Math 635 – Spring 2012

Homework 6

Due: April 18th, 2012, beginning of the class. Late homework will **not** be accepted.

1. (a) Suppose that $X(t) = (X_1(t), X_2(t)) \in \mathbb{R}^2$ satisfies

$$\begin{aligned} dX_1 &= -X_2 dB_t \\ dX_2 &= X_1 dB_t \end{aligned}$$

and $|X(0)| = 1$. Show that $X(t)$ does *not* stay on the circle $|X| = 1$, even though the vector field $(-X_2, X_1)$ is tangent to the circle at (X_1, X_2).
(Hint: if $X(t)$ stays on the unit circle, is there a function that should remain constant?)

   (b) Show that if $X(t) = (X_1(t), X_2(t))$ satisfies

$$\begin{aligned} dX_1 &= -X_2 dB_t - \frac{1}{2} X_1 dt \\ dX_2 &= X_1 dB_t - \frac{1}{2} X_2 dt \end{aligned}$$

with $|X(0)| = 1$ then $X(t)$ stays on the unit circle.

2. Show that if B_t is a standard BM then $M_t = (B_t + t)\exp(-B_t - t/2)$ is a martingale.

3. Exercise 8.2 (integration by parts). Use Itô's formula to prove that if $h \in C^1(\mathbb{R}^+)$, then

$$\int_0^t h(s) dB(s) = h(t) B_t - \int_0^t h'(s) B_s ds.$$

Note, do not use formula (8.31). Instead, I want you to verify the result directly using Itô.