Linear Difference Equations

Posted for Math 635, Spring 2012.

Consider the following second-order linear difference equation

$$f(n) = af(n-1) + bf(n+1), \quad K < n < N, \tag{1}$$

where $f(n)$ is a function defined on the integers $K \leq n \leq N$, the value N can be chosen to be infinity, and a and b are nonzero real numbers. Note that if f satisfies (1) and if the values $f(K)$, $f(K+1)$ are known then $f(n)$ can be determined for all $K \leq n \leq N$ recursively via the formula

$$f(n+1) = \frac{1}{b}\big[f(n) - af(n-1)\big].$$

Note also that if f_1 and f_2 are two solutions of (1), then $c_1 f_1 + c_2 f_2$ is a solution for any real numbers c_1, c_2. Therefore, the solution space of (1) is a two-dimensional vector space and one basis for the space is $\{f_1, f_2\}$ with $f_1(K) = 1, f_1(K+1) = 0$ and $f_2(K) = 0, f_2(K+1) = 1$.

We will solve (1) by looking for solutions of the form $f(n) = \alpha^n$, for some $\alpha \neq 0$. Plugging α^n into equation (1) yields

$$\alpha^n = a\alpha^{n-1} + b\alpha^{n+1}, \quad K < n < N,$$

or

$$\alpha = a + b\alpha^2 \quad \iff \quad b\alpha^2 - \alpha + a = 0.$$

Solving this quadratic gives

$$\alpha = \frac{1 \pm \sqrt{1-4ba}}{2b}. \tag{2}$$

There are two cases that need handling based upon whether or not the discriminant, $1 - 4ba$, is zero.

Case 1: If $1 - 4ba \neq 0$, we find two solutions, α_1 and α_2, and see that the general solution to the difference equation (1) is

$$c_1\alpha_1^n + c_2\alpha_2^n,$$

with c_1, c_2 found depending upon the boundary conditions. If $1 - 4ba < 0$, then the roots are complex and the general solution is found by switching to polar coordinates. That is, we let $\alpha = re^{i\theta}$, and find

$$f(n) = r^n e^{in\theta} = r^n \cos(n\theta) \pm ir^n \sin(n\theta),$$

are solutions, implying both the real and imaginary parts are solutions. Therefore, the general solution is

$$c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta),$$

with c_1, c_2 found depending upon the boundary conditions.

Case 2: If $1 - 4ba = 0$, we only find the one solution, $f_1(n) = (1/2b)^n$ by solving the quadratic.

However, let $f_2(n) = n(2b)^{-n}$. We have that

$$\begin{aligned} af_2(n-1) + bf_2(n+1) &= a(n-1)(2b)^{-(n-1)} + b(n+1)(2b)^{-(n+1)} \\ &= \left(\frac{1}{2b}\right)^n \left[a(n-1)2b + b(n+1)\frac{1}{2b}\right] \\ &= \left(\frac{1}{2b}\right)^n \left[(n-1)\frac{1}{2} + (n+1)\frac{1}{2}\right] \qquad \text{(remember, } 4ab = 1\text{)} \\ &= \left(\frac{1}{2b}\right)^n n \\ &= f_2(n). \end{aligned}$$

Note that f_2 is obviously linearly independent from f_1. Thus, when $4ab = 1$, the general form of the solution is

$$f(n) = c_1 \left(\frac{1}{2b}\right)^n + c_2 n \left(\frac{1}{2b}\right)^n.$$

with c_1, c_2 found depending upon the boundary conditions.

Example 1. Find a function $f(n)$ satisfying

$$f(n) = 2f(n-1) + \frac{1}{10}f(n+1), \qquad 0 < n < \infty,$$

with $f(0) = 8, f(1) = 2$.

Solution. Here, $a = 2$ and $b = 1/10$. Therefore, plugging into (2) gives

$$\alpha = 5 \pm \sqrt{5},$$

and the general solution is

$$f(n) = c_1 \left(5 + \sqrt{5}\right)^n + c_2 \left(5 - \sqrt{5}\right)^n.$$

Using the boundary conditions yields

$$\begin{aligned} 8 &= f(0) = c_1 + c_2 \\ 2 &= f(1) = c_1(5 + \sqrt{5}) + c_2(5 - \sqrt{5}), \end{aligned}$$

which has solution $c_1 = 4 - 19\sqrt{5}/5$, $c_2 = 4 + 19\sqrt{5}/5$. Thus, the solution to the problem is

$$f(n) = \left(4 - \frac{19\sqrt{5}}{5}\right)\left(5 + \sqrt{5}\right)^n + \left(4 + \frac{19\sqrt{5}}{5}\right)\left(5 - \sqrt{5}\right)^n.$$

□

Some of the most important difference equations we will see in this course are those of the form

$$f(n) = pf(n-1) + qf(n+1), \quad \text{with } p + q = 1, \quad p, q \geq 0.$$

These will arise when studying random walks with p and q interpreted as the associated probabilities of moving right or left. Supposing that $p \neq q$, the roots of the quadratic formula (2) can be found:

$$\frac{1 \pm \sqrt{1 - 4(1-p)p}}{2q} = \frac{1 \pm \sqrt{(1-2p)^2}}{2q} = \frac{1 \pm |q - p|}{2q} = \left\{1, \frac{p}{q}\right\}.$$

Thus, the general solution when $p \neq 1/2$ is

$$f(n) = c_1 + c_2 \left(\frac{p}{q}\right)^n .$$

For the case that $p = q = 1/2$, the only root is 1, hence the general solution is

$$f(n) = c_1 + c_2 n.$$

We analyzed only second-order linear difference equations above. However, and similar to the study of differential equations, higher order difference equations can be studied in the same manner. Consider the general kth order, homogeneous linear difference equation:

$$f(n+k) = a_0 f(n) + a_1 f(n+1) + \cdots + a_{k-1} f(n+k-1), \tag{3}$$

where we are given $f(0), f(1), \ldots, f(k-1)$. Then, again, we may solve for the general $f(n)$ recursively using (3). We look for solutions of the form $f(n) = \alpha^n$, which is a solution if and only if

$$\alpha^k = a_0 + a_1 \alpha + \cdots + a_{k-1} \alpha^{k-1}.$$

If there are k distinct roots of the above equation, then we automatically get k linearly independent solutions to (3). However, if a given root α is a root with a multiplicity of j, then

$$\alpha^n, n\alpha^n, \ldots, n^{j-1}\alpha^n,$$

are linearly independent solutions. We can then use the given initial conditions to find the desired particular solution.