Math 632 – Spring 2012

Homework 3

Due: Thursday, October 11th, 2012.

1. Exercise 1.48 from text.
2. Exercise 1.53 from text. Here x is the initial condition.
3. Exercise 1.55 from text. (If you need more practice, do Exercise 1.56 also.)
4. Exercise 1.65 from text. (If you need more practice, do Exercise 1.64 also.)
5. Consider a Markov chain on $S = \{0, 1, 2, \dots\}$ with

$$\begin{aligned} p(k, k+1) &= p_k, \text{ for } k \geq 0 \\ p(k, k-1) &= q_k, \text{ for } k \geq 1 \\ p(0, 0) &= q_0, \end{aligned}$$

where $p_k, q_k > 0$ and $p_k + q_k = 1$ for all k. This model is called a "birth and death" model since transitions can only be of size one. Note that the transition rates depend upon the state of the system k. We are going to look for a condition which guarantees positive recurrence. To do so, we will search for a stationary distribution.

(a) Show that a stationary distribution satisfies

$$q_{k+1}\pi(k+1) - p_k\pi(k) = q_k\pi(k) - p_{k-1}\pi(k-1),$$

and that each of the above terms is, in fact, zero. Conclude that for π to be a stationary measure, we must have

$$\pi(k+1) = \frac{p_k}{q_{k+1}}\pi_k$$

for all $k \geq 0$.

(b) Use part (a) to conclude that the chain has a stationary distribution if and only if

$$\sum_{k=1}^{\infty} \frac{p_0 p_1 \cdots p_{k-1}}{q_1 q_2 \cdots q_k} < \infty.$$

(c) Supposing that the above sum is finite. What is the expected return to to state 0? To state 17?