Notes on conditional expectations for Math 635 in Spring of 2012.
David Anderson
January 27, 2012

Recall that if X and Y are discrete random variables defined on the same probability space, then the conditional probability mass function of X given $Y = y$ is

$$p_{X|Y}(x|y) = P\{X = x \mid Y = y\} = \frac{p(x,y)}{p_Y(y)}.$$

We therefore define the conditional expectation of X given $Y = y$ for all y such that $p_Y(y) > 0$ to be

$$\begin{aligned}\mathbb{E}[X|Y = y] &= \sum_{x \in \text{Range}(X)} xP\{X = x|Y = y\}\\ &= \sum_x x p_{X|Y}(x|y).\end{aligned}$$

We denote by $\mathbb{E}[X|Y]$ the function of the random variable Y whose value at $Y = y$ is $\mathbb{E}[X|Y = y]$. Note that $\mathbb{E}[X|Y]$ is itself a random variable. Further, it is immediate that if X and Y are independent, then for any y

$$\mathbb{E}[X|Y = y] = \sum_{x \in \text{Range}(X)} xP\{X = x|Y = y\} = \sum_{x \in \text{Range}(X)} xP\{X = x\} = \mathbb{E}X.$$

Thus, if X and Y are independent, we have

$$\mathbb{E}[X|Y] = \mathbb{E}X.$$

The following is extremely important

Theorem 1.

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]].$$

Thus, we have

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y = y]P\{Y = y\}.$$

Proof.

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|Y]] &= \sum_y \mathbb{E}[X|Y = y]P\{Y = y\}\\ &= \sum_y \sum_x xP\{X = x|Y = y\}P\{Y = y\}\\ &= \sum_x x \sum_y P\{X = x, Y = y\}\\ &= \sum_x xP\{X = x\}\\ &= \mathbb{E}X.\end{aligned}$$

□

Let's also consider the fact that

$$\mathbb{E}[f(Y)X|Y] = f(Y)\mathbb{E}[X|Y].$$

This simply says that for all y in the range of Y,

$$\mathbb{E}[f(Y)X|Y=y] = f(y)\mathbb{E}[X|Y=y].$$

But this is immediate now:

$$\begin{aligned}
\mathbb{E}[f(Y)X|Y=y] &= \sum_{\tilde{y}} \sum_{x} f(\tilde{y}) x P\{Y=\tilde{y}, X=x|Y=y\} \\
&= \sum_{x} f(y) x P\{X=x|Y=y\} \\
&= f(y)\mathbb{E}[X|Y=y].
\end{aligned}$$