:m\le t<\infty\}} |M_t - M_m| > \lambda) \le \lambda^{-p}\mathbb{E}(|M_\infty - M_m|^p) \to 0, \text{ as } m \to \infty.$$

DCT then tells us can pass limit on probability to conclude

$$P(\lim_{m\to\infty} \sup_{\{t:m\le t<\infty\}} |M_t - M_m| > \lambda) = 0,$$

giving us convergence:

$$\begin{aligned} P(\lim_{m\to\infty} \sup_{\{t:m\le t<\infty\}} |M_t - M_m| = 0) &= 1 - P(\lim_{m\to\infty} \sup_{\{t:m\le t<\infty\}} |M_t - M_m| > 0) \\ &= 1 - P\left(\bigcup_{n=1}^{\infty} \left\{\lim_{m\to\infty} \sup_{\{t:m\le t<\infty\}} |M_t - M_m| > 1/n\right\}\right) \\ &= 1. \end{aligned}$$

Section 4.4: Martingales in Continuous Time

For L^p convergence: for all integers $m \le t$, we have

$$\|M_t - M_\infty\|_p \le \|M_t - M_m\|_p + \|M_m - M_\infty\|_p.$$

Since $S_t = |M_t - M_m|$ is a submartingale, we have for $t < n$,

$$\|M_t - M_m\|_p \le \|M_n - M_m\|_p,$$

yielding

$$\|M_t - M_\infty\|_p \le \|M_m - M_\infty\|_p + \sup_{\{n: n \ge m\}} \|M_n - M_m\|_p.$$

Above is independent of t, so:

$$\limsup_{t\to\infty} \|M_t - M_\infty\|_p \le \|M_m - M_\infty\|_p + \sup_{\{n: n \ge m\}} \|M_n - M_m\|_p \to 0, \text{ as } m \to \infty.$$

Section 4.4: Martingales in Continuous Time

L^1 proof:

- let τ_n be the hitting time of level n by $|M_t|$:

$$\tau_n = \inf\{t : |M_t| \geq n\}.$$

- The martingale $M_{t \wedge \tau_n}$ is bounded so it will converge by first part of theorem.
- In particular, for ω for which $\tau_n(\omega) = \infty$, and so

$$M_t(\omega) = M_{t \wedge \tau_n}(\omega),$$

  we have M_t converges.
- So we just have to prove that

$$\bigcup_{n=1}^{\infty} \{\tau_n = \infty\}.$$

  has probability one.
- This can be proved with the maximal inequality (next slide).
- Fatou's lemma again gives bound $\mathbb{E}|M_\infty| \leq \liminf_{t \to \infty} \mathbb{E}|M_t| \leq B$.

Section 4.4: Martingales in Continuous Time

So we just have to prove that

$$\bigcup_{n=1}^{\infty}\{\tau_n = \infty\}.$$

has probability one.

From Maximal:

$$P(\sup_{0\le t\le T}|M_t| \ge \lambda) \le \mathbb{E}(|M_T|)/\lambda \le \frac{B}{\lambda}.$$

Implying (DCT on $f(T) = 1(\sup_{0\le t\le T}|M_t| \ge \lambda)$),

$$P\left(\sup_{0\le t\le \infty}|M_t| \ge \lambda\right) \le \frac{B}{\lambda}.$$

Converting to τ_n this is

$$P(\tau_n = \infty) = 1 - P(\sup_{0\le t\le \infty}|M_t| \ge n) \ge 1 - \frac{B}{n}.$$

taking unions and using continuity of probability function (note: $\{\tau_m = \infty\} \subset \{\tau_{m+1} = \infty\}$):

$$P\left(\cup_{n=1}^{\infty}\{\omega : \tau_n = \infty\}\right) = P(\lim_{m\to\infty}\{\tau_m = \infty\}) = \lim_{m\to\infty} P(\{\tau_m = \infty\})$$
$$= 1.$$

Section 4.5: Classic Brownian Motion martingales

We now have:

1. Brownian motion.
2. Notion of martingale in continuous time.
3. Stopping time theorem: $M_{t\wedge\tau}$ is a Martingale if τ is a stopping time.
4. Convergence theorems: martingales converge! "Given ω, $M_t(\omega) \to M_\infty(\omega)$ in classical sense."

We can start using this to compute things pertaining to Brownian motion.

Section 4.5: Classic Brownian Motion martingales

Lemma

Each of the following process is a continuous martingale with respect to the standard Brownian filtration:

1. B_t,
2. $B_t^2 - t$,
3. $\exp(\alpha B_t - \alpha^2 t/2)$, *for* $\alpha \in \mathbb{R}$.

Proof: Continuity, adapted, integrability are immediate. Only really check Martingale identity. For example, if $s < t$,

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] = B_s.$$

$$\begin{aligned}
\mathbb{E}[B_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(B_t - B_s + B_s)^2 - t|\mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t|\mathcal{F}_s] \\
&= (t - s) + B_s^2 - t \\
&= B_s^2 - s.
\end{aligned}$$

Section 4.5: Classic Brownian Motion martingales

Finally, let

$$X_t = \exp(\alpha B_t - \alpha^2 t/2).$$

B_t is $N(0, t)$, so

$$\mathbb{E}X_t = e^{-\alpha^2 t/2} \int_{-\infty}^{\infty} e^{\alpha x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 1,$$

and for $s < t$,

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[\exp(\alpha B_t - \alpha^2 t/2)|\mathcal{F}_s] \\
&= \mathbb{E}[\exp(\alpha(B_t - B_s) - \alpha^2(t-s)/2) \exp(\alpha B_s - \alpha^2 s/2)|\mathcal{F}_s] \\
&= X_s \mathbb{E}[\exp(\alpha(B_t - B_s) - \alpha^2(t-s)/2)] \\
&= X_s.
\end{aligned}$$

Section 4.5: Classic Brownian Motion martingales

We have a similar theorem as in random walk.

Theorem

Let B_t be a standard Brownian motion. If $A, B > 0$ and

$$\tau = \min\{t : B_t = -B \text{ or } B_t = A\},$$

then $P(\tau < \infty) = 1$ and

$$P(B_\tau = A) = \frac{B}{A+B} \text{ and } \mathbb{E}(\tau) = AB.$$

Proofs are similar. To prove finiteness, use geometric random variable argument:

$$P(\sup_{n \le t \le n+1} |B_{n+1} - B_n| > A + B) = \epsilon < 1$$

Events $E_n = \{\sup_{n \le t \le n+1} |B_{n+1} - B_n\}$ are independent, so

$$P(\tau > n+1) \le (1-\epsilon)^n \implies P(\tau < \infty) = 1.$$

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Rest of proof is same too.

$$\begin{aligned}\mathbb{E}B_\tau &= A \cdot P(B_\tau = A) - B \cdot P(B_\tau = -B) \\ &= A \cdot P(B_\tau = A) - B \cdot (1 - P(\tau = A)).\end{aligned}$$

However,

1. $B_{t\wedge\tau}$ is a martingale.
2. $\mathbb{E}B_{t\wedge\tau} = 0$ for all t.
3. $|B_{t\wedge\tau}| \leq A + B$.

So, by dominated convergence theorem,

$$\mathbb{E}B_\tau = \mathbb{E}\lim_{t\to\infty} B_{t\wedge\tau} = \lim_{t\to\infty}\mathbb{E}B_{t\wedge\tau} = 0.$$

Solving yields

$$P(B_\tau = A) = \frac{B}{A+B}.$$

Section 4.5: Classic Brownian Motion martingales

Consider hitting time of one-sided boundary:

$$\tau_a = \inf\{t : B_t = a\}.$$

Will show $P(\tau_a < \infty) = 1$ and $\mathbb{E}\tau_a = \infty$ for **all** a.

Proof.

Suppose $a > 0$. Let $b > 0$ be arbitrary. Then,

$$P(\tau_a < \infty) \geq P(B_{\tau_a \wedge \tau_{-b}} = a) = \frac{b}{a+b}.$$

b is arbitrary and right hand side $\to 1$ as $b \to \infty$.

Next, and as before,

$$\mathbb{E}\tau_a \geq \mathbb{E}\tau_a \wedge \tau_{-b} = ab \to \infty, \text{ as } b \to \infty.$$

□

Section 4.5: Classic Brownian Motion martingales

Theorem

Let $f \in C_b^3(\mathbb{R})$ (the bounded continuous functions with three bounded continuous derivatives. If B_t is a standard Brownian motion with respect to $\{F_t\}$, then

$$f(B_t) - f(0) - \int_0^t \frac{1}{2} f''(B_s)ds$$

is a $\{\mathcal{F}_t\}$-martingale.

Notes:

1. This is a Riemannian integral (calculus) since B_t is continuous.
2. Taking $f(x) = x$ shows B_t is a martingale.
3. Taking $f(x) = x^2$ shows $B_t^2 - t$ is a martingale.
4. Taking $f(x) = x^3$ shows

$$B_t^3 - 3\int_0^t B_s ds,$$

is a martingale.

Section 4.5: Classic Brownian Motion martingales

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is a $\{\mathcal{F}_t\}$-martingale.

Proof.

Let $r < t$. And consider

$$\mathbb{E}[f(B_t) - f(0) - (f(B_r) - f(0))|\mathcal{F}_r] = \mathbb{E}[f(B_t) - f(B_r)|\mathcal{F}_r].$$

$\square$