Math 635 – Spring 2012

Homework 3

Due: March 12, 2012, beginning of the class. Late homework will **not** be accepted.

1. **(Exercise 4.1, a)** (Tower property) Use the definition of conditional expectation to prove that if $\mathcal{H}$ is a sub-σ field of $\mathcal{G}$ then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}).$$

2. **(Exercise 4.2, a)** Show that if $\mathbb{E}(|X_n - X|^\alpha) \to 0$ for some $\alpha > 0$, then X_n converges to X in probability.

3. **(Exercise 4.5)** Suppose that τ is a stopping time for the filtration $\{\mathcal{F}_n\}$ and that there is a constant N such that for all $n \geq 0$ we have

$$P(\tau \leq n + N \mid \mathcal{F}_n) \geq \epsilon > 0. \tag{1}$$

   Informally, equation (1) tells us that - no matter what has happened so far - there is at least an ϵ chance that we will stop sometime in the next N steps.

   Use induction and the trivial relation

$$P(\tau > kN) = P(\tau > kN \text{ and } \tau > (k-1)N)$$

   to show that $P(\tau > kN) \leq (1-\epsilon)^k$. Conclude that we have $P(\tau < \infty) = 1$ and that $\mathbb{E}\tau^p < \infty$ for all $p \geq 1$.

   Note: Here one should note that $P(A|\mathcal{F}_n)$ is interpreted as $\mathbb{E}(1_A|\mathcal{F}_n)$, and any honest calculation with $P(A|\mathcal{F}_n)$ must rest on the properties of conditional expectations.

4. Show that with probability one the standard BM is not strictly increasing on any interval. Hints: First show that it is enough to prove that $P(B_t \text{ is strictly increasing on } [a, b]) = 0$ for any $0 \leq a < b$. Then use the fact that

$$P(B_t \text{ is strictly increasing on } [a, b]) \leq P(B_{t_1} < B_{t_2} < \dots B_{t_k}), \quad \text{if} \quad a \leq t_1 < t_2 < \dots \leq b.$$