A SPECTRAL SZEGŐ THEOREM ON THE REAL LINE

R. V. BESSONOV, S. A. DENISOV

ABSTRACT. We characterize even measures $\mu = w\,dx + \mu_s$ on the real line $\mathbb{R}$ with finite entropy integral $\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2}\,dt > -\infty$ in terms of 2×2 Hamiltonians generated by μ in the sense of the inverse spectral theory. As a corollary, we obtain criterion for spectral measure of Krein string to have converging logarithmic integral.

1. INTRODUCTION

Each probability measure μ supported on an infinite subset of the unit circle $\mathbb{T} = \{z : |z| = 1\}$ of the complex plane, $\mathbb{C}$, gives rise to an infinite family $\{\Phi_n\}_{n\geqslant 0}$ of monic polynomials orthogonal with respect to μ. For integer $n \geqslant 0$, the polynomial Φ_n has degree n, unit coefficient in front of z^n, and $(\Phi_n, \Phi_k)_{L^2(\mu)} = 0$ for all $k \neq n$. The polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfy the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z), \qquad \Phi_0 = 1, \tag{1.1}$$

where $\{\Phi_n^*\}$ are the "reversed" polynomials defined by $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$. Recurrence coefficients $\{\alpha_n\}$ are completely determined by μ and we have $|\alpha_n| < 1$ for every $n \geqslant 0$. Given any sequence of complex numbers $\{\alpha_n\}$ with $|\alpha_n| < 1$, one can find the unique probability measure μ on $\mathbb{T}$ such that $\{\alpha_n\}$ is the sequence of the recurrence coefficients of μ, see [32], [34].

Szegő Theorem. *Let $\mu = w\,dm + \mu_s$ be a probability measure on $\mathbb{T}$ with density w and a singular part μ_s with respect to the Lebesgue measure m on $\mathbb{T}$. The following assertions are equivalent:*

(a) *the set* $\operatorname{span}\{z^n,\ n \geqslant 0\}$ *of analytic polynomials is not dense in $L^2(\mu)$;*
(b) *the entropy of μ is finite: $\int_{\mathbb{T}} \log w\,dm > -\infty$;*
(c) *the recurrence coefficients $\{\alpha_n\}$ of μ satisfy $\sum_{n\geqslant 0} |\alpha_n|^2 < \infty$.*

We refer the reader to [32], [33] for the historical account and an extended version of this result. Independent contributions to different aspects of its proof were done by Szegő, Verblunsky, and Kolmogorov. A partial counterpart of Szegő theorem for measures supported on the real line, $\mathbb{R}$, is due to Krein [24] and Wiener [36] (see also Section 4.2 in [13] or Theorem A.6 in [11] for modern expositions). Denote by $\Pi(\mathbb{R})$ the class of all Radon measures on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$.

Krein–Wiener Theorem. *Let $\mu = w\,dx + \mu_s$ be a measure in $\Pi(\mathbb{R})$ where w is the density with respect to the Lebesgue measure dx on $\mathbb{R}$ and μ_s is the singular part. The following assertions are equivalent:*

(a) *the set of functions whose Fourier transform is smooth and compactly supported on* $[0, +\infty)$ *is not dense in $L^2(\mu)$;*
(b) *the entropy of μ is finite: $\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2}\,dt > -\infty$.*

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The Szegő and Krein–Wiener theorems have a probabilistic interpretation. Roughly, it says that a stationary Gaussian sequence/process with the spectral measure μ is non-deterministic if and only if the entropy of μ is finite, see, e.g, Section II.2 in [18] or survey [6] for more details.

The aim of this paper is to complement assertions (a), (b) in the Krein–Wiener theorem with a necessary and sufficient condition similar to condition (c) in the Szegő theorem. Instead of the recurrence relation $\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z)$, we will consider the canonical Hamiltonian system $JM' = z\mathcal{H}M$ which naturally appears from μ via Krein – de Branges spectral theory.

Consider the Cauchy problem for a canonical Hamiltonian system on the half-axis $\mathbb{R}_+ = [0, +\infty)$,

$$JM'(t,z) = z\mathcal{H}(t)M(t,z), \qquad M(0,z) = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \quad t \geqslant 0, \quad z \in \mathbb{C}. \tag{1.2}$$

Here $J = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, the derivative of M is taken with respect to t, the Hamiltonian $\mathcal{H}$ is the mapping taking numbers $t \in \mathbb{R}_+$ into positive semi-definite matrices, the entries of $\mathcal{H}$ are real measurable functions on $\mathbb{R}_+$ absolutely integrable on compact subsets of $\mathbb{R}_+$. In addition, we assume that the trace of $\mathcal{H}$ does not vanish identically on any set of positive Lebesgue measure. A Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ is called singular if

$$\int_0^{+\infty} \operatorname{trace}\mathcal{H}(t)\,dt = +\infty.$$

Two Hamiltonians $\mathcal{H}_1$, $\mathcal{H}_2$ on $\mathbb{R}_+$ are called equivalent if there exists an increasing absolutely continuous function η defined on $\mathbb{R}_+$ such that $\eta(0) = 0$, $\lim_{t\to+\infty}\eta(t) = +\infty$, and $\mathcal{H}_2(t) = \eta'(t)\mathcal{H}_1(\eta(t))$ for Lebesgue almost every $t \in \mathbb{R}_+$. Clearly, $\eta(t)$ rescales the variable t. We say that Hamiltonian $\mathcal{H}$ is trivial if there is a non-negative matrix A with $\operatorname{rank} A = 1$, such that $\mathcal{H}$ is equivalent to A, i.e., $\mathcal{H}(t) = \eta'(t)A$ for a.e. $t \in \mathbb{R}_+$, where η is an increasing absolutely continuous function on $\mathbb{R}_+$, which satisfies $\eta(0) = 0$ and $\lim_{t\to+\infty}\eta(t) = +\infty$. If Hamiltonian is not trivial, it is called nontrivial.

Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$, and let M be the solution of (1.2). Fix a parameter $\omega \in \mathbb{R}\cup\{\infty\}$ and define the Weyl-Titchmarsh function m of (1.2) on $\mathbb{C}\setminus\mathbb{R}$ by

$$m(z) = \lim_{t\to+\infty}\frac{\omega\Phi^+(t,z) + \Phi^-(t,z)}{\omega\Theta^+(t,z) + \Theta^-(t,z)}, \qquad M(t,z) = \left(\begin{smallmatrix}\Theta^+(t,z) & \Phi^+(t,z)\\ \Theta^-(t,z) & \Phi^-(t,z)\end{smallmatrix}\right). \tag{1.3}$$

The fraction $\frac{\infty c_1 + c_2}{\infty c_3 + c_4}$ for non-zero numbers c_1, c_3 is interpreted as $\frac{c_1}{c_3}$. For the Weyl-Titchmarsh theory of canonical Hamiltonian systems see [17] or Section 8 in [31]. Theorem 2.1 in [17] implies that the denominator of the fraction in (1.3) is nonzero for large $t \geqslant 0$, the function m does not depend on the choice of the parameter ω, and $\operatorname{Im} m(z) > 0$ for z in $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Hence, there exists a measure $\mu \in \Pi(\mathbb{R})$, and numbers $a \in \mathbb{R}$, $b \geqslant 0$, such that

$$m(z) = \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{x-z} - \frac{x}{1+x^2}\right)d\mu(x) + bz + a, \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.4}$$

The measure μ in (1.4) is called the spectral measure of the system (1.2). It is easy to check that equivalent Hamiltonians have equal Weyl-Titchmarsh functions, see [38]. The following theorem is central to Krein – de Branges inverse spectral theory [19], [9].

De Branges Theorem. *For every analytic function m in $\mathbb{C}^+$ with positive imaginary part, there exists a singular nontrivial Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ such that m is the Weyl-Titchmarsh function* (1.3) *for $\mathcal{H}$. Moreover, any two singular nontrivial Hamiltonians $\mathcal{H}_1$, $\mathcal{H}_2$ on $\mathbb{R}_+$ generated by m are equivalent.*

See [31], [37] for proofs to this theorem. A measure μ on $\mathbb{R}$ is called even if $\mu(I) = \mu(-I)$ for every interval $I \subset \mathbb{R}_+$. It is well-known that a Hamiltonian $\mathcal{H}$ has the diagonal form $\mathcal{H} = \operatorname{diag}(h_1, h_2)$

almost everywhere on $\mathbb{R}_+$ if and only if its spectral measure μ is even and $a = 0$ in (1.4), see Lemma 2.2 below. Here $\operatorname{diag}(c_1, c_2) = \left(\begin{smallmatrix} c_1 & 0 \\ 0 & c_2 \end{smallmatrix}\right)$ for $c_1, c_2 \in \mathbb{R}_+$.

Szegő class $\mathrm{Sz}(\mathbb{R})$ on the real line $\mathbb{R}$ consists of measures $\mu \in \Pi(\mathbb{R})$ that satisfy equivalent assertions (a), (b) in Krein–Wiener theorem. Given a measure $\mu = w\,dx + \mu_s$ in $\mathrm{Sz}(\mathbb{R})$, define its normalized entropy by

$$\mathcal{K}(\mu) = \log \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\,dx.$$

By Jensen's inequality, we have $\mathcal{K}(\mu) \geqslant 0$, and, moreover, $\mathcal{K}(\mu) = 0$ if and only if μ is a non-zero scalar multiple of the Lebesgue measure on $\mathbb{R}$.

We say that a measure $\mu \in \Pi(\mathbb{R})$ generates a Hamiltonian $\mathcal{H}$ if the Weyl-Titchmarsh function (1.3) of $\mathcal{H}$ has the form $m : z \mapsto \frac{1}{\pi}\int_{\mathbb{R}}\bigl(\frac{1}{x-z} - \frac{x}{1+x^2}\bigr)\,d\mu(x)$. To every $\mathcal{H}$ with $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$ we associate the sequence of points $\{\eta_n\}$ by

$$\eta_n = \min\left\{t \geqslant 0 : \int_0^t \sqrt{\det \mathcal{H}(s)}\,ds = n\right\}, \qquad n \geqslant 0. \tag{1.5}$$

Our main result is the following theorem.

Theorem 1. *An even measure $\mu \in \Pi(\mathbb{R})$ belongs to the Szegő class $\mathrm{Sz}(\mathbb{R})$ if and only if some (and then every) Hamiltonian $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ generated by μ is such that $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$ and*

$$\widetilde{\mathcal{K}}(\mathcal{H}) = \sum_{n=0}^{+\infty}\left(\int_{\eta_n}^{\eta_{n+2}} h_1(s)\,ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s)\,ds - 4\right) < \infty, \tag{1.6}$$

where $\{\eta_n\}$ are given by (1.5). Moreover, we have $\widetilde{\mathcal{K}}(\mathcal{H}) \leqslant c\mathcal{K}(\mu)e^{c\mathcal{K}(\mu)}$ and $\mathcal{K}(\mu) \leqslant c\widetilde{\mathcal{K}}(\mathcal{H})e^{c\widetilde{\mathcal{K}}(\mathcal{H})}$ for an absolute constant c.

By definition, the terms in (1.6) are nonnegative:

$$\int_{\eta_n}^{\eta_{n+2}} h_1(s)\,ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s)\,ds - 4 \geqslant \left(\int_{\eta_n}^{\eta_{n+2}} \sqrt{\det \mathcal{H}(s)}\,ds\right)^2 - 4 = 0,$$

and the sum in (1.6) equals zero if and only if $\mathcal{H}$ is a constant Hamiltonian. Note that the spectral measure μ of a constant diagonal Hamiltonian $\mathcal{H}$ with $\det \mathcal{H} \neq 0$ is a scalar multiple of the Lebesgue measure on $\mathbb{R}$, in particular, we have $\mathcal{K}(\mu) = 0$ in this case.

Diagonal canonical Hamiltonian systems are closely related to the differential equation of a vibrating string:

$$-\frac{d}{dM(t)}\frac{d}{dt}\Bigl(y(t,z)\Bigr) = zy(t,z), \qquad t \in [0, L), \qquad z \in \mathbb{C}. \tag{1.7}$$

Here $0 < L \leqslant +\infty$ is the length of the string, $M : (-\infty, L) \to \mathbb{R}_+$ is an arbitrary non-decreasing and right-continuous function (mass distribution) that satisfies $M(t) = 0$ for $t < 0$. If M is smooth and strictly increasing on $\mathbb{R}_+$, then equation (1.7) takes the form $-y'' = zM'y$.

In this paper, we consider L and M that satisfy the following conditions:

$$L + \lim_{t\to L} M(t) = \infty \quad \text{and} \quad \lim_{t\to L} M(t) > 0\,, \tag{1.8}$$

where the last bound means that M is not identically equal to zero. If (1.8) holds, we say that M and L form $[M, L]$ pair. To every $[M, L]$ pair one can associate a string and Weyl-Titchmarsh function q with spectral measure σ supported on the positive half-axis $\mathbb{R}_+$. We discuss these objects in more detail in Section 6. Theorem 1 can be reformulated for Krein strings as follows.

Theorem 2. *Let $[M,L]$ satisfy (1.8) and $\sigma = v\,dx + \sigma_s$ be the spectral measure of the corresponding string. Then, we have $\int_0^\infty \frac{\log v(x)}{(1+x)\sqrt{x}}\,dx > -\infty$ if and only if $\sqrt{M'} \notin L^1(\mathbb{R}_+)$ and*

$$\widetilde{\mathcal{K}}[M,L] = \sum_{n=0}^{+\infty}\Big((t_{n+2}-t_n)(M(t_{n+2})-M(t_n)) - 4\Big) < \infty, \tag{1.9}$$

where $t_n = \min\bigl\{t \geqslant 0 :\ n = \int_0^t \sqrt{M'(s)}\,ds\bigr\}$.

Condition (1.8) guarantees that the string $[M,L]$ has a unique spectral measure. It does not restrict the generality of Theorem 2: if (1.8) is violated, then either $M = 0$ and $\int_0^\infty \frac{\log v(x)}{(1+x)\sqrt{x}}\,dx = -\infty$ because $v = 0$, or $L + \lim_{t\to L} M(t) < \infty$ in which case the Weyl-Titchmarsh function is meromorphic and real-valued on $\mathbb{R}$, so $v(x) = 0$ again and the logarithmic integral diverges. More details on Theorem 2 can be found in Section 6.

Strategy of the proof of Theorem 1. Our approach is based on the analysis of an entropy function $\mathcal{K}_{\mathcal{H}}$ of the Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ which we define as follows:

$$\mathcal{K}_{\mathcal{H}}(r) = \mathcal{K}(\mu_r), \qquad r \geqslant 0,$$

where μ_r is the spectral measure of the "shifted" Hamiltonian $\mathcal{H}_r : x \mapsto \mathcal{H}(x+r)$. To estimate $\widetilde{\mathcal{K}}(\mathcal{H})$ in terms of $\mathcal{K}(\mu) = \mathcal{K}_{\mathcal{H}}(0)$ for an arbitrary Hamiltonian $\mathcal{H}$, we first study the function $r \mapsto \mathcal{K}_{\mathcal{H}}(r)$ for "nice" $\mathcal{H}$, derive two-sided estimates for it, and then use an approximation argument to prove that these bounds hold for all $\mathcal{H}$. It turns out that the function $\mathcal{K}_{\mathcal{H}}$ has a number of remarkable properties. For example, $\mathcal{K}_{\mathcal{H}}$ is a non-negative absolutely continuous function on $\mathbb{R}_+$ that satisfies $\mathcal{K}_{\mathcal{H}}(0) = \mathcal{K}_{\widehat{\mathcal{H}}_r}(0) + \mathcal{K}_{\mathcal{H}}(r)$ where $\widehat{\mathcal{H}}_r$ is a suitable analog of Bernstein-Szegő approximation of $\mathcal{H}$. Moreover, $\mathcal{K}_{\mathcal{H}}$ is non-increasing on $\mathbb{R}_+$ and its derivative, $\mathcal{K}_{\mathcal{H}}'$, appears in a differential equation that involves coefficients h_1, h_2 of the Hamiltonian $\mathcal{H} = \operatorname{diag}(h_1, h_2)$, see Lemma 2.7. Hence, the problem of estimating $\mathcal{K}(\mu)$ is reduced to describing all functions h_1, h_2 for which the solution to this equation, $\mathcal{K}_{\mathcal{H}}$, is bounded on the half-axis $\mathbb{R}_+$. Analyzing this equation in the case when $\mathcal{H} = \operatorname{diag}(h, 1/h)$, we obtain two inequalities

$$\sum_{n\geqslant 0}\left(\frac{1}{t_n}\int_{4n}^{4n+t_n} h(s)\,ds \cdot \frac{1}{t_n}\int_{4n}^{4n+t_n}\frac{ds}{h(s)} - 1\right) \leqslant e^{10\mathcal{K}(\mu)} - 1, \qquad \{t_n\} \subset [3,4],$$

and

$$\mathcal{K}(\mu) \leqslant \int_0^\infty \left(\frac{1}{h(r)}\int_r^\infty h(s)e^{r-s}\,ds + h(r)\int_r^\infty \frac{1}{h(s)}e^{r-s}\,ds - 2\right)dr,$$

see Propositions 3.1 and 3.2. The first one is reminiscent of (1.6) and it is used to derive an estimate $\widetilde{\mathcal{K}}(\mathcal{H}) \leqslant c\mathcal{K}(\mu)e^{c\mathcal{K}(\mu)}$. Showing that the second bound implies $\mathcal{K}(\mu) \leqslant c\widetilde{\mathcal{K}}(\mathcal{H})e^{c\widetilde{\mathcal{K}}(\mathcal{H})}$ is more involved. In fact, to do that we need to introduce and study a new functional class $A_2(\mathbb{R}_+, \ell^1)$ which resembles the Muckenhoupt class of weights $A_2(\mathbb{R}_+)$. This is done in Section 5.

Historical remarks. Except for Krein–Wiener theorem, all previously known results on Szegő theorem in the continuous setting were proved for the so-called Krein systems, i.e., differential systems that appear as a result of "orthogonalization process with continuous parameter" invented by Krein in [26]. Krein systems with locally summable coefficients can be reduced to the canonical Hamiltonian systems with absolutely continuous Hamiltonians $\mathcal{H}$ (see, e.g, [2] for this reduction in the diagonal case). The class of Hamiltonians considered in Theorem 1 is considerably wider. Krein himself formulated a restricted version of Szegő theorem for Krein systems in [26]. In [10], the second author of this paper characterized Krein systems with coefficients from a Stummel class whose spectral measures belong to $\mathrm{Sz}(\mathbb{R})$. In [35], Teplyaev fixed an error in the original formulation of Szegő theorem in [26]. The reader interested in Szegő theory for Krein systems can find further information in monograph [11]. In [21] and [22], Killip and Simon proved analogs of Szegő theorem for Jacobi matrices and Schrödinger operators. See also the work [29] by Nazarov, Peherstorfer, Volberg, and Yuditskii for a closely related subject of sum rules for Jacobi matrices. Deep relations of various completeness problems to the theory of de Branges spaces and canonical Hamiltonian systems were utilized in [1], [7], [27], [28], [30]. The results of the present paper were used in [3], [4], [5], [14], [23].

The structure of the paper. We start by studying the basic properties of entropy function for diagonal canonical systems in Section 2. Section 3 contains the proof of upper and lower bounds for the entropy. Theorem 1 is proved in the fourth section. The new functional class which appears in the proof of Theorem 1 is studied in Section 5. We consider Krein strings and prove Theorem 2 in Section 6. The paper ends with an appendix which contains some auxiliary results.

Notation. In the text, we use the following standard notation. Given set $E \subset \mathbb{R}$ with positive Lebesgue measure $|E| > 0$ and nonnegative $f \in L^1(E)$, we denote $\langle f\rangle_E = \frac{1}{|E|}\int_E f\,dx$. Suppose $a \in \mathbb{R}$, $l > 0$, then $I_{a,l} = [a, a+l)$. The symbols C, c denote absolute constants which can change the value from formula to formula. For two non-negative functions f_1, f_2, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that $f_1 \leqslant Cf_2$ for all values of the arguments of f_1, f_2. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. Given a set $E \subset \mathbb{R}$, χ_E stands for the characteristic function of E. The norm of the space $L^p(\mathbb{R}_+)$ is denoted by $\|\cdot\|_p$. The space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ consists of functions that are absolutely integrable on compact subsets of $\mathbb{R}_+$. Symbol $[x]$ stands for the integer part of a real number x.

2. Entropy function of a canonical Hamiltonian system

In this section, we introduce the entropy function of a diagonal canonical Hamiltonian system and show that it has a number of remarkable properties.

Let $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ be a singular nontrivial diagonal Hamiltonian on $\mathbb{R}_+$, and let m, μ be its Weyl-Titchmarsh function and the spectral measure, so that

$$\operatorname{Im} m(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|x-z|^2}\,d\mu(x) + b\operatorname{Im} z, \qquad z \in \mathbb{C}^+. \tag{2.1}$$

For every $r \geqslant 0$ define $\mathcal{H}_r$ to be the Hamiltonian on $\mathbb{R}_+$ taking x into $\mathcal{H}(x+r)$. Let m_r, μ_r, b_r denote the Weyl-Titchmarsh function, the spectral measure, and the coefficient in (1.4) of system (1.2) for $\mathcal{H} = \mathcal{H}_r$. Each time we work with these objects later in the text we assume that $\mathcal{H}_r$ is nontrivial. Define

$$\mathcal{I}_{\mathcal{H}}(r) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu_r(x)}{1+x^2} + b_r = -i m_r(i), \qquad \mathcal{Y}_{\mathcal{H}}(r) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w_r(x)}{1+x^2}\,dx, \tag{2.2}$$

where w_r is the density of the absolutely continuous part of $\mu_r = w_r\,dx + \mu_{r,s}$. The second identity above follows from the fact that μ is even, hence m takes imaginary values on imaginary axis. If $\mu_r \notin \operatorname{Sz}(\mathbb{R})$, we put $\mathcal{Y}_{\mathcal{H}}(r) = -\infty$. Define the entropy function of $\mathcal{H}$ by

$$\mathcal{K}_{\mathcal{H}}(r) = \log \mathcal{I}_{\mathcal{H}}(r) - \mathcal{Y}_{\mathcal{H}}(r), \qquad r \geqslant 0.$$

Note again that Jensen's inequality and an estimate $b_r \geqslant 0$ give

$$\mathcal{K}_{\mathcal{H}}(r) \geqslant 0\,. \tag{2.3}$$

For the "dual" Hamiltonian $\mathcal{H}^d = J^*\mathcal{H}J = \operatorname{diag}(h_2, h_1)$ we denote the corresponding objects by $\mathcal{H}^d_r$, m^d_r, μ^d_r, b^d_r, w^d_r, $\mathcal{I}_{\mathcal{H}^d_r}$, $\mathcal{Y}_{\mathcal{H}^d_r}$, and $\mathcal{K}_{\mathcal{H}^d}$. Note that a Hamiltonian $\mathcal{H}$ is singular and nontrivial if and only if $\mathcal{H}^d$ is singular and nontrivial. We also will need the Hamiltonian

$$\widehat{\mathcal{H}}_r(t) = \begin{cases} \mathcal{H}(t), & t \in [0, r), \\ \operatorname{diag}(\mathcal{I}_{\mathcal{H}}^{-1}(r), \mathcal{I}_{\mathcal{H}}(r)), & t \in [r, +\infty), \end{cases} \tag{2.4}$$

which plays the role of "Bernstein-Szegő approximation" to $\mathcal{H}$. From formula (2.2) we see that the Hamiltonian $\widehat{\mathcal{H}}_r$ is correctly defined and nontrivial if and only if $m_r(i) \neq 0$, that is, $\mathcal{H}_r$ is nontrivial. Indeed, if $m_r(i) \neq 0$, then $0 < \mathcal{I}_{\mathcal{H}}(r) < \infty$ and $\widehat{\mathcal{H}}_r$ is nontrivial by definition. The converse statement also holds.

Later we will use notation $\widehat{\mu}_r$ for the spectral measure generated by $\widehat{\mathcal{H}_r}$.

An analytic function f in the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is said to have bounded type if $f = \frac{f_1}{f_2}$ for some bounded analytic functions f_1, f_2 in $\mathbb{C}^+$, where f_2 is not identically zero. Denote by $\mathcal{N}(\mathbb{C}^+)$ the class of all functions of bounded type in $\mathbb{C}^+$. For every function $f \in \mathcal{N}(\mathbb{C}^+)$ we have

$$\int_{\mathbb{R}} \frac{\left|\log |f(x)|\right|}{1 + x^2}\, dx < \infty, \tag{2.5}$$

see, e.g., Theorem 9 in [9]. The mean type of a function $f \in \mathcal{N}(\mathbb{C}^+)$ is defined by

$$\operatorname{type}_+(f) = \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y}.$$

The upper limit above is finite for every nonzero function $f \in \mathcal{N}(\mathbb{C}^+)$ by Theorem 10 in [9]. A remarkable fact of the spectral theory of canonical Hamiltonian systems is that for every $t \geqslant 0$ the entries of solution $M(t, z)$ to Cauchy problem (1.2) are entire functions in z of bounded type in $\mathbb{C}^+$ and their mean type in $\mathbb{C}^+$ equals

$$\xi_{\mathcal{H}}(t) = \int_0^t \sqrt{\det \mathcal{H}(s)}\, ds. \tag{2.6}$$

This formula has been found by Krein [25] in the setting of the string equation and then proved in full generality by de Branges, see Theorem X in [8]. A short proof of (2.6) is in Section 6 of [31]. As a consequence, we have the following result.

Proposition 2.1. *Let $\mathcal{H}$ be a Hamiltonian on $\mathbb{R}_+$ and let entire function $f(z)$ be one of the entries $\{\Theta^\pm(t, z), \Phi^\pm(t, z)\}$ of the matrix M in* (1.3). *Then, if f is not equal to zero identically in $\mathbb{C}_+$, we have*

$$\frac{1}{\pi} \int_{\mathbb{R}} \log |f(x)| \frac{\operatorname{Im} z}{|x - z|^2}\, dx = \log |f(z)| - \xi_{\mathcal{H}}(t) \operatorname{Im} z \tag{2.7}$$

for every $z \in \mathbb{C}_+$.

Proof. Let $M = \left(\begin{smallmatrix} \Theta^+ & \Phi^+ \\ \Theta^- & \Phi^- \end{smallmatrix}\right)$ be the matrix solution of (1.2), and let $\Theta = \left(\begin{smallmatrix} \Theta^+ \\ \Theta^- \end{smallmatrix}\right)$ denote its first column. Then

$$J\Theta'(t, z) = z\mathcal{H}(t)\Theta(t, z), \qquad \Theta(0, z) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \quad t \geqslant 0, \quad z \in \mathbb{C}.$$

Integration by parts gives

$$\int_0^t \langle J\Theta'(s, z), \Theta(s, z)\rangle_{\mathbb{C}^2}\, ds = \langle J\Theta(t, z), \Theta(t, z)\rangle_{\mathbb{C}^2} + \int_0^t \langle \Theta(s, z), J\Theta'(s, z)\rangle_{\mathbb{C}^2}\, ds,$$

where the inner product in $\mathbb{C}^2$ is given by $\langle \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} c_3 \\ c_4 \end{smallmatrix}\right)\rangle_{\mathbb{C}^2} = c_1\overline{c_3} + c_2\overline{c_4}$. It follows that

$$\operatorname{Im}(\Theta^+(t, z)\overline{\Theta^-(t, z)}) = \operatorname{Im} z \cdot \int_0^t \langle \mathcal{H}(s)\Theta(s, z), \Theta(s, z)\rangle_{\mathbb{C}^2}\, ds, \qquad z \in \mathbb{C}. \tag{2.8}$$

Take f as one of $\{\Theta^{\pm}\}$. If $f(z_0) = 0$ for some $z_0 \in \mathbb{C}_+$, then (2.8) implies that $\mathcal{H}(s)\Theta(s, z_0) = 0$ for almost all $s \in [0, t]$ due to the fact that $\mathcal{H} \geqslant 0$ on $\mathbb{R}_+$. Hence, $J\Theta'(s, z_0) = 0$ for almost all $s \in [0, t]$. This implies that $\Theta(s, z_0) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for all $s \in [0, t]$. This may happen only in the case when $\mathcal{H}$ has the form $\left(\begin{smallmatrix}0 & 0\\0 & h_{22}\end{smallmatrix}\right)$ on $[0, t]$. But then $\Theta(s, z) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for every $s \in [0, t]$, $z \in \mathbb{C}$, in particular, $f(z) = f(z_0) = 0$ in $\mathbb{C}_+$. Thus, we see that either f is identically zero in $\mathbb{C}_+$ or $f(z) \neq 0$ for $z \in \mathbb{C}_+$. Function f belongs to $\mathcal{N}(\mathbb{C}^+)$, it is smooth on $\mathbb{R}$, and has no zeros in $\mathbb{C}^+$. So, there exists an outer function F on $\mathbb{C}_+$ such that $f(z) = e^{-i\xi_{\mathcal{H}}(t)z}F(z)$, $z \in \mathbb{C}^+$, see Theorem 9 in [9]. Now (2.7) follows from the mean value theorem for the harmonic function $\log|F|$. The proof for $\Phi^{\pm}$ is similar. $\square$

Proposition 2.2. *Let f be an analytic function in $\mathbb{C}^+$ such that $\operatorname{Im} f(z) > 0$ for all $z \in \mathbb{C}^+$. Then for almost all $x \in \mathbb{R}$ there exists finite non-tangential limit $f(x) = \lim\limits_{\substack{|z-x|<2\operatorname{Im} z\\ z\to x}} f(z)$ and*

$$\frac{1}{\pi}\int_{\mathbb{R}} \log|f(x)|\frac{\operatorname{Im} z}{|x-z|^2}\,dx = \log|f(z)|$$

for every $z \in \mathbb{C}^+$, where integral in the left hand side converges absolutely.

Proof. Combine Corollary 4.8 in Section 4 with Exercise 13 in Section 7 of Chapter II in [16]. $\square$

For every $\varphi \in [0, \pi)$, set $e_\varphi = \left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)$. An open interval $I \subset \mathbb{R}_+$ is called indivisible for $\mathcal{H}$ of type φ if there is a function h on I such that $\mathcal{H}(x) = h(x)e_\varphi e_\varphi^\top$ for almost all $x \in I$, and I is the maximal open interval having this property. Note that a Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ is nontrivial if $(0, +\infty)$ is not an indivisible interval of some type φ for $\mathcal{H}$.

The following four lemmas are known. We give their proofs in Appendix for the reader's convenience.

Lemma 2.1. *Let $\mathcal{H}$ be a Hamiltonian on $\mathbb{R}_+$ such that $(0, \ell)$ is indivisible interval of type $\varphi \in [0, \pi)$ for $\mathcal{H}$. Then the solution M of (1.2) has the form $M(t, z) = \left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right) - zJ\int_0^t \mathcal{H}(\tau)d\tau$ for every $t \in [0, \ell]$. In particular, for $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ and $t \in [0, \ell]$ we have*

$$M(t,z) = \begin{cases} \begin{pmatrix} 1 & 0\\ -z\int_0^t h_1(s)\,ds & 1\end{pmatrix} & \text{if } \varphi = 0,\\ \begin{pmatrix} 1 & z\int_0^t h_2(s)\,ds\\ 0 & 1\end{pmatrix} & \text{if } \varphi = \pi/2\end{cases}.$$

Lemma 2.2. *Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$, and let m be its Weyl-Titchmarsh function (1.3). Then, $\mathcal{H}$ is diagonal if and only if the measure μ is even and $a = 0$ in the Herglotz representation (1.4) of m.*

Lemma 2.3. *Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ and let m be its Weyl-Titchmarsh function. Then, we have $b > 0$ in the Herglotz representation (1.4) of m if and only if $(0, \varepsilon)$ is indivisible interval for $\mathcal{H}$ of type $\pi/2$ for some $\varepsilon > 0$. Moreover, we have $b = \int_0^\varepsilon \langle \mathcal{H}(t)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\rangle\,dt$ in the latter case.*

Lemma 2.4. *Let $\mathcal{H} = \operatorname{diag}(a_1, a_2)$ be the constant Hamiltonian on $\mathbb{R}_+$ generated by positive numbers a_1, a_2. Then for all $r \geqslant 0$ we have $w_r = \sqrt{a_2/a_1}$ on $\mathbb{R}$ and*

$$\log \mathcal{I}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(r) = \log\sqrt{a_2/a_1}\,. \tag{2.9}$$

The following lemma is crucial for our paper.

Lemma 2.5. *Let $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ and let μ be the spectral measure of system (1.2) generated by $\mathcal{H}$. Assume that $\mu \in \operatorname{Sz}(\mathbb{R})$. Then for every $r \geqslant 0$ we have*

(a) $\mu_r \in \mathrm{Sz}(\mathbb{R})$ *and* $\mu_r^d \in \mathrm{Sz}(\mathbb{R})$,
(b) $\mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(0) - 2\xi_{\mathcal{H}}(r) + 2\log|\Theta^+(r,i) + i\mathcal{I}_{\mathcal{H}}(r)\Theta^-(r,i)|$,
(c) $\mathcal{I}_{\mathcal{H}}(r) = 1/\mathcal{I}_{\mathcal{H}^d}(r)$,
(d) $\mathcal{K}_{\mathcal{H}}(r) = \mathcal{K}_{\mathcal{H}^d}(r)$,
(e) $\widehat{\mu}_r \in \mathrm{Sz}(\mathbb{R})$ *and* $\mathcal{K}_{\mathcal{H}}(0) = \mathcal{K}_{\widehat{\mathcal{H}}_r}(0) + \mathcal{K}_{\mathcal{H}}(r)$,

where $\xi_{\mathcal{H}}$ *is defined in* (2.6).

Proof. Take $r \geqslant 0$ and consider solutions

$$M(t,z) = \begin{pmatrix} \Theta^+(t,z) & \Phi^+(t,z) \\ \Theta^-(t,z) & \Phi^-(t,z) \end{pmatrix}, \qquad M_r(t,z) = \begin{pmatrix} \Theta_r^+(t,z) & \Phi_r^+(t,z) \\ \Theta_r^-(t,z) & \Phi_r^-(t,z) \end{pmatrix}, \tag{2.10}$$

of Cauchy problem (1.2) for the Hamiltonians $\mathcal{H}$ and $\mathcal{H}_r : x \mapsto \mathcal{H}(r+x)$, respectively. We have

$$M_0(t,z) = M_r(t-r,z)M_0(r,z), \qquad t \geqslant r, \quad z \in \mathbb{C}. \tag{2.11}$$

Indeed, the right hand side of the above equality satisfies equation $JM' = z\mathcal{H}M$ on $[r,\infty)$ and coincides with $M_0(t,z)$ at $t = r$. Multiplying matrices in (2.11) and using (1.3) with $\omega = 0$, we obtain

$$m_0(z) = \lim_{t\to+\infty} \frac{\Theta_r^-(t-r,z)\Phi^+(r,z) + \Phi_r^-(t-r,z)\Phi^-(r,z)}{\Theta_r^-(t-r,z)\Theta^+(r,z) + \Phi_r^-(t-r,z)\Theta^-(r,z)}, \tag{2.12}$$

Suppose there is $c > 0$ such that $(c,+\infty)$ is the indivisible interval of type $\pi/2$ for $\mathcal{H}$. Then from Lemma 2.1 and formula (2.12) we see that $m_0(z) = \frac{\Phi^-(c,z)}{\Theta^-(c,z)}$ for all $z \in \mathbb{C}^+$. Since functions Φ^-, Θ^- are real on the real axis, this implies that μ is a discrete measure concentrated at zeros of entire function $z \mapsto \Theta^-(c,z)$. In particular, we cannot have $\mu \in \mathrm{Sz}(\mathbb{R})$. A similar argument applies in the case where $(c,+\infty)$ is the indivisible interval of type 0 for some $c > 0$. It follows that the Hamiltonian $\mathcal{H}_r$ is nontrivial for every $r \geqslant 0$, in particular, its Weyl-Titchmarsh function m_r is correctly defined and nonzero. Using (2.12) and (1.3) with $\omega = 0$ for m_r, we get the relation

$$m_0(z) = \frac{\Phi^+(r,z) + m_r(z)\Phi^-(r,z)}{\Theta^+(r,z) + m_r(z)\Theta^-(r,z)}, \qquad z \in \mathbb{C}^+, \quad r \geqslant 0. \tag{2.13}$$

Hence,

$$\begin{aligned} \operatorname{Im} m_0(z) = &\frac{\operatorname{Im}\big(\Phi^+(r,z)\overline{\Theta^+(r,z)} + |m_r(z)|^2\Phi^-(r,z)\overline{\Theta^-(r,z)}\big)}{|\Theta^+(r,z) + m_r(z)\Theta^-(r,z)|^2} \\ &+ \frac{\operatorname{Im}\big(m_r(z)\big(\overline{\Theta^+(r,z)}\Phi^-(r,z) - \Theta^-(r,z)\overline{\Phi^+(r,z)}\big)\big)}{|\Theta^+(r,z) + m_r(z)\Theta^-(r,z)|^2}. \end{aligned}$$

Since the analytic function m_r has positive imaginary part in $\mathbb{C}^+$ for every $r \geqslant 0$, we can take non-tangential limit as $z \to x$ in this formula for almost all $x \in \mathbb{R}$, see Proposition 2.2. The analytic functions $\Theta^\pm$, $\Phi^\pm$ are real on the real line. The Wronskian is constant in r, thus

$$\Theta^+(r,z)\Phi^-(r,z) - \Theta^-(r,z)\Phi^+(r,z) = \det M_0(r,z) = \det M(r,z) = \det M(0,z) = 1,$$

for all $r \geqslant 0$, $z \in \mathbb{C}$, hence we obtain

$$w_0(x) = \operatorname{Im} m_0(x) = \frac{\operatorname{Im} m_r(x)}{|F_r(x)|^2} = \frac{w_r(x)}{|F_r(x)|^2}, \tag{2.14}$$

for almost all $x \in \mathbb{R}$, where $F_r : z \mapsto \Theta^+(r,z) + m_r(z)\Theta^-(r,z)$ is the analytic function in $\mathbb{C}^+$ and $F_r(x)$, $x \in \mathbb{R}$, are the non-tangential boundary values of F_r. Denote the first column of the matrix-function M in (2.10) by $\Theta = \begin{pmatrix}\Theta^+ \\ \Theta^-\end{pmatrix}$. Assume for a moment that $(0,r)$ is not an indivisible interval of type $\pi/2$ for $\mathcal{H}$. Then formula (2.8) implies that $\Theta^-(r,z) \neq 0$ for every $z \notin \mathbb{R}$, and, moreover, $\operatorname{Im} \frac{\Theta^+(r,z)}{\Theta^-(r,z)} > 0$ for $z \in \mathbb{C}^+$. Thus, the function $\log|F_r|$ can be represented in the form

$$\log|F_r(z)| = \log|\Theta^-(r,z)| + \log\left|m_r(z) + \frac{\Theta^+(r,z)}{\Theta^-(r,z)}\right|, \qquad z \in \mathbb{C}^+.$$

Since the functions m_r, $\frac{\Theta^+(r,\cdot)}{\Theta^-(r,\cdot)}$ have positive imaginary parts in $\mathbb{C}^+$ and $\Theta^- \in \mathcal{N}(\mathbb{C}^+)$, we have $|\log|F_r(x)||\,dx \in \Pi(\mathbb{R})$, and, moreover,

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|F_r(x)|}{1+x^2}\,dx = \log|F_r(i)| - \xi_{\mathcal{H}}(r),$$

by Proposition 2.1 and Proposition 2.2. In particular, the measure μ_r belongs to the Szegő class $\mathrm{Sz}(\mathbb{R})$. Taking logarithms in (2.14) and integrating with $\frac{1}{1+x^2}$, we obtain assertion (b):

$$\mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(0) - 2\xi_{\mathcal{H}}(r) + 2\log|F_r(i)|. \tag{2.15}$$

Let us now prove (b) in the case where $\mathcal{H}$ has an indivisible interval $(0,\varepsilon)$ of type $\pi/2$ for some $\varepsilon > 0$ and $r \leqslant \varepsilon$. In that situation, we can use Lemma 2.1 to show that $F_r(z) = 1$ for all z, hence $w_0 = w_r$ on $\mathbb{R}$ by (2.14), yielding $\mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(0)$ for $r \in [0,\varepsilon]$. Since $\xi_{\mathcal{H}} = 0$ on $[0,\varepsilon]$ by definition, this gives us relation (b) in full generality.

Next, the solution $M^d(r,z)$ of the canonical Hamiltonian system generated by the dual Hamiltonian $\mathcal{H}^d = J^*\mathcal{H}J$ has the form

$$M^d(r,z) = J^*M(r,z)J = \begin{pmatrix} \Phi^-(r,z) & -\Theta^-(r,z) \\ -\Phi^+(r,z) & \Theta^+(r,z) \end{pmatrix}. \tag{2.16}$$

Note that $\mathcal{H}^d$, $\mathcal{H}_r^d$ are singular nontrivial Hamiltonians because $\mathcal{H}$, $\mathcal{H}_r$ are singular and nontrivial. Using formula (1.3) with $\omega = \infty$, we see that $m_r^d(z) = -\lim_{t\to+\infty} \frac{\Theta_r^+(t,z)}{\Phi_r^+(t,z)} = -\frac{1}{m_r(z)}$ for all $r \geqslant 0$ and all $z \in \mathbb{C}^+$. Taking the non-tangential values of imaginary parts gives $w_r^d(x) = \frac{\operatorname{Im} m_r(x)}{|m_r(x)|^2} = \frac{w_r(x)}{|m_r(x)|^2}$. This formula and Proposition 2.2 imply $\mu_r^d \in \mathrm{Sz}(\mathbb{R})$ thus completing the proof of (a). Since the measures μ_r, μ_r^d are even, we have

$$\mathcal{I}_{\mathcal{H}^d}(r) = \operatorname{Im} m_r^d(i) = \frac{1}{\operatorname{Im} m_r(i)} = \frac{1}{\mathcal{I}_{\mathcal{H}}(r)}, \tag{2.17}$$

as claimed in (c). Next, using the formula $w_r^d(x) = \frac{w_r(x)}{|m_r(x)|^2}, x \in \mathbb{R}$, the mean value formula in Proposition 2.2, formula (2.17), and identity $m_r(i) = i\mathcal{I}_{\mathcal{H}}(r)$, we obtain assertion (d):

$$\begin{aligned}\mathcal{K}_{\mathcal{H}^d}(r) &= \log\mathcal{I}_{\mathcal{H}^d}(r) - \mathcal{Y}_{\mathcal{H}}(r) + \log|m_r(i)|^2 \\ &= -\log\mathcal{I}_{\mathcal{H}}(r) - \mathcal{Y}_{\mathcal{H}}(r) + 2\log\mathcal{I}_{\mathcal{H}}(r) = \mathcal{K}_{\mathcal{H}}(r).\end{aligned}$$

Finally, consider the Hamiltonian $\widehat{\mathcal{H}}_r$ introduced in (2.4). Since $\mathcal{H}_r$ is nontrivial, we have $\mathcal{I}_{\mathcal{H}}(r) \neq 0$ and hence $\widehat{\mathcal{H}}_r$ is defined correctly. By definition and Lemma 2.4, we have $\mathcal{I}_{\widehat{\mathcal{H}}_r}(r) = \mathcal{I}_{\mathcal{H}}(r)$, $\mathcal{Y}_{\widehat{\mathcal{H}}_r}(r) = \log\mathcal{I}_{\mathcal{H}}(r)$, and $\widehat{F}_r(i) = F_r(i)$ for the corresponding function $\widehat{F}_r$. The proof of Lemma 2.4 shows that $\widehat{m}_t$ is a constant function for each $t \geqslant r$. Using this and the fact that $\Phi^\pm, \Theta^\pm \in \mathcal{N}(\mathbb{C}^+)$, from (2.13) we obtain $\widehat{\mu}_r \in \mathrm{Sz}(\mathbb{R})$. Comparing the right hand sides of formula (2.13) for m_0 and $\widehat{m}_0$ at $z = i$, we get $\mathcal{I}_{\widehat{\mathcal{H}}_r}(0) = \mathcal{I}_{\mathcal{H}}(0)$. Hence, relation (2.15) for $\widehat{H}_r$ can be written in the form

$$\mathcal{Y}_{\widehat{\mathcal{H}}_r}(r) = \mathcal{Y}_{\widehat{\mathcal{H}}_r}(0) - 2\xi_{\mathcal{H}}(r) + 2\log|F_r(i)| = \mathcal{Y}_{\widehat{\mathcal{H}}_r}(0) - \mathcal{Y}_{\mathcal{H}}(0) + \mathcal{Y}_{\mathcal{H}}(r).$$

On the other hand, we have $\log \mathcal{I}_{\mathcal{H}}(r) = \mathcal{Y}_{\widehat{\mathcal{H}}_r}(r)$ and $\mathcal{I}_{\widehat{\mathcal{H}}_r}(0) = \mathcal{I}_{\mathcal{H}}(0)$. This yields assertion (e):

$$\begin{aligned}\mathcal{K}_{\mathcal{H}}(r) &= \log \mathcal{I}_{\mathcal{H}}(r) - \mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\widehat{\mathcal{H}}_r}(r) - \mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\widehat{\mathcal{H}}_r}(0) - \mathcal{Y}_{\mathcal{H}}(0)\\ &= \mathcal{Y}_{\widehat{\mathcal{H}}_r}(0) - \log \mathcal{I}_{\mathcal{H}}(0) + \log \mathcal{I}_{\mathcal{H}}(0) - \mathcal{Y}_{\mathcal{H}}(0)\\ &= \mathcal{Y}_{\widehat{\mathcal{H}}_r}(0) - \log \mathcal{I}_{\widehat{\mathcal{H}}_r}(0) + \log \mathcal{I}_{\mathcal{H}}(0) - \mathcal{Y}_{\mathcal{H}}(0)\\ &= -\mathcal{K}_{\widehat{\mathcal{H}}_r}(0) + \mathcal{K}_{\mathcal{H}}(0).\end{aligned}$$

The lemma is proved. $\square$

Lemma 2.6. *Let $l > 0$ and $\mathcal{H}$ be a singular Hamiltonian on $\mathbb{R}_+$ satisfying $\mathcal{H}(t) = \operatorname{diag}(a_1, a_2)$ for all $t \in [\ell, +\infty)$ where a_1, a_2 are positive parameters. Then its spectral measure μ belongs to the Szegő class $\mathrm{Sz}(\mathbb{R})$.*

Proof. Formula (2.14) for $r = \ell$ says that the absolutely continuous part of μ coincides with $\frac{|w_\ell(x)|^2}{|F_\ell(x)|^2}$. Since $\mathcal{H}_\ell = \operatorname{diag}(a_1, a_2)$ on $\mathbb{R}_+$, we have $w_\ell(x) = \sqrt{a_2/a_1}$ for all $x \in \mathbb{R}$ by Lemma 2.4. It remains to use Proposition 2.1 for the function $F_\ell \neq 0$ of class $\mathcal{N}(\mathbb{C}^+)$. $\square$

Lemma 2.7. *Let $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ whose spectral measure belongs to the Szegő class $\mathrm{Sz}(\mathbb{R})$. Then the functions $\mathcal{Y}_{\mathcal{H}}(r), \mathcal{K}_{\mathcal{H}}(r)$ are absolutely continuous and*

$$\mathcal{Y}'_{\mathcal{H}}(r) = 2\mathcal{I}_{\mathcal{H}}(r)h_1(r) - 2\xi'_{\mathcal{H}}(r), \tag{2.18}$$

$$\mathcal{K}'_{\mathcal{H}}(r) = -\mathcal{I}_{\mathcal{H}}(r)h_1(r) - \frac{h_2(r)}{\mathcal{I}_{\mathcal{H}}(r)} + 2\xi'_{\mathcal{H}}(r), \tag{2.19}$$

for almost all $r \geqslant 0$.

Proof. At first, assume additionally that h_1, h_2 belong to $\mathcal{C}^1(\mathbb{R}_+)$, the space of continuously differentiable functions on $(0, +\infty)$ whose derivatives have a finite limit at 0. Then the entries of the the solution $M(\cdot, i)$ of (1.2) at $z = i$ belong to the space $\mathcal{C}^1(\mathbb{R}_+)$ as well. From formula (2.13) and identity $m_r(i) = i\mathcal{I}_{\mathcal{H}}(r)$, $r \geqslant 0$, we also have $\mathcal{I}_{\mathcal{H}} \in \mathcal{C}^1(\mathbb{R}_+)$. Assertion (b) of Lemma 2.5 says that

$$\mathcal{Y}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(0) - 2\xi_{\mathcal{H}}(r) + 2\log|\Theta^+(r,i) + i\mathcal{I}_{\mathcal{H}}(r)\Theta^-(r,i)|, \qquad r \geqslant 0. \tag{2.20}$$

Differentiating the above formula with respect to r at $r = 0$ and using the equation

$$\left.\begin{pmatrix}\Theta^+(r,i)' & \Phi^+(r,i)'\\ \Theta^-(r,i)' & \Phi^-(r,i)'\end{pmatrix}\right|_{r=0} = M'(0,i) = iJ^*\mathcal{H}(0)M(0,i) = \begin{pmatrix}0 & ih_2(0)\\ -ih_1(0) & 0\end{pmatrix},$$

we obtain

$$\begin{aligned}\mathcal{Y}'_{\mathcal{H}}(0) &= -2\xi'_{\mathcal{H}}(0) + 2\,\mathrm{Re}\left.\left(\frac{\Theta^+(r,i)' + i\mathcal{I}'_{\mathcal{H}}(r)\Theta^-(r,i) + i\mathcal{I}_{\mathcal{H}}(r)\Theta^-(r,i)'}{\Theta^+(r,i) + i\mathcal{I}_{\mathcal{H}}(r)\Theta^-(r,i)}\right)\right|_{r=0}\\ &= -2\xi'_{\mathcal{H}}(0) + 2\mathcal{I}_{\mathcal{H}}(0)h_1(0).\end{aligned}$$

For $r > 0$ we have

$$\mathcal{Y}'_{\mathcal{H}}(r) = \mathcal{Y}'_{\mathcal{H}_r}(0) = -2\xi'_{\mathcal{H}_r}(0) + 2\mathcal{I}_{\mathcal{H}_r}(0)h_1(r) = -2\xi'_{\mathcal{H}}(r) + 2\mathcal{I}_{\mathcal{H}}(r)h_1(r).$$

Thus, relation (2.18) holds in the case when $h_1, h_2 \in \mathcal{C}^1(\mathbb{R}_+)$. Now let $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ be an arbitrary singular nontrivial Hamiltonian on $\mathbb{R}_+$ with spectral measure in $\mathrm{Sz}(\mathbb{R})$. By Lemma 2.5, the functions $\mathcal{I}_{\mathcal{H}}(r)$, $\mathcal{Y}_{\mathcal{H}}(r)$ are correctly defined on $\mathbb{R}_+$. Find a sequence of positive smooth functions $\{h_{1,n}\}$, $\{h_{2,n}\}$ such that

$$\lim_{n\to\infty}\int_0^T |h_j(s) - h_{j,n}(s)|\,ds = 0$$

for every $T > 0$ and $j = 1, 2$. Solutions of the equations $JM'_{(n)} = i\mathcal{H}_{(n)}M_{(n)}$, $M_{(n)}(0,i) = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, generated by the Hamiltonians $\mathcal{H}_{(n)} = \operatorname{diag}(h_{1,n}, h_{2,n})$ will then converge uniformly on compact subsets of $\mathbb{R}_+$ to the solution $M(\cdot, i)$ of the equation $JM' = i\mathcal{H}M$, $M(0,i) = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. From formulas (2.13) and (2.20) we see that continuous functions $\mathcal{I}_{\mathcal{H}_{(n)}}(r)$, $\mathcal{Y}_{\mathcal{H}_{(n)}}(r)$ converge uniformly on compact subsets of $\mathbb{R}_+$ to the functions $\mathcal{I}_{\mathcal{H}}(r)$, $\mathcal{Y}_{\mathcal{H}}(r)$, respectively. Thus, we have

$$\begin{aligned}\mathcal{Y}_{\mathcal{H}}(r) - \mathcal{Y}_{\mathcal{H}}(0) &= \lim_{n\to\infty}(\mathcal{Y}_{\mathcal{H}_{(n)}}(r) - \mathcal{Y}_{\mathcal{H}_{(n)}}(0))\\ &= -2\xi_{\mathcal{H}}(r) + \lim_{n\to\infty}\int_0^r \mathcal{I}_{\mathcal{H}_{(n)}}(s)h_1(s)\,ds\\ &= -2\xi_{\mathcal{H}}(r) + \int_0^r \mathcal{I}_{\mathcal{H}}(s)h_1(s)\,ds,\end{aligned}$$

for every $r > 0$. This formula shows that $\mathcal{Y}_{\mathcal{H}}$ is absolutely continuous and satisfies relation (2.18). Relation (2.19) follows by adding (2.18) written for $\mathcal{H}$ and $\mathcal{H}_d = \operatorname{diag}(h_2, h_1)$ and using identity

$$\mathcal{K}_{\mathcal{H}} = -(\mathcal{Y}_{\mathcal{H}} + \mathcal{Y}_{\mathcal{H}_d})/2 \tag{2.21}$$

which is immediate from Lemma 2.5.$(c), (d)$. $\square$

Lemma 2.8. *Let $\ell > 0$, $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ be a singular Hamiltonian on $\mathbb{R}_+$ such that $\mathcal{H}(t) = \mathcal{H}(\ell)$ for $t \in [\ell, +\infty)$, and $\det\mathcal{H}(\ell) \neq 0$. Then, for every $r \geqslant 0$ we have*

$$e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(r) - \xi_{\mathcal{H}}(r)} = \int_r^\infty h_1(s)e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(s) - \xi_{\mathcal{H}}(s)}\,ds, \tag{2.22}$$

$$e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(r) - \xi_{\mathcal{H}}(r)} = \int_r^\infty h_2(s)e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(s) - \xi_{\mathcal{H}}(s)}\,ds. \tag{2.23}$$

Proof. The right hand side of (2.22) at $r_0 \geqslant \ell$ is equal to

$$h_1(\ell)e^{-\xi_{\mathcal{H}}(r_0) - \frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(r_0)}\int_{r_0}^\infty e^{(r_0 - s)\sqrt{h_1(\ell)h_2(\ell)}}\,ds = \sqrt{\frac{h_1(\ell)}{h_2(\ell)}}\cdot e^{-\xi_{\mathcal{H}}(r_0) - \frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(r_0)}.$$

Substituting $\mathcal{Y}_{\mathcal{H}}(r_0) = \log\sqrt{\frac{h_2(\ell)}{h_1(\ell)}}$, $\mathcal{Y}_{\mathcal{H}^d}(r_0) = \log\sqrt{\frac{h_1(\ell)}{h_2(\ell)}}$ into the formula above, we see that (2.22) holds for all $r \geqslant \ell$. Next, differentiating the left hand side of (2.22) and using Lemma 2.5 and Lemma 2.7, we obtain

$$\begin{aligned}-\left(\frac{\mathcal{Y}'_{\mathcal{H}}(r)}{2} + \xi'_{\mathcal{H}}(r)\right)e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(r) - \xi_{\mathcal{H}}(r)} &= -h_1(r)\mathcal{I}_{\mathcal{H}}(r)e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(r) - \xi_{\mathcal{H}}(r)}\\ &= -h_1(r)e^{\frac{1}{2}\log\mathcal{I}_{\mathcal{H}}(r) + \frac{1}{2}\mathcal{K}_{\mathcal{H}}(r) - \xi_{\mathcal{H}}(r)}\\ &= -h_1(r)e^{\frac{1}{2}\log\mathcal{I}_{\mathcal{H}}(r) + \frac{1}{2}(\log\mathcal{I}_{\mathcal{H}^d}(r) - \mathcal{Y}_{\mathcal{H}^d}(r)) - \xi_{\mathcal{H}}(r)}\\ &= -h_1(r)e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(r) - \xi_{\mathcal{H}}(r)}.\end{aligned}$$

This agrees with the derivative of the right hand side of (2.22) for almost all $r \geqslant 0$. It follows that (2.22) holds for all $r \geqslant 0$. Formula (2.23) can be proved in a similar way. $\square$

3. Some estimates of the entropy function

In this section we consider Hamiltonians $\mathcal{H}$ such that $\det\mathcal{H} = 1$ almost everywhere on $\mathbb{R}_+$. In the notations of Section 2, we have $\mathcal{K}(\mu) = \mathcal{K}_{\mathcal{H}}(0)$ for such Hamiltonians. Indeed, the coefficient b_0 in (2.2) is non-zero if and only if there exists $\varepsilon > 0$ such that $(0, \varepsilon)$ is the indivisible interval of type $\pi/2$ for $\mathcal{H}_0 = \mathcal{H}$, see Lemma 2.3. The latter never happens for Hamiltonians $\mathcal{H}$ with $\det\mathcal{H} = 1$ almost everywhere on $\mathbb{R}_+$.

3.1. **A lower bound for the entropy.** We first obtain a local estimate for the entropy $\mathcal{K}(\mu) = \mathcal{K}_{\mathcal{H}}(0)$ in terms of $\mathcal{H}$ and then use assertion (e) of Lemma 2.5 to improve it.

Lemma 3.1. *Let $h \geqslant 0$ be a function on $\mathbb{R}_+$ such that $h, 1/h \in L^1_{\rm loc}(\mathbb{R}_+)$ and assume that h equals to some positive constant on $[\ell, +\infty)$ for some $\ell \geqslant 0$. Then, for the Hamiltonian $\mathcal{H} = \operatorname{diag}(h, 1/h)$, we have*

$$e^{\frac{1}{2}\mathcal{K}_{\mathcal{H}}(0)} \geqslant \int_0^\infty \sqrt{\zeta_h(t)} \cdot te^{-t}dt,$$

where $\zeta_h(t) = \frac{1}{t}\int_0^t h(s)\, ds \cdot \frac{1}{t}\int_0^t \frac{1}{h(s)}\, ds$ for $t > 0$.

Proof. Using Lemma 2.8 twice, we get

$$\begin{aligned} e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(0)} &= \int_0^\infty h(s) e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(s) - s}\, ds \\ &= \int_0^\infty h(s) \left(\int_s^\infty \frac{1}{h(\tau)} e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(\tau)} e^{s-\tau}\, d\tau \right) e^{-s}\, ds \\ &= \int_0^\infty \frac{1}{h(\tau)} e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}}(\tau)} \left(\int_0^\tau h(s)\, ds \right) e^{-\tau}\, d\tau. \end{aligned} \tag{3.1}$$

Analogous formula holds for $\mathcal{Y}_{\mathcal{H}^d}$:

$$e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(0)} = \int_0^\infty h(\tau) e^{-\frac{1}{2}\mathcal{Y}_{\mathcal{H}^d}(\tau)} \left(\int_0^\tau \frac{1}{h(s)}\, ds \right) e^{-\tau}\, d\tau. \tag{3.2}$$

We have $2\mathcal{K}_{\mathcal{H}}(r) = -\mathcal{Y}_{\mathcal{H}}(r) - \mathcal{Y}_{\mathcal{H}^d}(r)$ for all $r \geqslant 0$ (see (2.21)). We also have $\mathcal{K}_{\mathcal{H}} \geqslant 0$ on $\mathbb{R}_+$ (check, e.g., (2.3)). Multiplying formulas (3.1), (3.2) and using Cauchy-Schwarz inequality, we obtain

$$e^{\frac{1}{2}\mathcal{K}_{\mathcal{H}}(0)} \geqslant \int_0^\infty e^{\frac{1}{2}\mathcal{K}_{\mathcal{H}}(\tau)} e^{-\tau} \sqrt{\int_0^\tau h(s)\, ds \int_0^\tau \frac{1}{h(s)}\, ds}\, d\tau \geqslant \int_0^\infty \sqrt{\zeta_h(t)} \cdot te^{-t}dt,$$

as required. $\square$

Remark. We can write $\zeta_h(t) = \langle h \rangle_{[0,t]} \langle 1/h \rangle_{[0,t]}$ and $\zeta_h(t) \geqslant 1$, as follows from Cauchy-Schwarz inequality.

This lemma and additivity of the entropy $\mathcal{K}_{\mathcal{H}}$ imply the following estimate.

Proposition 3.1. *Let $h \geqslant 0$ be a function on $\mathbb{R}_+$ such that $h, 1/h \in L^1_{\rm loc}(\mathbb{R}_+)$ and $\mathcal{H} = \operatorname{diag}(h, 1/h)$. Then, there exists a sequence of numbers $\{t_n\}$ such that $t_n \in [3, 4]$ and*

$$\sum_{n \geqslant 0} \left(\frac{1}{t_n} \int_{4n}^{4n+t_n} h(s)\, ds \cdot \frac{1}{t_n} \int_{4n}^{4n+t_n} \frac{ds}{h(s)} - 1 \right) \leqslant e^{10\mathcal{K}_{\mathcal{H}}(0)} - 1.$$

Proof. Iteratively applying assertion (e) of Lemma 2.5, we can find a sequence of Hamiltonians $\mathcal{H}_{(n)} = \operatorname{diag}(h_n, 1/h_n)$ such that $\mathcal{H}_{(n)}(x) = \mathcal{H}(4n + x)$ for $x \in [0, 4]$, $\mathcal{H}_{(n)}(x) = \operatorname{diag}(a_n, 1/a_n)$ for almost all $x > 4$ and some constant $a_n > 0$, and

$$\mathcal{K}_{\mathcal{H}}(0) \geqslant \sum_{n \geqslant 0} \mathcal{K}_{\mathcal{H}_{(n)}}(0). \tag{3.3}$$

Take $n \geqslant 0$ and apply Lemma 3.1 for the Hamiltonian $\mathcal{H}_{(n)}$. Making note of

$$\int_0^\infty te^{-t}dt = 1$$

and applying Jensen's inequality, we get

$$\mathcal{K}_{\mathcal{H}_{(n)}}(0) \geqslant \int_0^\infty \log(\zeta_{h,n}(t)) \cdot te^{-t}\, dt,$$

where $\zeta_{h,n}(t) = \frac{1}{t}\int_{4n}^{4n+t} h(s)\, ds \cdot \frac{1}{t}\int_{4n}^{4n+t} \frac{1}{h(s)}\, ds$ for $t \in [0,4]$, $\zeta_{h,n}(t) \geqslant 1$ for all $t > 0$. Since $\int_I te^{-t} dt \geqslant 1/10$ for $I = [3,4]$, we have $10\mathcal{K}_{\mathcal{H}_{(n)}}(0) \geqslant \min_{t\in I} \log \zeta_{h,n}(t)$. Define t_n to be a point in I such that $\zeta_{h,n}(t_n) = \min_{t\in I} \zeta_{h,n}(t)$. Since $e^{x+y} - 1 \geqslant e^x - 1 + e^y - 1$ for all $x, y \geqslant 0$, we notice that (3.3) implies

$$\begin{aligned} e^{10\mathcal{K}_{\mathcal{H}}(0)} - 1 &\geqslant \sum_{n\geqslant 0} \left(e^{10\mathcal{K}_{\mathcal{H}_{(n)}}(0)} - 1\right) \geqslant \sum_{n\geqslant 0} \left(\zeta_{h,n}(t_n) - 1\right) \\ &= \sum_{n\geqslant 0} \left(\frac{1}{t_n}\int_{4n}^{4n+t_n} h(s)\, ds \cdot \frac{1}{t_n}\int_{4n}^{4n+t_n} \frac{1}{h(s)}\, ds - 1\right), \end{aligned}$$

which is the desired estimate. □

3.2. **An upper bound for the entropy.**

Proposition 3.2. *Let h be a function as in Lemma 3.1, and let $\mathcal{H} = \operatorname{diag}(h, 1/h)$ be the corresponding Hamiltonian. Then,*

$$\mathcal{K}_{\mathcal{H}}(0) \leqslant \int_0^\infty (\kappa(s) + \kappa_d(s) - 2)\, ds,$$

where $\kappa(r) = \frac{1}{h(r)}\int_r^\infty h(s)e^{r-s}\, ds$ and $\kappa_d(r) = h(r)\int_r^\infty \frac{1}{h(s)}e^{r-s}\, ds$ for $r \geqslant 0$.

Proof. Consider the functions

$$u(r) = \int_r^\infty \frac{1}{h(s)} e^{-\mathcal{Y}_{\mathcal{H}}(s)-s}\, ds, \qquad u_d(r) = \int_r^\infty h(s)e^{-\mathcal{Y}_{\mathcal{H}^d}(s)-s}\, ds,$$

defined on $\mathbb{R}_+$. By Lemma 2.8, we have

$$\begin{aligned} e^{-\mathcal{Y}_{\mathcal{H}}(r)} &= \left(\int_r^\infty h(s)e^{-\frac{\mathcal{Y}_{\mathcal{H}^d}(s)}{2}} e^{r-s}\, ds\right)^2 \\ &\leqslant \left(\int_r^\infty h(s)e^{r-s}\, ds\right)\left(\int_r^\infty h(s)e^{-\mathcal{Y}_{\mathcal{H}^d}(s)}e^{r-s}\, ds\right) \\ &= h(r)e^r\kappa(r)u_d(r). \end{aligned}$$

Dividing by he^r, we obtain $-u'(r) \leqslant \kappa(r)u_d(r)$ for almost all $r \geqslant 0$. Analogously, we have $-u_d'(r) \leqslant \kappa_d(r)u(r)$, $r \geqslant 0$ for the function u_d. It follows that

$$0 \leqslant -(u^2 + u_d^2)'(r) \leqslant 2(\kappa(r) + \kappa_d(r))u(r)u_d(r) \leqslant (\kappa(r) + \kappa_d(r))(u^2 + u_d^2)(r),$$

for almost all $r \geqslant 0$. Thus, we have

$$-\frac{\partial}{\partial r}\log\bigl(u^2(r) + u_d^2(r)\bigr) \leqslant \kappa(r) + \kappa_d(r).$$

Taking into account that $u(r) = u_d(r) = e^{-r}$ for $r \geqslant \ell$ by (2.9), we get

$$u^2(0) + u_d^2(0) \leqslant (u^2(\ell) + u_d^2(\ell))e^{\int_0^\ell (\kappa(s)+\kappa_d(s))\, ds} = 2e^{\int_0^{+\infty}(\kappa(s)+\kappa_d(s)-2)\, ds}. \tag{3.4}$$

On the other hand, we have

$$u(0) = \int_0^\infty \frac{1}{\mathcal{I}_{\mathcal{H}}(s)h(s)} e^{\mathcal{K}_{\mathcal{H}}(s)-s}\, ds, \qquad u_d(0) = \int_0^\infty \mathcal{I}_{\mathcal{H}}(s)h(s)e^{\mathcal{K}_{\mathcal{H}}(s)-s}\, ds,$$

by assertions (c), (d) of Lemma 2.5. From (2.19) for $h_1 = h = 1/h_2$ we now get

$$\begin{aligned} u(0) + u_d(0) &= -\int_0^\infty \mathcal{K}'_{\mathcal{H}}(s) e^{\mathcal{K}_{\mathcal{H}}(s)-s}\,ds + 2\int_0^\infty e^{\mathcal{K}_{\mathcal{H}}(s)-s}\,ds \\ &= e^{\mathcal{K}_{\mathcal{H}}(0)} + \int_0^\infty e^{\mathcal{K}_{\mathcal{H}}(s)-s}\,ds \\ &\geqslant e^{\mathcal{K}_{\mathcal{H}}(0)} + 1 \geqslant 2e^{\mathcal{K}_{\mathcal{H}}(0)/2}, \end{aligned}$$

using integration by parts and the fact that $\mathcal{K}_{\mathcal{H}}(s) \geqslant 0$ for all s. Last estimate and (3.4) imply

$$e^{\mathcal{K}_{\mathcal{H}}(0)} \leqslant \left(\frac{u(0) + u_d(0)}{2}\right)^2 \leqslant \frac{u^2(0) + u_d^2(0)}{2} \leqslant e^{\int_0^{+\infty}(\kappa(s)+\kappa_d(s)-2)\,ds}.$$

Taking the logarithms, we arrive to the statement of the proposition. □

4. Proof of Theorem 1

The classical Muckenhoupt class $A_2(\mathbb{R})$ is defined as the set of measurable functions $h \geqslant 0$ on $\mathbb{R}$ with finite characteristic

$$[h]_2 \equiv \sup_{I\subset\mathbb{R}} \langle h\rangle_I \langle h^{-1}\rangle_I,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$. Recall that $I_{x,y}$ denotes $[x, x + y)$ for $x, y \in \mathbb{R}_+$. For a function $h \geqslant 0$ on $\mathbb{R}_+$ and a sequence $\alpha = \{\alpha_n\}$ of positive numbers, put

$$[h, \alpha] = \sum_{n=0}^{\infty}\Bigl(\langle h\rangle_{I_{n,\alpha_n}} \langle h^{-1}\rangle_{I_{n,\alpha_n}} - 1\Bigr). \tag{4.1}$$

Each term in the sum above is nonnegative, hence $[h, \alpha] \in \mathbb{R}_+ \cup \{+\infty\}$ is correctly defined. Denote by $\mathbf{2}$ the constant sequence $2, 2, \ldots$ indexed by non-negative integers.

Definition. Let $A_2(\mathbb{R}_+, \ell^1)$ be the set of functions $h \geqslant 0$ on $\mathbb{R}_+$ such that the characteristic $[h]_{2,\ell^1} = [h, \mathbf{2}]$ is finite.

Note that $[h]_{2,\ell^1} = 0$ if and only if the function h is constant. Next, for a function $h \geqslant 0$ on $\mathbb{R}_+$ define

$$[h]_{int} = \int_0^\infty (\kappa(s) + \kappa_d(s) - 2)\,ds, \tag{4.2}$$

where $\kappa(r) = \frac{1}{h(r)}\int_r^\infty h(s)e^{r-s}\,ds$ and $\kappa_d(r) = h(r)\int_r^\infty \frac{1}{h(s)}e^{r-s}\,ds$ for $r \geqslant 0$. Since $h \geqslant 0$ on $\mathbb{R}_+$, we have $\frac{h(s)}{h(r)} + \frac{h(r)}{h(s)} \geqslant 2$, hence the quantity $[h]_{int} \in \mathbb{R}_+ \cup \{+\infty\}$ is correctly defined.

Proposition 4.1. *Let $h \geqslant 0$ be a measurable function on $\mathbb{R}_+$. Assume that $[h, \alpha]$ is finite for a sequence $\alpha = \{\alpha_n\}$ where $\alpha_n \in [3, 4], \forall n \in \mathbb{Z}^+$. Then $h \in A_2(\mathbb{R}_+, \ell^1)$ and, moreover, we have $[h]_{2,\ell^1} \leqslant c[h, \alpha]$ with absolute constant c.*

Proposition 4.2. *There exists an absolute constant c such that $[h]_{int} \leqslant c[h]_{2,\ell^1} e^{c[h]_{2,\ell^1}}$ for every function $h \in A_2(\mathbb{R}_+, \ell^1)$.*

Propositions 4.1, 4.2 will be proved in the next section. Later, in the proof of the theorem, we will need the following lemma.

Lemma 4.1. *Let $\mathcal{H}$, $\mathcal{H}_{(k)}$ be singular diagonal Hamiltonians on $\mathbb{R}_+$ such that $\mathcal{H}_{(k)}(x) = \mathcal{H}(x)$ for every $k \geqslant 0$ and all $x \in [0, k]$. Suppose that the spectral measure of $\mathcal{H}_{(k)}$ belongs to $\mathrm{Sz}(\mathbb{R})$ for every $k \geqslant 0$ and $\sup_{k\geqslant 0} \mathcal{K}_{\mathcal{H}_{(k)}}(0) < \infty$. Then, the spectral measure of $\mathcal{H}$ belongs to $\mathrm{Sz}(\mathbb{R})$ and $\mathcal{K}_{\mathcal{H}}(0) \leqslant \limsup_{k\to\infty} \mathcal{K}_{\mathcal{H}_{(k)}}(0)$.*

Proof. Let $\mathcal{H}$ be a singular Hamiltonian on $\mathbb{R}_+$ and let m be its Weyl-Titchmarsh function. As usual, denote by $\Theta^\pm$, $\Phi^\pm$ the corresponding entries of the solution M of Cauchy problem (1.2). Then, by the nesting circles analysis (see page 42 in Section 8 of [31] or page 475 in Section 7 of [17]), we have

$$\left|m(z) - \frac{\Phi^-(k,z)}{\Theta^-(k,z)}\right| \leqslant \frac{1}{\operatorname{Im}\bigl(\Theta^+(k,z)\overline{\Theta^-(k,z)}\bigr)}, \qquad z \in \mathbb{C}^+, \quad k \geqslant 0, \tag{4.3}$$

where the right hand side tends to zero as $k \to +\infty$ uniformly on compacts in $\mathbb{C}^+$. Let $m_{(k)}$ be the Weyl-Titchmarsh function of the Hamiltonian $\mathcal{H}_{(k)}$. Since $\mathcal{H}_{(k)}$ coincides with $\mathcal{H}$ on $[0,k]$, we have estimate (4.3) with m replaced by $m_{(k)}$ and the same right hand side. The triangle inequality now implies that $m - m_{(k)}$ tends to zero uniformly on compact subsets of $\mathbb{C}^+$.

Let us consider the measures $\widetilde{\mu}$, $\widetilde{\mu}_{(k)}$ supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ whose Poisson extensions to the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ coincide with positive harmonic functions $\operatorname{Im} m(\omega)$, $\operatorname{Im} m_{(r)}(\omega)$ in $\mathbb{D}$, respectively, where $\omega : w \mapsto i\frac{1-w}{1+w}$ is the conformal mapping from $\mathbb{D}$ onto $\mathbb{C}^+$. Since the difference $m - m_{(k)}$ tends to zero uniformly on compacts in $\mathbb{C}^+$, the measures $\widetilde{\mu}_{(k)}$ converge weakly to the measure $\widetilde{\mu}$. Recall that the the relative entropy of two positive finite measures ν_1, ν_2 on $\mathbb{T}$ is defined by

$$S(\nu_1|\nu_2) = \begin{cases} -\infty & \text{if } \nu_1 \text{ is not } \nu_2 \text{ a.c.,} \\ -\int_{\mathbb{T}} \log\left(\frac{d\nu_1}{d\nu_2}\right) d\nu_1 & \text{if } \nu_1 \text{ is } \nu_2 \text{ a.c..} \end{cases}$$

It is known (see Section 2.2.3 in [32]) that the relative entropy is weakly upper-semicontinuous, which means $\limsup_{k\to+\infty} S(\nu_1|\nu_{2,k}) \leqslant S(\nu_1|\nu_2)$ for every sequence of finite measures $\nu_{2,k}$ on $\mathbb{T}$ converging weakly to a measure ν_2. This implies that $\widetilde{\mu}$ belongs to the Szegő class on $\mathbb{T}$ and

$$-\infty < \limsup_{k\to\infty} \int_{\mathbb{T}} \log \widetilde{w}_{(k)}(\xi)\, dm(\xi) \leqslant \int_{\mathbb{T}} \log \widetilde{w}(\xi)\, dm(\xi), \tag{4.4}$$

where m is the Lebesgue measure on $\mathbb{T}$ normalized by $m(\mathbb{T}) = 1$, and $\widetilde{w}$, $\widetilde{w}_{(k)}$ are the densities on $\widetilde{\mu}$, $\widetilde{\mu}_{(k)}$ with respect to m. Changing variables in (4.4), we see that the spectral measure of $\mathcal{H}$ lies in the class $\operatorname{Sz}(\mathbb{R})$, and, moreover,

$$\limsup_{k\to+\infty} \mathcal{Y}_{\mathcal{H}_{(k)}}(0) \leqslant \mathcal{Y}_{\mathcal{H}}(0).$$

From the relation $\lim_{k\to\infty} m_{(k)}(i) = m(i)$ we get $\mathcal{I}_{\mathcal{H}}(0) = \lim_{k\to+\infty} \mathcal{I}_{\mathcal{H}_{(k)}}(0)$. The lemma now follows. $\square$

The next result establishes the key two-sided estimates for a special class of Hamiltonians. Recall that the quantity $\widetilde{\mathcal{K}}(\mathcal{H})$ is defined in (1.6).

Lemma 4.2. *Let h be a function as in Lemma 3.1, and let $\mathcal{H} = \operatorname{diag}(h, 1/h)$. Then, we have $\mathcal{K}_{\mathcal{H}}(0) \leqslant c\widetilde{\mathcal{K}}(\mathcal{H})e^{c\widetilde{\mathcal{K}}(\mathcal{H})}$ and $\widetilde{\mathcal{K}}(\mathcal{H}) \leqslant c\mathcal{K}_{\mathcal{H}}(0)e^{c\mathcal{K}_{\mathcal{H}}(0)}$ for an absolute constant c.*

Proof. By Lemma 2.6, the spectral measure of $\mathcal{H}$ belongs to $\operatorname{Sz}(\mathbb{R})$. From Proposition 3.2 we know that $\mathcal{K}_{\mathcal{H}}(0) \leqslant [h]_{int}$. Proposition 4.2 implies $[h]_{int} \leqslant c[h]_{2,\ell^1} e^{c[h]_{2,\ell^1}}$ with $[h]_{2,\ell^1} = \frac{1}{4}\widetilde{\mathcal{K}}(\mathcal{H})$. Combining these estimates, we obtain inequality $\mathcal{K}_{\mathcal{H}}(0) \leqslant c\widetilde{\mathcal{K}}(\mathcal{H})e^{c\widetilde{\mathcal{K}}(\mathcal{H})}$. To prove the second inequality, observe that Proposition 3.1, when applied to $\mathcal{H}$, provides a sequence $\{t_n\} \subset [3,4]$ such that

$$\sum_{n\geqslant 0} \left(\frac{1}{t_n}\int_{4n}^{4n+t_n} h(s)\, ds \cdot \frac{1}{t_n}\int_{4n}^{4n+t_n} \frac{ds}{h(s)} - 1\right) \leqslant e^{10\mathcal{K}_{\mathcal{H}}(0)} - 1.$$

The same proposition applied to three "translated" Hamiltonians $\mathcal{H}_k : x \mapsto \mathcal{H}(x+k), k = 1,2,3$, gives

$$\sum_{n\geqslant 0}\left(\frac{1}{t_n^{(k)}}\int_{4n}^{4n+t_n^{(k)}} h(s+k)\,ds\cdot\frac{1}{t_n^{(k)}}\int_{4n}^{4n+t_n^{(k)}}\frac{ds}{h(s+k)}-1\right)\leqslant e^{10\mathcal{K}_{\mathcal{H}_k}(0)}-1.$$

for three new sequences $\{t_n^{(k)}\}\subset[3,4]$ where $k=1,2,3$. Summing up the above four formulas, we obtain $[h,\alpha]\leqslant e^{10\mathcal{K}_{\mathcal{H}}(0)}-1+\sum_{k=1}^{3}(e^{10\mathcal{K}_{\mathcal{H}_k}(0)}-1)$ for the sequence $\alpha=\{\alpha_n\}$ defined by $\alpha_{4n}=t_n$, $\alpha_{4n+k}=t_n^{(k)}$, $n\geqslant 0, k=1,2,3$. By Lemma 2.5.(e), we have $\mathcal{K}_{\mathcal{H}_k}(0)\leqslant\mathcal{K}_{\mathcal{H}}(0)$, hence $[h,\alpha]\leqslant 4(e^{10\mathcal{K}_{\mathcal{H}}}(0)-1)\leqslant c\mathcal{K}_{\mathcal{H}}(0)e^{10\mathcal{K}_{\mathcal{H}}(0)}$. Proposition 4.1 says that $[h]_{2,\ell^1}\leqslant c[h,\alpha]$ for an absolute constant c. By definition, we have $\widetilde{\mathcal{K}}(\mathcal{H})=4[h]_{2,\ell^1}$, hence $\widetilde{\mathcal{K}}(\mathcal{H})\leqslant c\mathcal{K}_{\mathcal{H}}(0)e^{10\mathcal{K}_{\mathcal{H}}(0)}$. □

In the next lemma, we will show that the condition that the determinant equals to one can be dropped.

Lemma 4.3. *Let $\mathcal{H}=\operatorname{diag}(h_1,h_2)$ be a singular Hamiltonian on $\mathbb{R}_+$ such that h_1, h_2 are equal to positive constants on $[\ell,+\infty)$ for some $\ell\geqslant 0$. Then, we have $\widetilde{\mathcal{K}}(\mathcal{H})\leqslant c\mathcal{K}_{\mathcal{H}}(0)e^{c\mathcal{K}_{\mathcal{H}}(0)}$ and $\mathcal{K}_{\mathcal{H}}(0)\leqslant c\widetilde{\mathcal{K}}(\mathcal{H})e^{c\widetilde{\mathcal{K}}(\mathcal{H})}$ with an absolute constant c.*

Proof. For every $\varepsilon>0$ define $\mathcal{H}_{(\varepsilon)}: t\mapsto\mathcal{H}(t)+\varepsilon\chi_{[0,\ell]}(t)I_2$, $t\in\mathbb{R}_+$, where $I_2=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ is the 2×2 identity matrix and $\chi_{[0,\ell]}$ denotes the characteristic function of $[0,\ell]$. Set $\xi_\varepsilon=\xi_{\mathcal{H}_{(\varepsilon)}}$, and let η_ε denote the inverse function to ξ_ε, so that $\eta_\varepsilon(\xi_\varepsilon(t))=t$ for all $t\geqslant 0$. Since $\xi_{\mathcal{H}_{(\varepsilon)}}$ maps $\mathbb{R}_+$ onto $\mathbb{R}_+$, the function η_ε is defined correctly. Moreover, we have $\det\mathcal{H}_{(\varepsilon)}>0$ almost everywhere on $\mathbb{R}_+$, hence η_ε is absolutely continuous on $\mathbb{R}_+$ and we can define the Hamiltonian $\widetilde{\mathcal{H}}_{(\varepsilon)}: t\mapsto\eta_\varepsilon'(t)\mathcal{H}_{(\varepsilon)}(\eta_\varepsilon(t))$. By construction, $\eta_\varepsilon'(t)=1/\sqrt{\det\mathcal{H}_{(\varepsilon)}(\eta_\varepsilon(t))}$ almost everywhere on $\mathbb{R}_+$, so the Hamiltonian $\widetilde{\mathcal{H}}_{(\varepsilon)}$ has determinant equal to one almost everywhere on $\mathbb{R}_+$. By Lemma 2.6, the spectral measures μ, $\mu_{(\varepsilon)}$, $\widetilde{\mu}_{(\varepsilon)}$ of $\mathcal{H}$, $\mathcal{H}_{(\varepsilon)}$, $\widetilde{\mathcal{H}}_{(\varepsilon)}$, respectively, belong to $\operatorname{Sz}(\mathbb{R})$. By Lemma 4.2,

$$\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(\varepsilon)})\leqslant c\mathcal{K}_{\widetilde{\mathcal{H}}_{(\varepsilon)}}(0)e^{c\mathcal{K}_{\widetilde{\mathcal{H}}_{(\varepsilon)}}(0)},\qquad \mathcal{K}_{\widetilde{\mathcal{H}}_{(\varepsilon)}}(0)\leqslant c\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(\varepsilon)})e^{c\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(\varepsilon)})},\tag{4.5}$$

for an absolute constant c. Let $h_{1,\varepsilon}$, $h_{2,\varepsilon}$, h_ε be defined by $\mathcal{H}_{(\varepsilon)}=\operatorname{diag}(h_{1,\varepsilon},h_{2,\varepsilon})$, $\widetilde{\mathcal{H}}_{(\varepsilon)}=\operatorname{diag}(h_\varepsilon,1/h_\varepsilon)$. Then, for every $t\geqslant 0$, we have

$$\int_{\eta_\varepsilon(t)}^{\eta_\varepsilon(t+2)} h_{1,\varepsilon}(s)\,ds\cdot\int_{\eta_\varepsilon(t)}^{\eta_\varepsilon(t+2)} h_{2,\varepsilon}(s)\,ds=\int_t^{t+2}h_\varepsilon(s)\,ds\cdot\int_t^{t+2}\frac{1}{h_\varepsilon(s)}\,ds,$$

by a change of variables. This shows that $\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(\varepsilon)})=\widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)})$. It is also not difficult to see that the spectral measures $\mu_{(\varepsilon)}$, $\widetilde{\mu}_{(\varepsilon)}$ of $\mathcal{H}_{(\varepsilon)}$, $\widetilde{\mathcal{H}}_{(\varepsilon)}$ coincide. Indeed, solutions $M_{(\varepsilon)}$, $\widetilde{M}_{(\varepsilon)}$ of Cauchy problem (1.2) for $\mathcal{H}_{(\varepsilon)}$, $\widetilde{\mathcal{H}}_{(\varepsilon)}$ satisfy $\widetilde{M}_{(\varepsilon)}(x)=M_{(\varepsilon)}(\eta_\varepsilon(x))$, $x\in\mathbb{R}_+$. Hence the limit in the right hand side of (2.1) defines the same harmonic function for $\mathcal{H}_{(\varepsilon)}$ and $\widetilde{\mathcal{H}}_{(\varepsilon)}$. Thus, from (4.5) we get

$$\widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)})\leqslant c\mathcal{K}_{\mathcal{H}_{(\varepsilon)}}(0)e^{c\mathcal{K}_{\mathcal{H}_{(\varepsilon)}}(0)},\qquad \mathcal{K}_{\mathcal{H}_{(\varepsilon)}}(0)\leqslant c\widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)})e^{c\widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)})},\tag{4.6}$$

for every $\varepsilon>0$. Next, by construction, we have $\xi_{\mathcal{H}_{(\varepsilon)}}(t)>\xi_{\mathcal{H}}(t)$ for all $t>0$ and $\varepsilon>0$. Moreover, the difference $\xi_{\mathcal{H}_{(\varepsilon)}}-\xi_{\mathcal{H}}$ tends to zero uniformly on $\mathbb{R}_+$ as ε tends to zero. Hence $\eta_\varepsilon(t)<\eta(t)$ for all $t>0$, $\varepsilon>0$ and $\eta(t)-\eta_\varepsilon(t)$ tends to zero for each $t\in\mathbb{R}_+$ as ε tends to zero. Since $\mathcal{H}$, $\mathcal{H}_{(\varepsilon)}$ are constant on $[\ell, +\infty)$, we have

$$0 = \int_{\eta_n}^{\eta_{n+2}} h_1(s)\,ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s)\,ds - 4,$$
$$0 = \int_{\eta_\varepsilon(n)}^{\eta_\varepsilon(n+2)} h_{1,\varepsilon}(s)\,ds \cdot \int_{\eta_\varepsilon(n)}^{\eta_\varepsilon(n+2)} h_{2,\varepsilon}(s)\,ds - 4,$$

for all $n \geqslant n_0$ and all sufficiently small $\varepsilon > 0$, where n_0 can be chosen independently of ε. Hence, the sums in (1.6) which define $\widetilde{\mathcal{K}}(\mathcal{H})$, $\widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)})$ contain at most n_0 nonzero terms for small $\varepsilon > 0$. It follows that $\lim_{\varepsilon\to 0} \widetilde{\mathcal{K}}(\mathcal{H}_{(\varepsilon)}) = \widetilde{\mathcal{K}}(\mathcal{H})$. It remains to show that $\lim_{\varepsilon\to 0} \mathcal{K}_{\mathcal{H}_{(\varepsilon)}}(0) = \mathcal{K}_{\mathcal{H}}(0)$. To do that, one can use formula (2.13) with $r = \ell$ for $\mathcal{H}$ and $\mathcal{H}_{(\varepsilon)}$. Since the matrix norm of $\mathcal{H} - \mathcal{H}_{(\varepsilon)}$ tends to zero uniformly on $[0, \ell]$ and $\mathcal{H} = \mathcal{H}_{(\varepsilon)}$ on $[\ell, +\infty)$, we have

$$\mathcal{Y}_{\mathcal{H}}(\ell) = \mathcal{Y}_{\mathcal{H}_{(\varepsilon)}}(\ell), \quad \lim_{\varepsilon\to 0} \xi_{\mathcal{H}_{(\varepsilon)}}(\ell) = \xi_{\mathcal{H}}(\ell), \quad \lim_{\varepsilon\to 0} |F_{\ell,\varepsilon}(i)| = |F_\ell(i)|. \tag{4.7}$$

To show that the last equality holds, we notice that the Hamiltonians $\mathcal{H}_\ell$ and $\mathcal{H}_{(\varepsilon)}(\cdot + \ell)$ coincide on $\mathbb{R}_+$ and thus have the same Weyl-Titchmarsh functions which we denote by m_ℓ. Hence, the corresponding functions $F_{\ell,\varepsilon} : z \mapsto \Theta^+_{(\varepsilon)}(l, z) + m_\ell(z)\Theta^-_{(\varepsilon)}(l, z)$ tend to F_ℓ uniformly on compact subsets of $\mathbb{C}^+$ as $\varepsilon \to 0$. From (4.7) and Lemma 2.5.(b) for $r = \ell$, we get $\lim_{\varepsilon\to 0} \mathcal{Y}_{\mathcal{H}_{(\varepsilon)}}(0) = \mathcal{Y}_{\mathcal{H}}(0)$. Using again formula (2.13) with $r = \ell$, we obtain $\lim_{\varepsilon\to 0} \mathcal{I}_{\mathcal{H}_{(\varepsilon)}}(0) = \mathcal{I}_{\mathcal{H}}(0)$. This completes the proof of the lemma. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{H}$ be a nontrivial singular diagonal Hamiltonian on $\mathbb{R}_+$ such that its spectral measure μ lies in the class $\mathrm{Sz}(\mathbb{R})$ and $b = 0$ in the Herglotz representation (1.4) of its Weyl-Tichmarsh function m. Note that we have $\mathcal{K}(\mu) = \mathcal{K}_{\mathcal{H}}(0)$ and no positive ε exists such that $(0, \varepsilon)$ is the indivisible interval for $\mathcal{H}$ of type $\pi/2$, see Lemma 2.3. Consider the family of Bernstein-Szegő Hamiltonians $\widehat{\mathcal{H}}_r = \mathrm{diag}(\widehat{h}_{1r}, \widehat{h}_{2r})$, $r \geqslant 0$, generated by $\mathcal{H}$ (see (2.4) for their definition). By Lemma 2.6, the spectral measure $\widehat{\mu}_r$ of $\widehat{\mathcal{H}}_r$ belongs to $\mathrm{Sz}(\mathbb{R})$ for every $r \geqslant 0$. Since the Hamiltonians $\widehat{\mathcal{H}}_r$ have no indivisible intervals $(0, \varepsilon)$ of type $\pi/2$, we have $\mathcal{K}(\widehat{\mu}_r) = \mathcal{K}_{\widehat{\mathcal{H}}_r}(0)$. From Lemma 2.5.$(e)$ we now get $\mathcal{K}(\widehat{\mu}_r) \leqslant \mathcal{K}(\mu)$. Let us first show that $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$. Since $2\sqrt{\det \mathcal{H}} \leqslant \operatorname{trace} \mathcal{H}$, the function $\sqrt{\det \mathcal{H}}$ is integrable on compact subsets of $\mathbb{R}_+$. Suppose that $\sqrt{\det \mathcal{H}} \in L^1(\mathbb{R}_+)$. Then the function $\xi_{\mathcal{H}}$ in (2.6) is bounded, hence there exists $n_0 \geqslant 0$ and $r_0 \geqslant \eta_{n_0} \geqslant 0$, such that for every $r \geqslant r_0$ the last *nonzero* term in the sum defining $\widetilde{\mathcal{K}}(\widehat{\mathcal{H}}_r)$ equals

$$c_{r,n_0} = \int_{\eta_{n_0}}^{\widehat{\eta}_{n_0+2}(r)} \widehat{h}_{1r}(s)\,ds \cdot \int_{\eta_{n_0}}^{\widehat{\eta}_{n_0+2}(r)} \widehat{h}_{2r}(s)\,ds - 4,$$

where $\eta_{n_0} = \min\{t \geqslant 0 : \xi_{\mathcal{H}}(t) = n_0\}$, and $\widehat{\eta}_{n_0+2}(r) = \min\{t \geqslant 0 : \xi_{\widehat{\mathcal{H}}_r}(t) = n_0 + 2\}$ increases infinitely with r. By Lemma 4.3 and Lemma 2.5.(e), we have $c_{r,n_0} \leqslant \widetilde{\mathcal{K}}(\widehat{\mathcal{H}}_r) \leqslant c\mathcal{K}(\widehat{\mu}_r)e^{c\mathcal{K}(\widehat{\mu}_r)} \leqslant c\mathcal{K}(\mu)e^{c\mathcal{K}(\mu)}$ for every r. From $\operatorname{trace} \mathcal{H} \notin L^1(\mathbb{R}_+)$ (recall that the Hamiltonian $\mathcal{H}$ is singular) and the uniform boundedness of c_{r,n_0}, $r \geqslant r_0$, we get

$$\int_{\eta_{n_0}}^{\infty} h_1(s)ds \int_{\eta_{n_0}}^{\infty} h_2(s)ds \leqslant \limsup_{r\to\infty} c_{r,n_0} + 4 < \infty, \qquad \int_0^\infty (h_1(s) + h_2(s))ds = \infty,$$

which implies that either $\int_{\eta_{n_0}}^\infty h_1(s)ds = 0$ or $\int_{\eta_{n_0}}^\infty h_2(s)ds = 0$. We see that ether $h_1 = 0$ or $h_2 = 0$ almost everywhere on $[r_0, +\infty)$ and the Hamiltonian $\mathcal{H}_{r_0}$ is trivial. The first part of the proof of

Lemma 2.5 shows that this is not the case, hence $\int_0^\infty \sqrt{\det \mathcal{H}(s)}\,ds = +\infty$[1] and the function η_x in the statement of Theorem 1 is correctly defined on $\mathbb{R}_+$. For every $r \geqslant \eta_2$ the first $[\xi_{\mathcal{H}}(r)] - 2$ terms defining $\widetilde{\mathcal{K}}(\mathcal{H})$ and $\widetilde{\mathcal{K}}(\widehat{\mathcal{H}}_r)$ in (1.6) are identical. Hence,

$$\widetilde{\mathcal{K}}(\mathcal{H}) \leqslant \limsup_{r\to\infty} \widetilde{\mathcal{K}}(\widehat{\mathcal{H}}_r) \leqslant \limsup_{r\to\infty} c\mathcal{K}(\widehat{\mu}_r)e^{c\mathcal{K}(\widehat{\mu}_r)} \leqslant c\mathcal{K}(\mu)e^{c\mathcal{K}(\mu)},$$

where the second and the third inequalities follow from Lemma 4.3 and Lemma 2.5.(e), respectively.

Conversely, suppose that $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ is a singular Hamiltonian on $\mathbb{R}_+$, $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$, and the sum defining $\widetilde{\mathcal{K}}(\mathcal{H})$ in (1.6) converges. For every integer $k \geqslant 0$, fix some positive constants a_{1k}, a_{2k} to be specified later, and consider

$$\widetilde{\mathcal{H}}_{(k)}(t) = \operatorname{diag}(h_{1k}, h_{2k}) = \begin{cases} \mathcal{H}(t) & \text{if } t \in [0, \eta_{k+2}], \\ \operatorname{diag}(a_{1k}, a_{2k}) & \text{if } t \in (\eta_{k+2}, +\infty). \end{cases}$$

For every $t > 0$, set $\widetilde{\eta}_t = \min\{s \geqslant 0 : \xi_{\mathcal{H}_{(k)}}(s) = t\}$, where $\xi_{\mathcal{H}_{(k)}}(s) = \int_0^s \sqrt{\det \mathcal{H}_{(k)}(\tau)}\,d\tau$. Then we have $\widetilde{\eta}_t = \eta_t$ for every $t \in [0, \eta_{k+2}]$. By construction,

$$\begin{aligned} \widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(k)}) = \sum_{n=0}^{k} \left(\int_{\eta_n}^{\eta_{n+2}} h_1(s)\,ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s)\,ds - 4 \right) \\ + \int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{1k}(s)\,ds \cdot \int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{2k}(s)\,ds - 4\,. \end{aligned} \tag{4.8}$$

Indeed, $\widetilde{\mathcal{H}}_{(k)}$ is constant on $[\eta_{k+2}, +\infty) = [\widetilde{\eta}_{k+2}, +\infty)$ and $\mathcal{H} = \widetilde{\mathcal{H}}_{(k)}$ on $[0, \eta_{n+2}]$, hence the terms with indexes $n \geqslant k+2$ in formula (1.6) for $\widetilde{\mathcal{H}}_{(k)}$ vanish, while the terms with indexes $n \leqslant k$ coincide with the corresponding terms in (1.6) for the Hamiltonian $\mathcal{H}$. Since $\widetilde{\mathcal{H}}_{(k)} = \operatorname{diag}(a_{1k}, a_{2k})$ on $[\eta_{k+2}, +\infty)$, we have

$$\int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{1k}\,ds \cdot \int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{2k}\,ds = \prod_{j=1}^{2} \left(\int_{\eta_{k+1}}^{\eta_{k+2}} h_j\,ds + a_{jk}(\widetilde{\eta}_{k+3} - \widetilde{\eta}_{k+2}) \right).$$

A short calculation gives $\widetilde{\eta}_{k+3} - \widetilde{\eta}_{k+2} = 1/\sqrt{a_{1k}a_{2k}}$. Thus, we have

$$\int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{1k}\,ds \cdot \int_{\widetilde{\eta}_{k+1}}^{\widetilde{\eta}_{k+3}} h_{2k}\,ds = \left(x_1 + \sqrt{\tfrac{a_{1k}}{a_{2k}}} \right) \left(x_2 + \sqrt{\tfrac{a_{2k}}{a_{1k}}} \right),$$

where $x_j = \int_{\eta_{k+1}}^{\eta_{k+2}} h_j\,ds$ for $j = 1, 2$. Denoting $y_j = \int_{\eta_{k+2}}^{\eta_{k+3}} h_j\,ds$, $j = 1, 2$, we get

$$\left(x_1 + \sqrt{\tfrac{a_{1k}}{a_{2k}}} \right) \left(x_2 + \sqrt{\tfrac{a_{2k}}{a_{1k}}} \right) \leqslant (x_1 + y_1)(x_2 + y_2) = \int_{\eta_{k+1}}^{\eta_{k+3}} h_1\,ds \cdot \int_{\eta_{k+1}}^{\eta_{k+3}} h_2\,ds, \tag{4.9}$$

for the following special choice of parameters a_{1k} and a_{2k}: $a_{1k} = y_1^2$, $a_{2k} = 1$, where the inequality in (4.9) follows from $y_1 y_2 \geqslant \left(\int_{\eta_{k+2}}^{\eta_{k+3}} \sqrt{h_1 h_2} ds\right)^2 = (\xi_{\mathcal{H}}(\eta_{k+3}) - \xi_{\mathcal{H}}(\eta_{k+2}))^2 = 1$. Combining (4.8) and (4.9), we see that $\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(k)}) \leqslant \widetilde{\mathcal{K}}(\mathcal{H})$ for every k and

$$\lim_{k\to\infty} \widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(k)}) = \widetilde{\mathcal{K}}(\mathcal{H}). \tag{4.10}$$

[1] There is a different way to prove this fact. One needs to check that the supremum of the function $\xi_{\mathcal{H}}$ in (2.6) determines the exponential type of the measure μ and then apply Krein-Wiener completeness theorem. See Section 6 in [31].

By Lemma 2.6, the spectral measure of the Hamiltonian $\widetilde{\mathcal{H}}_{(k)}$ belongs to $\mathrm{Sz}(\mathbb{R})$ for every k. From Lemma 4.1, Lemma 4.3, and (4.10) we obtain $\mu \in \mathrm{Sz}(\mathbb{R})$ and

$$\mathcal{K}(\mu) \leqslant \limsup_{k\to\infty} \mathcal{K}_{\widetilde{\mathcal{H}}_{(k)}}(0) \leqslant c \limsup_{r\to\infty} \widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(k)}) e^{c\widetilde{\mathcal{K}}(\widetilde{\mathcal{H}}_{(k)})} \leqslant c\widetilde{\mathcal{K}}(\mathcal{H}) e^{c\widetilde{\mathcal{K}}(\mathcal{H})},$$

with an absolute constant c. The theorem is proved. $\square$

5. Functions with summable fixed-scale Muckenhoupt characteristic

In this section, we study functions from the class $A_2(\mathbb{R}_+, \ell^1)$ defined in Section 4 and prove Propositions 4.1, 4.2.

Lemma 5.1. *Let $I = I^- \cup I^+$ be a splitting of an interval $I \subset \mathbb{R}$ into the union of two disjoint subintervals $I^{\pm}$. Let $h \geqslant 0$ be a function on I such that $h, 1/h \in L^1(I)$, and let $\gamma = \langle h\rangle_I \langle 1/h\rangle_I - 1$. Assume that $|I^-|/|I| \geqslant \frac{1}{5}$, then*

$$\left|\frac{\langle h\rangle_I}{\langle h\rangle_{I^-}} - 1\right| \lesssim \sqrt{\gamma(1+\gamma)}, \qquad \left|\frac{\langle h\rangle_{I^-}}{\langle h\rangle_I} - 1\right| \lesssim \min(1, \sqrt{\gamma}), \tag{5.1}$$

and, moreover,

$$\langle h\rangle_{I^-}\langle 1/h\rangle_{I^-} - 1 \lesssim \gamma. \tag{5.2}$$

Proof. The number γ and all bounds are invariant with respect to multiplying h with a positive constant, thus we can assume that $\langle h\rangle_I = 1$. Next, put $\upsilon = |I^-|/|I|$, $a^{\pm} = \langle h\rangle_{I^{\pm}}$, $b^{\pm} = \langle h^{-1}\rangle_{I^{\pm}}$. We have

$$\upsilon a^- + (1-\upsilon)a^+ = 1, \qquad \upsilon b^- + (1-\upsilon)b^+ = \langle h^{-1}\rangle_I = 1+\gamma, \qquad a^{\pm}b^{\pm} \geqslant 1. \tag{5.3}$$

Adding the first two estimates and using the bounds $1/a^{\pm} \leqslant b^{\pm}$, one gets $\upsilon\,(a^- + 1/a^-) + (1-\upsilon)\,(a^+ + 1/a^+) \leqslant 2+\gamma$. Since $x + 1/x \geqslant 2$ for all $x > 0$, this yields $\upsilon(a^- + 1/a^-) \leqslant 2\upsilon + \gamma$. Dividing by 2υ, we get the inequality

$$\frac{1}{2}\left(a^- + \frac{1}{a^-}\right) \leqslant 1 + \frac{\gamma}{2\upsilon}. \tag{5.4}$$

It can be rewritten in the form $(1/a^- - 1)^2 \leqslant \gamma/(\upsilon a^-)$. Since $\upsilon \in [\frac{1}{5}, 1]$ and $1/a^- \lesssim (1+\gamma)$ by (5.4), this gives the first bound in (5.1). To get the second bound in (5.1), rewrite (5.4) in the form $(a^- - 1)^2 \leqslant a^-\gamma/\upsilon$ and use the fact that $\upsilon a^- \leqslant 1$. Thus,

$$|a^- - 1| \leqslant \frac{\sqrt{\gamma}}{\upsilon}, \qquad |a^- - 1| \leqslant 1 + \upsilon^{-1},$$

which implies the second inequality in (5.1). Next, let us prove (5.2). Since $a^{\pm} + b^{\pm} \geqslant 2$, we get $\upsilon(a^- + b^-) \leqslant 2\upsilon + \gamma$ by summing up the first two identities in (5.3). Hence $\sqrt{a^-b^-} \leqslant 1 + \gamma/(2\upsilon)$ and $a^-b^- \leqslant 1 + \gamma/\upsilon + \gamma^2/(4\upsilon^2)$. This gives the inequality $\langle h\rangle_{I^-}\langle 1/h\rangle_{I^-} - 1 \lesssim \gamma$ in the case where $\gamma \leqslant \upsilon$. For $\gamma \geqslant \upsilon$ we can use (5.3) to get $a^- \leqslant 1/\upsilon \leqslant 5$ and $b^- \leqslant 5(1+\gamma)$. This gives $\langle h\rangle_{I^-}\langle 1/h\rangle_{I^-} - 1 \leqslant 25(1+\gamma) - 1 \lesssim \gamma$ since $\gamma \geqslant 1/5$. $\square$

Proof of Proposition 4.1. Apply Lemma 5.1 to the function h and the intervals $I = I_{n,\alpha_n}$, $I_- = [n, n+2]$, $n \geqslant 0$. Since $\{\alpha_n\} \subset [3,4]$, this will give the estimate $[h]_{2,\ell^1} \leqslant c[h,\alpha]$ with an absolute constant c. $\square$

Lemma 5.2. *For $h \in A_2(\mathbb{R}_+, \ell^1)$, define $\gamma_n = \langle h\rangle_{I_{n,2}}\langle h^{-1}\rangle_{I_{n,2}} - 1$ and $\theta_n = \langle h\rangle_{I_{n,1}}$. Then,*

$$(1+\gamma_n)^{-1} \lesssim \frac{\theta_{n+1}}{\theta_n} \lesssim 1 + \gamma_n, \tag{5.5}$$

$$\left|\frac{\theta_{n+1}}{\theta_n}-1\right| \leqslant c\sqrt{\gamma_n}, \quad \textit{if} \;\; \gamma_n \leqslant 1\,. \tag{5.6}$$

Moreover, we have $\|\widetilde{h}+\widetilde{h}^{-1}-2\|_1 \lesssim [h]_{2,\ell^1}=\sum_{n=0}^{\infty}\gamma_n$ *for the function* $\widetilde{h}$ *defined by*

$$\widetilde{h}(x)=h(x)/\langle h\rangle_{I_{n,1}}, \quad x\in I_{n,1}, \quad n\geqslant 0. \tag{5.7}$$

Proof. Represent θ_{n+1}/θ_n in the form

$$\frac{\theta_{n+1}}{\theta_n}=\frac{\langle h\rangle_{I_{n+1,1}}}{\langle h\rangle_{I_{n,2}}}\frac{\langle h\rangle_{I_{n,2}}}{\langle h\rangle_{I_{n,1}}}\,. \tag{5.8}$$

We write

$$\frac{1}{2}\leqslant \frac{\langle h\rangle_{I_{n,2}}}{\langle h\rangle_{I_{n,1}}}\leqslant 1+c\sqrt{\gamma_n(\gamma_n+1)}\lesssim 1+\gamma_n, \tag{5.9}$$

where the first inequality is immediate and the second one follows from the first estimate in (5.1). Similarly, we get

$$\frac{1}{2}\leqslant \frac{\langle h\rangle_{I_{n,2}}}{\langle h\rangle_{I_{n+1,1}}}\leqslant 1+c\sqrt{\gamma_n(\gamma_n+1)}\lesssim 1+\gamma_n$$

and

$$(1+\gamma_n)^{-1}\lesssim \frac{\langle h\rangle_{I_{n+1,1}}}{\langle h\rangle_{I_{n,2}}}\leqslant 2\,. \tag{5.10}$$

It is now sufficient to multiply (5.10) with (5.9) and substitute into (5.8) to get (5.5). Take $n\geqslant 0$ such that $\gamma_n\leqslant 1$. By Lemma 5.1, we have

$$\left|\frac{\langle h\rangle_{I_{n,2}}}{\langle h\rangle_{I_{n,1}}}-1\right|\lesssim\sqrt{\gamma_n}, \qquad \left|\frac{\langle h\rangle_{I_{n+1,1}}}{\langle h\rangle_{I_{n,2}}}-1\right|\lesssim\sqrt{\gamma_n}\,. \tag{5.11}$$

Substituting these bounds into (5.8) gives (5.6). Finally, observe that for every $n\geqslant 0$ we have $\langle h\rangle_{I_{n,1}}\langle h^{-1}\rangle_{I_{n,1}}-1\lesssim\gamma_n$ by (5.2). Using the identity

$$\sum_{n=0}^{\infty}\|\widetilde{h}+\widetilde{h}^{-1}-2\|_{L^1(I_{n,1})}=2\sum_{n=0}^{\infty}\left(\langle h\rangle_{I_{n,1}}\langle h^{-1}\rangle_{I_{n,1}}-1\right),$$

we complete the proof of the lemma. □

Remark. Notice that (5.5) and (5.6) imply

$$|\log(\theta_{n+1}/\theta_n)|\lesssim\begin{cases}\sqrt{\gamma_n}, & \gamma_n<2\,,\\ \log\gamma_n, & \gamma_n>2.\end{cases} \tag{5.12}$$

Proof of Proposition 4.2. Define $\widetilde{h}$ as in (5.7) and consider the function $f_1=(\widetilde{h}-1)\chi_{\frac{1}{2}<\widetilde{h}<\frac{3}{2}}$. Notice that $[h]_{2,\ell^1}=\sum_{n=0}^{\infty}\gamma_n$ where γ_n is defined in the previous lemma. Since the function $\widetilde{h}+\widetilde{h}^{-1}-2\in L^1(\mathbb{R}_+)$, we have $f_1\in L^2(\mathbb{R}_+)$ and $\|f_1\|_2^2\lesssim[h]_{2,\ell^1}$. Indeed, this follows from the fact that $x+x^{-1}-2\sim(x-1)^2$ for $x\in[\frac{1}{2},\frac{3}{2}]$ and the estimate $\|\widetilde{h}+\widetilde{h}^{-1}-2\|_1\lesssim[h]_{2,\ell^1}$ in Lemma 5.2. Similarly, the function $f_2=(\widetilde{h}-1)\chi_{|\widetilde{h}-1|\geqslant\frac{1}{2}}$ belongs to $L^1(\mathbb{R}_+)$ and $\|f_2\|_1\lesssim[h]_{2,\ell^1}$. Thus, we see that $\widetilde{h}$ can be represented in the form $\widetilde{h}=f_0+f_1+f_2$, where $f_0=1$, $f_1\in L^2(\mathbb{R}_+)$, $f_2\in L^1(\mathbb{R}_+)$, and $\|f_1\|_2^2+\|f_2\|_1\lesssim[h]_{2,\ell^1}$. Function $\widetilde{h}^{-1}$ admits similar representation $\widetilde{h}^{-1}=\widehat{f_0}+\widehat{f_1}+\widehat{f_2}$, where $\widehat{f_0}=1$, $\widehat{f_1}=-f_1$ and $\widehat{f_2}\in L^1(\mathbb{R}_+)$ is such that $\|\widehat{f_2}\|_1\lesssim[h]_{2,\ell^1}$. Notice that we have got $\widehat{f_1}=-f_1$ from

$$\frac{\chi_{|\widetilde{h}-1|<1/2}}{\widetilde{h}}=\frac{\chi_{|\widetilde{h}-1|<1/2}}{1+f_1}=\chi_{|\widetilde{h}-1|<1/2}(1-f_1+O(f_1^2))$$

and $\widehat{f}_2 \in L^1(\mathbb{R}_+)$ because $\widehat{f}_2 = \chi_{|\widetilde{h}-1|<1/2} O(f_1^2) + \chi_{|\widetilde{h}-1|>1/2}(\widetilde{h}^{-1} - 1) \in L^1(\mathbb{R}_+)$.

Let g_0 be the function on $\mathbb{R}_+$ such that $g_0 = \log \theta_n$ on each $I_{n,1}$, then $h = e^{g_0}\widetilde{h}$ on $\mathbb{R}_+$. Define also the function $g : x \mapsto g_0(x) - g_0(0)$ on $\mathbb{R}_+$. Then, for κ and κ_d from Proposition 3.2, we have

$$\kappa = \sum_{0\leqslant k,j\leqslant 2} p_{kj}, \qquad p_{kj} : x \mapsto \int_x^\infty \widehat{f}_k(x) f_j(\xi) e^{g(\xi)-g(x)+x-\xi}\, d\xi,$$

$$\kappa_d = \sum_{0\leqslant k,j\leqslant 2} p_{d,kj}, \qquad p_{d,kj} : x \mapsto \int_x^\infty f_k(x) \widehat{f}_j(\xi) e^{g(x)-g(\xi)+x-\xi}\, d\xi.$$

We will need some estimates for the function g. Let again γ_j, θ_j be defined as in Lemma 5.2 and let $v_n = \log\bigl(\theta_n/\theta_{n-1}\bigr), n \in \mathbb{N}$, $v_0 = 0$. Observe that $g(x) = \sum_{n=0}^{[x]} v_n$ on $\mathbb{R}_+$ by construction. Here, as usual, $[x]$ stands for the integer part of a number $x \in \mathbb{R}_+$. We can estimate

$$\|\{v_n\}\|_2^2 = \sum_{n:\,\gamma_{n-1}<2} v_n^2 + \sum_{n:\,\gamma_{n-1}\geqslant 2} v_n^2 \lesssim \sum_{n:\,\gamma_{n-1}<2} \gamma_n + \sum_{n:\,\gamma_{n-1}\geqslant 2} \log^2 \gamma_n \lesssim [h]_{2,\ell^1}\,, \tag{5.13}$$

where we used (5.12) and the trivial bound: $\log^2 \gamma \lesssim \gamma$ which holds for all $\gamma \geqslant 2$. Bound (5.12) also yields

$$\|\{v_n\}\|_\infty \lesssim \log(2 + [h]_{2,\ell^1})\,. \tag{5.14}$$

For $x < y$, we can apply (5.12) to write

$$\begin{aligned} |g(x) - g(y)| &\leqslant \left|\sum_{j=[x]}^{[y]} v_j\right| \leqslant \sum_{j=[x],\,\gamma_{j-1}<2}^{[y]} |v_j| + \sum_{j=[x],\,\gamma_{j-1}\geqslant 2}^{[y]} |v_j| \\ &\lesssim \sum_{j=[x],\,\gamma_{j-1}<2}^{[y]} \sqrt{|\gamma_{j-1}|} + \sum_{j=[x],\,\gamma_{j-1}\geqslant 2}^{[y]} \log \gamma_{j-1} \\ &\lesssim \Bigl((|x-y|+1)\sum_{j\geqslant 0}\gamma_j\Bigr)^{1/2} + \sum_{j\geqslant 0}\gamma_j \\ &\lesssim \sqrt{(|x-y|+1)[h]_{2,\ell^1}} + [h]_{2,\ell^1}\,. \end{aligned} \tag{5.15}$$

It follows that there is an absolute constant C such that for all $x, y \in \mathbb{R}_+$ we have

$$|g(x) - g(y)| \leqslant \tfrac{1}{2}|x-y| + C(1 + [h]_{2,\ell^1}). \tag{5.16}$$

Now, for indexes k, j such that $k+j \geqslant 2$, we can use (5.16) and the Young inequality for convolutions to estimate

$$\begin{aligned} \|p_{d,kj}\|_1 &\lesssim e^{C[h]_{2,\ell^1}} \int_0^\infty\!\!\int_0^\infty |f_k(x)|\chi_{\mathbb{R}_+}(\xi - x)e^{-(\xi-x)/2}|\widehat{f}_j(\xi)|\, d\xi\, dx \\ &\lesssim e^{C[h]_{2,\ell^1}} \|f_k\|_{q_k} \cdot \|\chi_{\mathbb{R}_+} e^{-x}\|_{r_{k,j}} \cdot \|\widehat{f}_j\|_{q_j} \lesssim [h]_{2,\ell^1} e^{C[h]_{2,\ell^1}}, \end{aligned}$$

where $q_0 = +\infty$, $q_1 = 2$, $q_2 = 1$, and the parameter $r_{k,j}$ is chosen so that $\frac{1}{q_k} + \frac{1}{r_{k,j}} + \frac{1}{q_j} = 2$. The estimate on p_{kj} for $k + j \geqslant 2$ is similar. To prove that $\kappa + \kappa_d - 2 \in L^1(\mathbb{R}_+)$, it remains to estimate the $L^1(\mathbb{R}^+)$–norms of functions

$$p_{00}+p_{d,00}-2=2\int_x^\infty e^{x-\xi}\left(\cosh G(x,\xi)-1\right)\,d\xi,$$
$$p_{01}+p_{d,01}=2\int_x^\infty \widehat{f}_1(\xi)e^{x-\xi}\sinh G(x,\xi)\,d\xi,$$
$$p_{10}+p_{d,10}=2\int_x^\infty f_1(x)e^{x-\xi}\sinh G(x,\xi)\,d\xi,$$

where $G(x,\xi)=g(x)-g(\xi)$. Let us define the function $\widetilde{g}$ on $[-1,\infty)$ to be continuous, linear on $I_{j,1}$ for each $j\geqslant -1$, and so that $\widetilde{g}(-1)=0$, $\widetilde{g}(j)=\sum_{n=0}^{j}|v_n|$ for $j\geqslant 0$. Clearly, $\widetilde{g}$ is non-decreasing on $[-1,\infty)$. Put $\widetilde{G}(x,\xi)=\widetilde{g}(\xi+1)-\widetilde{g}(x-1)$ for every $0<x<\xi$. Then $|G(x,\xi)|\leqslant \widetilde{G}(x,\xi)$ and so $\cosh G(x,\xi)\leqslant\cosh\widetilde{G}(x,\xi)$. By construction and (5.13), we have

$$\|\widetilde{g}'\|_2^2\lesssim\sum_{n\geqslant 0}|v_n|^2\lesssim[h]_{2,\ell^1}. \tag{5.17}$$

The bound (5.13) also implies

$$\|\widetilde{G}(x,x)\|_2^2\lesssim\|\{v_n\}\|_2^2\lesssim[h]_{2,\ell^1}\,. \tag{5.18}$$

The estimate (5.14) gives

$$\|\widetilde{G}(x,x)\|_\infty\lesssim\sup_{n\geqslant 0}|v_n|\lesssim\log(2+[h]_{2,\ell^1}) \tag{5.19}$$

and argument given in (5.15) yields

$$\widetilde{G}(x,\xi)\lesssim\sqrt{(|x-\xi|+1)[h]_{2,\ell^1}}+[h]_{2,\ell^1},\quad \widetilde{G}(x,\xi)\leqslant\tfrac{1}{2}|x-\xi|+C(1+[h]_{2,\ell^1}) \tag{5.20}$$

for all $x<\xi$. Integrate by parts to get

$$\begin{aligned}\|p_{00}+p_{d,00}-2\|_1&\leqslant 2\int_0^\infty\int_x^\infty e^{x-\xi}(\cosh\widetilde{G}(x,\xi)-1)\,d\xi\,dx\\&\leqslant 2\int_0^\infty\int_x^\infty\widetilde{g}'(\xi+1)e^{x-\xi}\sinh\widetilde{G}(x,\xi)\,d\xi\,dx+2R_1,\end{aligned}$$

where $R_1=\int_0^\infty(\cosh\widetilde{G}(x,x)-1)\,dx$. Using the inequality $\cosh t-1\lesssim t^2e^{|t|}$, we obtain $R_1\leqslant\|\widetilde{G}(x,x)\|_2^2\exp(\|\widetilde{G}(x,x)\|_\infty)\lesssim[h]_{2,\ell^1}e^{C[h]_{2,\ell^1}}$ by (5.18) and (5.19). To estimate the double integral, let us change the order of integration and integrate by parts once again:

$$\int_0^\infty\widetilde{g}'(\xi+1)\int_0^\xi e^{x-\xi}\sinh\widetilde{G}(x,\xi)\,dxd\xi=\int_0^\infty\widetilde{g}'(\xi+1)\int_0^\xi\widetilde{g}'(x-1)e^{x-\xi}\cosh\widetilde{G}(x,\xi)\,dx\,d\xi+R_2, \tag{5.21}$$

where $R_2=\int_0^\infty\widetilde{g}'(\xi+1)(\sinh\widetilde{G}(\xi,\xi)-e^{-\xi}\sinh\widetilde{G}(0,\xi))\,d\xi\leqslant\int_0^\infty\widetilde{g}'(\xi+1)\sinh\widetilde{G}(\xi,\xi)d\xi$ because $\widetilde{g}'\geqslant 0$. Let us estimate the integral first using the second bound in (5.20)

$$\begin{aligned}\int_0^\infty\widetilde{g}'(\xi+1)\int_0^\xi\widetilde{g}'(x-1)e^{x-\xi}\cosh\widetilde{G}(x,\xi)\,dx\,d\xi&\lesssim e^{C[h]_{2,\ell^1}}\int_0^\infty\widetilde{g}'(\xi+1)\int_0^\xi\widetilde{g}'(x-1)e^{(x-\xi)/2}\,dx\,d\xi\\&\lesssim e^{C[h]_{2,\ell^1}}\|\widetilde{g}'\|_2^2\lesssim[h]_{2,\ell^1}e^{C[h]_{2,\ell^1}},\end{aligned}$$

as follows from Young's inequality for convolution and (5.17). We are left with estimating R_2. Using inequality $|\sinh t|\leqslant|t|e^{|t|}$ we obtain

$$\int_0^\infty\widetilde{g}'(\xi+1)\sinh\widetilde{G}(\xi,\xi)d\xi\leqslant\|\widetilde{g}'(\xi+1)\|_2\cdot\|\widetilde{G}(\xi,\xi)\|_2\exp(\|\widetilde{G}(\xi,\xi)\|_\infty)\lesssim[h]_{2,\ell^1}e^{C[h]_{2,\ell^1}}\,.$$

Collecting the bounds, we get $\|p_{00}+p_{d,00}-2\|_1 \lesssim [h]_{2,\ell^1} e^{C[h]_{2,\ell^1}}$. It remains to bound the $L^1(\mathbb{R}_+)$–norms of $p_{01}+p_{d,01}$ and $p_{10}+p_{d,10}$. First, we write

$$\|p_{01}+p_{d,01}\|_1 \leqslant 2\int_0^\infty |\widehat{f}_1(\xi)| \Big|\int_0^\xi e^{x-\xi}\sinh \widetilde{G}(x,\xi)\,dx\,d\xi \lesssim [h]_{2,\ell^1} e^{C[h]_{2,\ell^1}}$$

since the integral has the form similar to the left hand side in (5.21) and the estimates for (5.21) can be repeated. Finally,

$$\begin{aligned}\|p_{10}+p_{d,10}\|_1 \leqslant & 2\int_0^\infty\int_x^\infty |f_1(x)| e^{x-\xi}\sinh\widetilde{G}(x,\xi)\,d\xi\,dx \\ \leqslant & 2\int_0^\infty |f_1(x)|\sinh\widetilde{G}(x,x)\,dx \\ & + 2\int_0^\infty\int_x^\infty |f_1(x)|\widetilde{g}'(\xi+1)e^{x-\xi}\cosh\widetilde{G}(x,\xi)\,d\xi\,dx,\end{aligned}$$

where the first term can be estimated similarly to R_2, while the second one is dominated by $Ce^{C[h]_{2,\ell^1}}\|f_1\|_2\cdot\|\widetilde{g}'(t-1)\|_2 \lesssim [h]_{2,\ell^1} e^{C[h]_{2,\ell^1}}$. Thus, we see that $\kappa+\kappa_d-2$ belongs to $L^1(\mathbb{R}_+)$ and $[h]_{int} \lesssim [h]_{2,\ell^1} e^{c[h]_{2,\ell^1}}$ with an absolute constant c. □

6. Krein strings and proof of Theorem 2

In this section, we introduce the spectral measure for Krein string and show how Theorem 1 and some results obtained in [20] imply Theorem 2. Let $0 < L \leqslant \infty$. Recall that M and L form $[M,L]$ pair if (1.8) holds, i.e., $L+\lim_{t\to L} M(t)=\infty$ and $\lim_{t\to L} M(t) > 0$. Define the Lebesgue–Stieltjes measure $\mathfrak{m}$ by $\mathfrak{m}[0,t]=M(t)$. Next, define the increasing function $N: t\mapsto t+M(t)$ on $[0,L)$ and let $\mathfrak{n}$ denote the corresponding measure, $\mathfrak{n}[0,t]=N(t)$ for $t\geqslant 0$. Define also the function $N^{(-1)}$ on $\mathbb{R}_+$ by $N^{(-1)}: y\mapsto \inf\{t\geqslant 0: N(t)\geqslant y\}$. The set under the last infimum is non-empty for every $y\geqslant 0$ because of the assumptions we made on M and L. Using the fact that N is strictly increasing, one can show that $N^{(-1)}$ is continuous on $\mathbb{R}_+$, and we have $N^{(-1)}(N(t))=t$ for every $t\in[0,L)$. Let M' be the density of the absolutely continuous part of $\mathfrak{m}$, so that $\mathfrak{m}=M'(t)\,dt+\mathfrak{m}_s$. Denote by E_s the support of the singular part $\mathfrak{m}_s$ of the measure $\mathfrak{m}$. Define two functions on $\mathbb{R}_+$,

$$h_1(x)=\begin{cases}0, & \text{if } N^{(-1)}(x)\in E_s,\\ \frac{1}{1+M'(N^{(-1)}(x))}, & \text{otherwise},\end{cases} \tag{6.1}$$

and

$$h_2(x)=\begin{cases}1, & \text{if } N^{(-1)}(x)\in E_s,\\ \frac{M'(N^{(-1)}(x))}{1+M'(N^{(-1)}(x))}, & \text{otherwise}.\end{cases} \tag{6.2}$$

The proof of Lemma 6.1 below shows that functions h_1, h_2 defined by different representatives of the function M' differ on a set of zero Lebesgue measure. Notice that h_1, h_2 are non-negative Lebesgue measurable functions and we have $h_1(x)+h_2(x)=1$ for all $x\in\mathbb{R}_+$. We are going to prove the following result from [20], pp. 1527–1528.

Lemma 6.1. *Formulas (6.1), (6.2) establish the bijection $[M,L]\mapsto \operatorname{diag}(h_1,h_2)$ between $[M,L]$ pairs and nontrivial diagonal Hamiltonians $\mathcal{H}=\operatorname{diag}(h_1,h_2)$ with unit trace almost everywhere on $\mathbb{R}_+$.*

Proof. Fix any pair $[M,L]$ and consider the corresponding function $N^{(-1)}$ and the measure $\mathfrak{n}$. For every function $f\in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathfrak{n})$ we have $f(N^{(-1)}(x))\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and, moreover,

$$\int_{[0,L)} f(t)\,d\mathfrak{n}(t)=\int_{\mathbb{R}_+} f(N^{(-1)}(x))\,dx, \tag{6.3}$$

if f is compactly supported in $[0, L)$. This result is known as the change of variables in the Lebesgue–Stieltjes integral (see, e.g., Exercise 5 in Section III.13 of [12]) but we give its proof for completeness. Without loss of generality we can assume that $f \geqslant 0$. Then (see, e.g., [20], Proposition 6.24), we have

$$\int_{[0,L)} f(t)\, d\mathfrak{n}(t) = \int_{\mathbb{R}_+} \Lambda_1(\lambda)\, d\lambda, \qquad \int_{[0,L)} f(N^{(-1)}(x))\, dx = \int_{\mathbb{R}_+} \Lambda_2(\lambda)\, d\lambda,$$

where $\Lambda_1(\lambda) = \mathfrak{n}\{t : f(t) > \lambda\}$ and $\Lambda_2(\lambda) = \left|\{x : f(N^{(-1)}(x)) > \lambda\}\right|$. For all $0 \leqslant a < b$ we have

$$\mathfrak{n}((a,b)) = N(b-) - N(a) = |(N(a), N(b-))|, \tag{6.4}$$

where $N(b-)$ denotes the left limit of N at the point b. In fact, $(N(a), N(b-))$ is preimage of (a, b) under the continuous map $N^{(-1)}$. Thus, the preimage under $N^{(-1)}$ of any open cover $\cup(a_j, b_j)$ for $\mathfrak{n}$-measurable set E will be an open cover for the set $\{x : N^{(-1)}(x) \in E\}$. Conversely, every open cover $\cup_j(c_j, d_j)$ for $\{x : N^{(-1)}(x) \in E\}$ is the preimage of some open cover for E. Indeed, for each j we get $(c_j, d_j) = (N(a_j), N(b_j))$, where a_j and b_j are points of continuity for N (to see this, note that the preimage of $\mathfrak{n}$'s atom under $N^{(-1)}$ is a closed segment). For every regular measure ν we have

$$\nu(E) = \inf\Big\{\sum_j \nu(I_j),\ E \subset \cup_j I_j,\ \{I_j\} \text{ are disjoint open intervals}\Big\}, \tag{6.5}$$

see, e.g., Lemma 1.17 in [15]. From (6.4) and (6.5) we now get $\Lambda_1(\lambda) = \Lambda_2(\lambda)$ and, consequently, relation (6.3) follows. Next, take a number $y \geqslant 0$. Since $h_1(x) = 0$ for all x such that $N^{(-1)}(x) \in E_s$, we have

$$\chi_{[0,y]}(x) h_1(x) = f_y(N^{(-1)}(x)), \quad x \in [0, L),$$

where $f_y : t \mapsto \frac{\chi_{[0,N^{(-1)}(y)]\setminus E_s}(t)}{1+M'(t)}$ is the compactly supported function from $L^1([0, L), \mathfrak{n})$. Applying formula (6.3) to the function f_y, we get

$$\int_0^y h_1(x)dx = \int_{[0,L)} \frac{\chi_{[0,N^{(-1)}(y)]\setminus E_s}(t)}{1 + M'(t)}\, d\mathfrak{n}(t) = \int_{[0,N^{(-1)}(y)]\setminus E_s} dt = N^{(-1)}(y), \tag{6.6}$$

where we used the fact that the singular part of $\mathfrak{n}$ is supported on E_s and the absolutely continuous part of $\mathfrak{n}$ has density $M' + 1$ with respect to the Lebesgue measure on $[0, L)$. If y is a point of growth for the function $N^{(-1)}$ (that is, there is no open interval I containing y such that $N^{(-1)}$ is constant on I), we have $\chi_{[0,y]}(x) = \chi_{[0,N^{(-1)}(y)]}(N^{(-1)}(x))$ for all $x \geqslant 0$, hence we can apply (6.3) to get

$$\int_0^y h_2(x)dx = \int_{[0,N^{(-1)}(y)]\setminus E_s} \frac{M'(t)(1 + M'(t))}{1 + M'(t)} dt + \int_{[0,N^{(-1)}(y)]\cap E_s} d\mathfrak{m}_s = \mathfrak{m}[0, N^{(-1)}(y)]. \tag{6.7}$$

From here we see that h_1, h_2 define M, L uniquely, in particular, these functions, as elements of $L^1_{\mathrm{loc}}(\mathbb{R}_+)$, do not depend on the choice of the representative of M'. Moreover, we cannot have $h_1 = 0$ or $h_2 = 0$ almost everywhere on $\mathbb{R}_+$ for any M, L satisfying (1.8). Hence, $[M, L] \mapsto \operatorname{diag}(h_1, h_2)$ is the injective mapping from a set of pairs $[M, L]$ to nontrivial diagonal Hamiltonians with unit trace. Now take a nontrivial Hamiltonian $\operatorname{diag}(h_1, h_2)$ with unit trace almost everywhere on $\mathbb{R}_+$, and consider the function

$$\Psi : y \mapsto \int_0^y h_1(x)\, dx.$$

Put $L = \sup_{y \geqslant 0} \Psi(y)$. Note that $|\Psi(y_1) - \Psi(y_2)| \leqslant |y_1 - y_2|$ for all y_1, y_2 in $\mathbb{R}_+$, hence there exists a measure $\mathfrak{m}$ on $[0, L)$ such that $\Psi(y) = \inf\{x \geqslant 0 : x + M(x) \geqslant y\}$ for every $y \geqslant 0$, where $M(x) = \mathfrak{m}[0, x]$. Using (6.6) and (6.7), it is easy to check that formulas (6.1), (6.2) for $[M, L]$ generate the singular Hamiltonian $\mathcal{H} = \operatorname{diag}(h_1, h_2)$ and it is nontrivial. The lemma is proved. $\square$

For any pair $[M, L]$, one can define the Krein string as the differential operator [13, 19]. In [20], the authors considered two functions $\varphi(x,z)$ and $\psi(x,z)$ that satisfy

$$\varphi(x,z) = 1 - z\int_{[0,x]}(x-s)\varphi(s,z)\,d\mathfrak{m}(s), \quad x \in [0,L)\,,$$

$$\psi(x,z) = x - z\int_{[0,x]}(x-s)\psi(s,z)\,d\mathfrak{m}(s), \quad x \in [0,L)\,.$$

These functions are uniquely determined by the string $[M, L]$ and they define the principal Weyl-Titchmarsh function q of $[M, L]$ by

$$q(z) = \lim_{x\to L}\frac{\psi(x,z)}{\varphi(x,z)}, \quad z \in \mathbb{C}\backslash[0,\infty),$$

see formula (2.21) in [20]. This function q has the unique integral representation

$$q(z) = b + \int_{\mathbb{R}_+}\frac{d\sigma(x)}{x-z}\,,$$

where $b \geqslant 0$ and σ, the spectral measure of the string $[M, L]$, is a measure on $\mathbb{R}_+ = [0, +\infty)$ satisfying condition

$$\int_{\mathbb{R}_+}\frac{d\sigma(x)}{1+x} < \infty\,.$$

The authors of [20] established, among other things, connection between q and the Weyl-Titchmarsh function of a canonical system. It is worth to mention that the definition of the Weyl-Titchmarsh function m we used in (1.3) was taken from [31]. The authors of [17], [20] deal with the canonical system written differently, i.e., they write the Cauchy problem

$$W'(t,z)J = zW(t,z)\mathcal{H}(t), \qquad W(0,z) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \qquad t \in \mathbb{R}_+, \quad z \in \mathbb{C},$$

and define the Weyl-Titchmarsh function Q^+ for $z \in \mathbb{C}\setminus\mathbb{R}$ by

$$Q^+(z) = \lim_{t\to+\infty}\frac{w_{11}(t,z)\tilde{\omega} + w_{12}(t,z)}{w_{21}(t,z)\tilde{\omega} + w_{22}(t,z)}, \qquad W(t,z) = \begin{pmatrix} w_{11}(t,z) & w_{12}(t,z)\\ w_{21}(t,z) & w_{22}(t,z)\end{pmatrix}. \tag{6.8}$$

It is not difficult to see that $W(t,z) = M(t,-z)^\top$ for the solution M of (1.2). If we let $\sigma_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and denote by M_{σ_1} the solution of Cauchy problem $JM'_{\sigma_1} = z\mathcal{H}_{\sigma_1}M_{\sigma_1}$, $M_{\sigma_1}(0,z) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ for the dual Hamiltonian $\mathcal{H}^d = \mathcal{H}_{\sigma_1} = \sigma_1\mathcal{H}\sigma_1$, then the function m_{σ_1} from formula (1.3) for $\mathcal{H}_{\sigma_1}$ will coincide with the function Q^+ in (6.8) for $\mathcal{H}$ and $\tilde{\omega} = 1/\omega$. Indeed, we have

$$M_{\sigma_1}(t,z) = \sigma_1 M(t,-z)\sigma_1 = \sigma_1 W(t,z)^\top\sigma_1 = \begin{pmatrix} w_{22}(t,z) & w_{12}(t,z)\\ w_{21}(t,z) & w_{11}(t,z)\end{pmatrix}. \tag{6.9}$$

We will need the following lemma from [20].

Lemma 6.2. *Suppose* $[M, L] \mapsto \operatorname{diag}(h_1, h_2)$ *is the bijection given by* (6.1) *and* (6.2), *q is the Weyl-Titchmarsh function for the string given by $[M, L]$, and m, m_{σ_1} are the Weyl-Titchmarsh functions for* $\operatorname{diag}(h_1, h_2)$ *and* $\operatorname{diag}(h_2, h_1)$, *respectively. Then, we have*

$$zq(z^2) = m_{\sigma_1}(z) = -m^{-1}(z), \quad z \in \mathbb{C}^+\,. \tag{6.10}$$

Proof. In [20], formula (4.20), it is proved that

$$Q^+(z) = zq(z^2), \quad z \in \mathbb{C}^+\,, \tag{6.11}$$

where Q^+ is defined in (6.8) and $\mathcal{H}$ is obtained from $[M, L]$ by bijection discussed in Lemma 6.1. On the other hand, $Q^+(z) = m_{\sigma_1}(z) = m^{-1}(-z) = -m^{-1}(z)$, where the first equality follows from discussion right before formula (6.9), the second one follows from (6.9) and (1.3), and the last one is the corollary of the spectral measure of $\mathrm{diag}(h_1, h_2)$ being even. $\square$

Proof of Theorem 2. Let $[M, L]$ be a string with Weyl-Titchmarsh function q and the spectral measure σ. Using Lemma 6.1, define the Hamiltonians $\mathcal{H}$ and $\mathcal{H}^d = \mathcal{H}_{\sigma_1} = \sigma_1\mathcal{H}\sigma_1$ on $\mathbb{R}_+$. Let m_{σ_1}, $\mu_{\sigma_1} = w_{\sigma_1}\,dx + \mu_{\sigma_1,s}$ be the Weyl-Titchmarsh function and the spectral measure of $\mathcal{H}^d$. Recall that $\sigma = v\,dx + \sigma_s$ for spectral measure of the string. In (6.10), taking the nontangential limits of $\mathrm{Im}(m_{\sigma_1}(z))$ and $\mathrm{Im}(zq(z^2))$ as $z \to x$, we get $w_{\sigma_1}(x)$ and $xv(x^2)$ for almost all $x \in \mathbb{R}_+$, respectively. Thus, $w_{\sigma_1}(x) = xv(x^2)$ for almost every $x \geqslant 0$, and, since μ_{σ_1} is even by Lemma 2.2, we get

$$\int_{\mathbb{R}} \frac{\log w_{\sigma_1}(x)}{1+x^2}\,dx = 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + 2\int_0^\infty \frac{\log v(x^2)}{x^2+1}\,dx = \int_0^\infty \frac{\log v(x)}{\sqrt{x}(x+1)}\,dx,$$

where we used the fact that $\int_0^\infty \frac{\log x}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{y}{e^y+e^{-y}}\,dy = 0$. This implies that $\int_0^\infty \frac{\log v(x)}{\sqrt{x}(x+1)}\,dx$ is finite if and only if $\mu_{\sigma_1} \in \mathrm{Sz}(\mathbb{R})$. On the other hand, formula (6.3) and the defintion of h_1, h_2 imply

$$\int_0^y \sqrt{h_1(x)h_2(x)}\,dx = \int_{[0,N^{(-1)}(y)]\setminus E_s} \frac{\sqrt{M'(t)}}{1+M'(t)}\,d\mathfrak{n}(t) = \int_0^{N^{(-1)}(y)} \sqrt{M'(t)}\,dt$$

if y is a point of growth of the function $N^{(-1)}$. For every $n \geqslant 1$ the points $\{\eta_n\}$ defined in (1.5) are the points of growth for $N^{(-1)}$. Indeed, this is clear from the formula (6.6) that was proved for all $y \geqslant 0$. Hence we have $t_n = N^{(-1)}(\eta_n)$ for all $n \geqslant 0$. It follows that

$$t_{n+2} - t_n = N^{(-1)}(\eta_{n+2}) - N^{(-1)}(\eta_n) = \int_{\eta_n}^{\eta_{n+2}} h_1(x)\,dx,$$

where we used (6.6) again. We also have

$$M(t_{n+2}) - M(t_n) = \mathfrak{m}(t_n, t_{n+2}] = \mathfrak{m}(N^{(-1)}(\eta_n), N^{(-1)}(\eta_{n+2})] = \int_{\eta_n}^{\eta_{n+2}} h_2(x)\,dx,$$

by the definition of M and (6.7). Thus, $\widetilde{\mathcal{K}}[M, L] = \widetilde{\mathcal{K}}(\mathcal{H}) = \widetilde{\mathcal{K}}(\mathcal{H}_{\sigma_1})$ and $\sqrt{\det \mathcal{H}} \in L^1(\mathbb{R}_+)$ if and only if $\sqrt{M'} \in L^1(\mathbb{R}_+)$. Now the result follows from Theorem 1. $\square$

Remark. If $[M, L] \mapsto \mathrm{diag}(h_1, h_2)$, then the string $[M_d, L_d]$ for which $[M_d, L_d] \mapsto \mathrm{diag}(h_2, h_1)$ is called the dual string. One can easily see that $\widetilde{\mathcal{K}}[M, L] = \widetilde{\mathcal{K}}[M_d, L_d]$ so the logarithmic integral for the string converges if and only it converges for the dual string.

We give two applications of Theorem 2.

Proposition 6.1. *Suppose that the mass distribution M of a string $[M, \infty]$ satisfies $M' = 1$ almost everywhere on $\mathbb{R}_+$. Let $\mathfrak{m}_s$ be the singular measure on $\mathbb{R}_+$ such that $M(t) = t + \mathfrak{m}_s[0, t]$ for all $t \geqslant 0$. Then we have*

$$\int_0^\infty \frac{\log v(x)}{\sqrt{x}(x+1)}\,dx > -\infty$$

for the spectral measure $\sigma = v\,dx + \sigma_s$ of $[M, \infty]$ if and only if $\mathfrak{m}_s(\mathbb{R}_+) < \infty$.

Proof. For given M, we have $t_n = n$ and $M(t_{n+2}) - M(t_n) = 2 + \mathfrak{m}_s(n, n+2]$, hence

$$\widetilde{\mathcal{K}}[M, \infty] = \sum_{n\geqslant 0}(2 \cdot (2 + \mathfrak{m}_s(n, n+2]) - 4) = 2\sum_{n\geqslant 0} \mathfrak{m}_s(n, n+2].$$

It remains to use Theorem 2. $\square$

The next result shows that logarithmic integral can converge even if $\mathfrak{m}_s(\mathbb{R}_+) = \infty$.

Proposition 6.2. *There exists a string $[M, L]$ with $L < \infty$ and $\mathfrak{m}_s[0, L) = +\infty$ such that*

$$\int_0^\infty \frac{\log v(x)}{\sqrt{x}(x+1)}\, dx > -\infty$$

for its spectral measure $\sigma = v\, dx + \sigma_s$.

Proof. Consider any sequence $\{\varepsilon_n\} \subset (-1, 1)$, and define $\delta_{t_n} = \prod_{j=0}^{n}(1+\varepsilon_j)$, $t_0 = 0, t_n = \sum_{j=0}^{n-1} \delta_{t_j}$ for integer $n \geqslant 0$, and let $L = \sup_{n \geqslant 0} t_n$. Consider the function

$$M'(t) = M_n = (\delta_{t_n})^{-2}, \qquad t \in [t_n, t_{n+1}], \qquad n \geqslant 0.$$

Define the measure $\mathfrak{m}$ by $\mathfrak{m} = M' dt + \mathfrak{m}_s$, where $\mathfrak{m}_s$ is some singular measure, and let $M(t) = \mathfrak{m}[0, t]$ for $t \geqslant 0$. Then, the condition (1.9) for $[M, L]$ is satisfied if and only if

$$\left\{(\delta_{t_n} + \delta_{t_{n+1}})\left(\frac{1}{\delta_{t_n}} + \frac{1}{\delta_{t_{n+1}}}\right) - 4\right\} \in \ell^1 \tag{6.12}$$

and

$$\left\{(\delta_{t_n} + \delta_{t_{n+1}})(\Delta\mathfrak{m}_s)_n\right\} \in \ell^1, \tag{6.13}$$

where $(\Delta\mathfrak{m}_s)_n = \mathfrak{m}_s(t_n, t_{n+2}]$ for $n \geqslant 0$. Condition (6.12) is satisfied if and only if

$$\left\{(1+\varepsilon_n) + (1+\varepsilon_n)^{-1} - 2\right\} \in \ell^1,$$

or, equivalently, $\{\varepsilon_n\} \in \ell^2$. If we choose $\varepsilon_n = -(n+1)^{-\alpha}, \alpha \in (\frac{1}{2}, 1)$, then $\sum_{n=1}^{\infty}(t_{n+2} - t_n) < \infty$ and we have $L < \infty$. Condition (6.13) in that case can be satisfied even if $\sum_n (\Delta\mathfrak{m}_s)_n$ diverges, that is, $\mathfrak{m}_s[0, L) = \infty$. For instance, we can take a singular measure $\mathfrak{m}_s$ such that $(\Delta\mathfrak{m}_s)_n = 1$ for all integers $n \geqslant 0$. $\square$

7. Appendix

Proof of Lemma 2.1. Differentiate the function $M : r \mapsto \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) - zJ\int_0^t \mathcal{H}(\tau) d\tau$ and use the fact that the solution to Cauchy problem (1.2) is unique. $\square$

Proof of Lemma 2.2. Put $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ and $M_{\sigma_1} = \sigma_1 M \sigma_1$, where M is the solution of (1.2). Using identity $\sigma_1 \mathcal{H} \sigma_1 = J^*\mathcal{H}J = \mathcal{H}_d$ and $J\sigma_1 = -\sigma_1 J$, it is easy to check that $JM'_{\sigma_1} = -z\mathcal{H}_d M_{\sigma_1}$. It follows that $M_{\sigma_1}(t, z) = M^d(t, -z)$ for all $t \geqslant 0$, $z \in \mathbb{C}$. Using (2.16), we get

$$\begin{pmatrix}\Phi^-(t,z) & \Theta^-(t,z)\\ \Phi^+(t,z) & \Theta^+(t,z)\end{pmatrix} = \begin{pmatrix}\Phi^-(t,-z) & -\Theta^-(t,-z)\\ -\Phi^+(t,-z) & \Theta^+(t,-z)\end{pmatrix}$$

for all $t \geqslant 0$ and $z \in \mathbb{C}$. From (1.3), one has $m(z) = -m(-z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$, hence

$$\frac{1}{\pi}\int_{\mathbb{R}_+} \frac{\operatorname{Im} z}{|x-z|^2}\, d\mu(x) + b \operatorname{Im} z = \frac{1}{\pi}\int_{\mathbb{R}_+} \frac{\operatorname{Im} z}{|x+z|^2}\, d\mu(x) + b \operatorname{Im} z, \qquad z \in \mathbb{C}^+.$$

This implies that μ is even. Using $m(i+1) = -m(-i-1)$, we conclude that $a = 0$.

Conversely, suppose that μ is even and $a = 0$. The approximation procedure in Section 9 of [31] gives a sequence of even measures μ_N supported at finitely many points such that the corresponding Hamiltonians, $\mathcal{H}_N$, constructed in Theorem 7 of [31] are diagonal and $\lim_{N\to\infty}\left\|\int_0^t (\mathcal{H}_N(s) - \mathcal{H}(s))\, ds\right\| = 0$ for every $t \geqslant 0$. It follows that $\mathcal{H}$ is diagonal, as required. $\square$

Proof of Lemma 2.3. Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ such that $(0, \varepsilon)$ is the indivisible interval of type $\pi/2$ for some $\varepsilon > 0$. Then, for all $z \in \mathbb{C}^+$, we have

$$m(z) = \frac{\Phi^+(\varepsilon, z) + m_\varepsilon(z)\Phi^-(\varepsilon, z)}{\Theta^+(\varepsilon, z) + m_\varepsilon(z)\Theta^-(\varepsilon, z)} = z\int_0^\varepsilon \langle \mathcal{H}(t)\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)\rangle\, dt + m_\varepsilon(z), \tag{7.1}$$

by formula (2.13) for $r = \varepsilon$ and Lemma 2.1. So, we have $b \geqslant \int_0^\varepsilon \langle \mathcal{H}(t)\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)\rangle\, dt$ in this situation.

Conversely, assume that $b > 0$ in (1.4). Consider a Hamiltonian $\mathcal{H}_{(b)}$ whose Weyl-Titchmarsh function $m_{\mathcal{H}_{(b)}}$ coincides with $m - bz$. Define

$$\widetilde{\mathcal{H}}(x) = \begin{cases} \operatorname{diag}(0,1), & x \in [0,b], \\ \mathcal{H}_{(b)}(x-b), & x > b. \end{cases}$$

Let $m_{\widetilde{\mathcal{H}}}$ denote the Weyl-Titchmarsh function of $\widetilde{\mathcal{H}}$. Then, a variant of (7.1) for $\widetilde{\mathcal{H}}$, $\varepsilon = b$, gives

$$m_{\widetilde{\mathcal{H}}} = bz + m_{\mathcal{H}_{(b)}} = bz + m - bz = m.$$

Thus, the Weyl-Titchmarsh functions of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ coincide. It follows from de Branges theorem formulated in the Introduction that the Hamiltonians $\mathcal{H}$, $\widetilde{\mathcal{H}}$ are equivalent. Hence, there is an absolutely continuous strictly increasing function $\eta \geqslant 0$ such that $\widetilde{\mathcal{H}}(t) = \eta'(t)\mathcal{H}(\eta(t))$ almost everywhere on $\mathbb{R}_+$. In particular, the interval $(0, \eta(b))$ is indivisible of type $\pi/2$ for $\mathcal{H}$. It follows that for $\varepsilon = \eta(b)$ we have

$$b = \int_0^b \operatorname{trace} \widetilde{\mathcal{H}}(t)\, dt = \int_0^{\eta(b)} \operatorname{trace} \mathcal{H}(s)\, ds = \int_0^{\varepsilon} \langle \mathcal{H}(s) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\rangle\, ds,$$

completing the proof of the lemma. $\square$

Proof of Lemma 2.4. The matrix-function

$$M(t,z) = \begin{pmatrix} \cos(t\sqrt{a_1 a_2} z) & \sqrt{a_2/a_1}\sin(t\sqrt{a_1 a_2} z) \\ -\sqrt{a_1/a_2}\sin(t\sqrt{a_1 a_2} z) & \cos(t\sqrt{a_1 a_2} z) \end{pmatrix}$$

solves Cauchy problem (1.2) for $\mathcal{H} = \operatorname{diag}(a_1, a_2)$. It follows from (1.3) that the Weyl-Titchmarsh function of $\mathcal{H}$ is given by $m(z) = i\sqrt{a_2/a_1}$ for all $z \in \mathbb{C}^+$. Taking imaginary part, we get $w_r(x) = \sqrt{a_2/a_1}$, $x \in \mathbb{R}$, and $\log \mathcal{I}_{\mathcal{H}}(r) = \mathcal{Y}_{\mathcal{H}}(r) = \log \sqrt{a_2/a_1}$ for all $r \geqslant 0$, as required. $\square$

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Roman Bessonov: bessonov@pdmi.ras.ru

St. Petersburg State University
Universitetskaya nab. 7/9, 199034 St. Petersburg, RUSSIA

St. Petersburg Department of Steklov Mathematical Institute
Russian Academy of Sciences
Fontanka 27, 191023 St.Petersburg, RUSSIA

Sergey Denisov: denissov@wisc.edu

University of Wisconsin–Madison
Department of Mathematics
480 Lincoln Dr., Madison, WI, 53706, USA

Keldysh Institute of Applied Mathematics
Russian Academy of Sciences
Miusskaya pl. 4, 125047 Moscow, RUSSIA