Math 635 – Spring 2012

Homework 4

Due: March 19, 2012, beginning of the class. Late homework will **not** be accepted.

1. Let B_t be a standard Brownian motion. In this exercise we will prove that with a probability of one the standard Brownian motion started from 0 has infinitely many zeroes in $[0, \delta]$ for any $\delta > 0$. Let
$$B_t^- = \min\{B_s, 0 \le s \le t\}.$$
   (a) Noting that $-B_t$ is also a standard Brownian motion, argue that $-B_t^-$ has the same distribution as $B_t^* = \max\{B_s, 0 \le s \le t\}$.
   (b) Show that $P(B_t^* > 0) = P(B_t^- < 0) = 1$ for any $t > 0$.
   (c) Argue that for any n,
$$P(B_t \text{ has no zeros in } (0, 1/n)) = 0.$$
   (d) Prove that with probability one, for any $\delta > 0$ Brownian motion started from zero has infinitely many zeros in $[0, \delta]$.

2. Suppose that g is globally Lipschitz on $[0, 1]$ (Holder continuous with $\alpha = 1$). Suppose that $H : C[0, 1] \to \mathbb{R}$ is continuous. Then the function $H \circ g : C[0, 1] \to \mathbb{R}$, defined via
$$(H \circ g)(f) = H(g \circ f)$$
is continuous.

3. Prove that for $f \in \mathcal{H}^2[0, T]$
$$\mathbb{E}I(f) = \mathbb{E}\int_0^T f(s)dB_s = 0.$$
Hint: show that the result holds for $f_n \in \mathcal{H}_0^2$, and then make the correct arguments.