Math 635 – Spring 2012

Homework 1

Due: February 6th, 2012, beginning of the class. Late homework will **not** be accepted.

1. Exercise 1.3 from Steele (page 10). Consider a simple random walk beginning at 0 and show that for any $k \neq 0$ the expected number of visits to level k before returning to 0 is exactly 1. Anyone who wants a hint might consider the number N_k of visits to level k before the first return to 0. We have $N_0 = 1$ and can use the results on hitting probabilities to show that for all $k \geq 1$ we have

$$P(N_k > 0) = \frac{1}{2k} \quad \text{and} \quad P(N_k > j+1 | N_k > j) = \frac{1}{2} + \frac{1}{2}\frac{k-1}{k}.$$

2. Suppose that X_i are i.i.d random variables with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1, \mathbb{E}X_i^3 = 0$ and let $S_n = \sum_{i=1}^n X_i$. Find a cubic polynomial $f_n(x)$ (with n-dependent coefficients) such that $\{f_n(S_n)\}$ is a martingale with respect to $\{X_n\}$.

3. Suppose that τ_1 and τ_2 are stopping times with respect to $\{\mathcal{F}_n\}$. Show that $\tau_1 + \tau_2$ is also a stopping time.

4. This is a variant of Exercise 2.1 from Steele (page 27). Consider a gambling game with multiple payouts: the player loses \$1 with probability α, wins \$1 with probability β, and wins \$2 with probability γ. Specifically, we assume that $\alpha = 0.52$, $\beta = 0.45$, and $\gamma = 0.03$, so the expected value of each round of the game is only \$-0.01.

   (a) Suppose the gambler bets one dollar on each round of the game and that $\{X_k\}$ is the amount won or lost on the kth round. Find two real numbers x_1, x_2 such that $M_n = x_i^{S_n}$ is a martingale for each of $i \in \{1, 2\}$ where $S_n = X_1 + X_2 + \cdots + X_n$ tallies the gambler's winnings. Note: you will need to find the numerical solutions to a cubic equation but $x = 1$ is one solution so the cubic can be reduced to a quadratic.

   (b) Let p denote the probability of winning \$100 or more before losing \$100. Using *both* martingales from part (a), find p. To do so, you will need to use a high precision software package like Maple or Mathematica. You should be sure to take proper account of the fact that the gambler's fortune may skip over \$100 if a win of \$2 takes place when the gambler's fortune is \$99.