Math 635 – Spring 2012

# Homework 2

**Due: Friday, February 24, 2012**, beginning of the class. Late homework will **not** be accepted.

1. Exercise 2.4 in text, page 28. If $\{M_n, \mathcal{F}_n\}$ is a martingale with
$$\mathbb{E}[M_n^2] < \infty \quad \text{for all } n \geq 0,$$
show that we can write
$$M_n^2 = N_n + A_n,$$
where (1) $\{N_n, \mathcal{F}_n\}$ is a martingale; (2) $A_n$ is monotone (so, $A_n \geq A_{n-1}$); and (3) $A_n$ is nonanticipating (so $A_n \in \mathcal{F}_{n-1}$). Those who want a hint might try defining the $A_n$ by first taking $A_0 = 0$ and then setting
$$A_{n+1} = A_n + \mathbb{E}\left[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n\right], \text{ for } n \geq 0.$$
(Why may we care about something like this? Note that it implies that if the increments, $(M_{n+1} - M_n)^2$ are "small", then $\mathbb{E}M_n^2$ is also small. This is an example of using local dynamics (one time increment) to prove something about the global dynamics.)

2. Recall Theorem 2.8, the $L^1$ Bounded convergence theorem. Here we will demonstrate that we need not have $X_n \to X_\infty$ in $L^1$.

   Let $X_1, X_2, \ldots$ be independent random variables with
$$P\left(X_i = \frac{3}{2}\right) = P\left(X_i = \frac{1}{2}\right) = \frac{1}{2}.$$
Let $M_0 = 1$, and for $n > 0$ let $M_n = X_1 \cdots X_n$.

   (a) Show that $M_n$ is a martingale with respect to $\mathcal{F}_n = \{X_1, \ldots, X_n\}$ and that it satisfies all the hypothesis of Theorem 2.8.

   (b) By considering $\ln M_n$, us the law of large numbers to conclude that $M_n \to 0$ a.s. as $n \to \infty$.

   (c) Conclude that $\|M_n - M_\infty\|_1$ does not converge to zero as $n \to \infty$.

   (d) Explain why this example does not contradict the $L^2$ bounded convergence theorem.

3. For $a > 0$, let
$$X_t = \frac{1}{\sqrt{a}} B_{at},$$
where $B_t$ is a Brownian motion. Show that $X_t$ is a Brownian motion.

4. Let $X_t$ and $Y_t$ be independent standard Brownian motions. Show that
$$Z_t = \frac{1}{\sqrt{2}}(X_t - Y_t)$$
is also a standard Brownian motion.

5. **(Exercise 3.1)** Let $U_t$ be a standard Brownian bridge (see page 41 for the definition).

   (a) Show that we can write $U_t = B_t - tB_1$ for $0 \leq t \leq 1$.

   (b) Show that $Cov(U_s, U_t) = s(1-t)$ for $0 \leq s \leq t \leq 1$.

   (c) Let $X_t = g(t)B_{h(t)}$, and find functions $g$ and $h$ such that $X_t$ has the same covariance as the Brownian bridge. ($B_t$ is a standard BM.) Hint: narrow your search by taking $g(t) = t$ and assume $h$ is monotone decreasing.

   (d) Show that $Y_t = (1+t)U_{t/(1+t)}$ is a BM on $[0, \infty)$.