JACOBI MATRICES ON TREES GENERATED BY ANGELESCO SYSTEMS: ASYMPTOTICS OF COEFFICIENTS AND ESSENTIAL SPECTRUM

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Abstract. We continue studying the connection between Jacobi matrices defined on a tree and multiple orthogonal polynomials (MOPs) that was discovered in [8]. In this paper, we consider Angelesco systems formed by two analytic weights and obtain asymptotics of the recurrence coefficients and strong asymptotics of MOPs along all directions (including the marginal ones). These results are then applied to show that the essential spectrum of the related Jacobi matrix is the union of intervals of orthogonality.

Contents

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1. Introduction

It is well-known [1] that the spectral theory of one-sided self-adjoint Jacobi matrices can be naturally studied in the context of orthogonal polynomials on the real line and, conversely, many results in the latter topic find an operator-theoretic interpretation. In [8], we discovered that a wide class of multiple orthogonal polynomials (MOPs), e.g., celebrated Angelesco systems, is connected to self-adjoint Jacobi matrices defined on rooted Cayley trees. The present paper makes a further step in this direction. We perform a case study of Angelesco systems with two measures of orthogonality given by analytic weights. Our analysis of the related matrix Riemann-Hilbert problem provides the asymptotics of the recurrence coefficients and strong asymptotics of MOPs for all large indices. One application of this precise asymptotic analysis is a characterization of the essential spectrum of the associated Jacobi matrix.

We start this introduction by recalling some definitions and main relations connecting Jacobi matrices on trees and MOPs and then state the main results of the paper. In what follows, we let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geqslant 0} := \{0, 1, 2, \ldots\}$. We write $|\vec{n}| := n_1 + \cdots + n_d$ for $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$, and let $\vec{e}_1 = (1, 0, \ldots, 0), \ldots, \vec{e}_d = (0, \ldots, 0, 1)$, $\vec{1} = (1, \ldots, 1) = \vec{e}_1 + \cdots + \vec{e}_d$. Given an operator $\mathcal{A}$ in the Banach space, the symbols $\sigma(\mathcal{A})$ and $\sigma_{\text{ess}}(\mathcal{A})$ will denote its spectrum and essential spectrum, respectively [41]. In a metric space, $B_r(X)$ denotes the closed ball with center at X and radius r. For a complex number z, $\Re z$ and $\Im z$ are its real and imaginary parts, respectively. For a function $f(z)$, holomorphic in $\mathbb{C}^+$, the upper half-plane, its boundary values on $\mathbb{R}$ are denoted by $f_+(x)$.

1.1. Jacobi matrices on trees.

Denote by $\mathcal{T}$ an infinite $(d+1)$-homogeneous rooted tree (rooted Cayley tree) and by $\mathcal{V}$ the set of its vertices with O being the root. On the lattice $\mathbb{N}^d$, consider an infinite path $\{\vec{n}^{(1)}, \vec{n}^{(2)}, \ldots\}$ that starts at $\vec{1}$ (i.e., $\vec{n}^{(1)} = \vec{1}$) and satisfies $\vec{n}^{(j+1)} = \vec{n}^{(j)} + \vec{e}_{k_j}$, $k_j \in \{1, \ldots, d\}$ for every $j = 0, 1, \ldots$. Clearly, these are paths for which, as we move from $\vec{1}$ to infinity, the multi-index of each next vertex is increasing by 1 at exactly one position. Each such path can be mapped to non-selfintersecting path in $\mathcal{T}$ that starts at O (see Figure 1 for $d = 2$) and this map is one-to-one. This construction defines a projection $\Pi : \mathcal{V} \to \mathbb{N}^d$ as follows: given $X \in \mathcal{V}$ we consider a path from O to X, map it to a path on $\mathbb{N}^d$ and let $\Pi(X)$ be the endpoint of the mapped path. Every vertex $Y \in \mathcal{V}, Y \neq O$, has the unique parent, which we denote by $Y_{(p)}$. This allows us to define the following index function:

$$\iota : \mathcal{V} \to \{1, \ldots, d\}, \quad Y \mapsto \iota_Y \text{ such that } \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\iota_Y}, \tag{1.1}$$

and therefore to distinguish the "children" of each vertex $Y \in \mathcal{V}$ by denoting $Z = Y_{(ch),\iota_Z}$ when $Y = Z_{(p)}$, see Figure 1 (for $d = 2$).

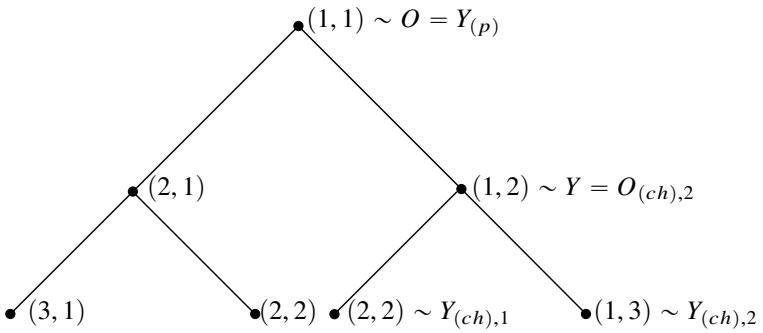

Figure 1. Three generations of $\mathcal{T}$ (for $d = 2$).

Let $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n}\in\mathbb{Z}^d_{\geqslant 0},\, i\in\{1,\ldots,d\}}$ be a collection of real parameters satisfying conditions

$$\begin{cases} 0 < a_{\vec{n},i} \text{ for all } \vec{n} \in \mathbb{N}^d, \quad i \in \{1, \ldots, d\}, \\ \sup\limits_{\vec{n}\in\mathbb{N}^d, i\in\{1,\ldots,d\}} a_{\vec{n},i} < \infty, \quad \sup\limits_{\vec{n}\in\mathbb{Z}^d_{\geqslant 0}, i\in\{1\ldots,d\}} |b_{\vec{n},i}| < \infty. \end{cases} \tag{1.2}$$

For a function f on $\mathcal{V}$, we denote by f_Y its value at a vertex $Y \in \mathcal{V}$. Given P satisfying (1.2) and $\vec{\kappa} \in \mathbb{R}^d$ with $|\vec{\kappa}| = 1$, we define the corresponding Jacobi operator $\mathcal{J}_{\vec{\kappa}}$ by

$$\begin{cases} (\mathcal{J}_{\vec{\kappa}} f)_Y := a^{1/2}_{\Pi(Y_{(p)}),\iota_Y} f_{Y_{(p)}} + b_{\Pi(Y_{(p)}),\iota_Y} f_Y + \sum_{i=1}^d a^{1/2}_{\Pi(Y),i} f_{Y_{(ch),i}}, & Y \neq O, \\ (\mathcal{J}_{\vec{\kappa}} f)_O := \sum_{i=1}^d \kappa_i b_{\vec{1}-\vec{e}_i,i} f_O + \sum_{i=1}^d a^{1/2}_{\vec{1},i} f_{O_{(ch),i}}, & Y = O. \end{cases} \tag{1.3}$$

Thus defined operator $\mathcal{J}_{\vec{\kappa}}$ is bounded and self-adjoint on $\ell^2(\mathcal{V})$.

The spectral theory of Jacobi matrices on trees enjoyed considerable progress in the last decade, see, e.g., [3, 14, 19, 22, 23, 25, 30, 31, 32, 33]. In this paper, we study Jacobi matrices on trees that are generated by multiple orthogonality conditions. For this class of Jacobi matrices, one can study subtle questions of spectral analysis, such as the spatial asymptotics of Green's function, by employing the powerful asymptotical methods developed in the context of multiple orthogonal polynomials (see, e.g, formulas (4.30) and (4.31) in [8]). In the current work, we focus on characterizing the so-called $\mathcal{R}$-limits and on detecting the essential spectrum in the case, when the multiple orthogonal polynomials are given by the Angelesco system with analytic weights.

1.2. **Multiple orthogonal polynomials and recurrence relations.** In [8], we investigated properties of the operator $\mathcal{J}_{\vec{\kappa}}$ in the case when the coefficients P are the recurrence coefficients for MOPs. We now recall some basic facts about multiple orthogonal polynomials.

Let $\vec{\mu} := (\mu_1, \dots, \mu_d)$, $d \in \mathbb{N}$, be a vector of positive finite Borel measures defined on $\mathbb{R}$ and $\vec{n}$ be a given a multi-index in $\mathbb{Z}^d_{\geqslant 0}$, $|\vec{n}| \geqslant 1$. Type I MOPs $\{A^{(j)}_{\vec{n}}\}_{j=1}^d$ are not identically zero polynomial coefficients of the linear form

$$Q_{\vec{n}}(x) := \sum_{j=1}^d A^{(j)}_{\vec{n}}(x) \mathrm{d}\mu_j(x), \quad \deg A^{(i)}_{\vec{n}} < n_i, \quad i \in \{1, \dots, d\},$$

defined by the conditions

$$\int x^l Q_{\vec{n}}(x) = 0, \quad l < |\vec{n}| - 1, \quad A^{(i)}_{\vec{1}-\vec{e}_i} \equiv 0. \tag{1.4}$$

Type II MOPs $P_{\vec{n}}(x)$, $\deg\left(P_{\vec{n}}\right) \leqslant |\vec{n}|$, are not identically zero and defined by

$$\int P_{\vec{n}}(x) x^l \mathrm{d}\mu_i(x) = 0, \quad l < n_i, \quad i \in \{1, \dots, d\}. \tag{1.5}$$

The polynomials of both types always exist, but their uniqueness is not guaranteed. If $\deg(P_{\vec{n}}) = |\vec{n}|$ for every non-identically zero polynomial $P_{\vec{n}}(x)$ satisfying (1.5), then the multi-index $\vec{n}$ is called normal. In this case $P_{\vec{n}}(x)$ is unique up to a multiplicative factor and we normalize it to be monic, i.e., $P_{\vec{n}}(x) = x^{|\vec{n}|} + \cdots$. It turns out that $\vec{n}$ is normal if and only if the linear form $Q_{\vec{n}}(x)$ is defined uniquely up to multiplication by a constant. In this case, we will normalize it by

$$\int x^{|\vec{n}|-1} Q_{\vec{n}}(x) = 1\,. \tag{1.6}$$

We will say that a vector $\vec{\mu}$ is called *perfect* if all the multi-indices $\vec{n} \in \mathbb{Z}^d_{\geqslant 0}$ are normal.

When $\vec{\mu}$ is perfect, it is known [43] that the polynomials $P_{\vec{n}}(x)$ and the forms $Q_{\vec{n}}(x)$ satisfy the following nearest-neighbor recurrence relations (NNRRs):

$$\begin{cases} zP_{\vec{n}}(z) = P_{\vec{n}+\vec{e}_j}(z) + b_{\vec{n},j} P_{\vec{n}}(z) + \sum_{i=1}^d a_{\vec{n},i} P_{\vec{n}-\vec{e}_i}(z), \\ zQ_{\vec{n}}(z) = Q_{\vec{n}-\vec{e}_j}(z) + b_{\vec{n}-\vec{e}_j,j} Q_{\vec{n}}(z) + \sum_{i=1}^d a_{\vec{n},i} Q_{\vec{n}+\vec{e}_i}(z), \end{cases} \quad \text{for each } j \in \{1, \dots, d\}. \tag{1.7}$$

For the coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$, we have representations [8]:

$$a_{\vec{n},j} = \frac{\int P_{\vec{n}}(x)\, x^{n_j} \mathrm{d}\mu_j(x)}{\int P_{\vec{n}-\vec{e}_j}(x)\, x^{n_j-1} \mathrm{d}\mu_j(x)}, \qquad b_{\vec{n}-\vec{e}_j,j} = \int x^{|\vec{n}|} Q_{\vec{n}}(x) - \int x^{|\vec{n}|-1} Q_{\vec{n}-\vec{e}_j}(x)\,. \tag{1.8}$$

If $d > 1$, unlike in one-dimensional case, we can not prescribe $\{a_{\vec{n},j}\}$ and $\{b_{\vec{n},j}\}$ arbitrarily. In fact, these coefficients satisfy the so-called "consistency conditions" which is a system of nonlinear difference equations. This discrete integrable system and the associated Lax pair were studied in [9, 43].

1.3. **Angelesco systems and ray limits of NNRR coefficients.** We recall that $\vec{\mu}$ is an *Angelesco* system of measures if

$$\operatorname{supp}\mu_j = \Delta_j := [\alpha_j, \beta_j]: \qquad \Delta_i \cap \Delta_j = \varnothing, \quad i \neq j, \quad i, j \in \{1, \dots, d\}, \tag{1.9}$$

i.e., the supports of measures form a system of d closed segments separated by $d-1$ nonempty open intervals. We can always assume without loss of generality that $\beta_j < \alpha_{j+1}$, $j \in \{1, \dots, d-1\}$.

Angelesco systems form an important subclass of the perfect systems. They were studied by Angelesco already in 1919, [4]. It is not difficult to see [8] that the corresponding NNRR coefficients satisfy conditions (1.2) and thus define the Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$ by (1.3).

The asymptotic behavior of these coefficients $\{a_{\vec{n},j}, b_{\vec{n},j}\}$ for the *ray sequences regime*, namely when

$$\mathcal{N}_{\vec{c}} = \{\vec{n}\}: \qquad n_i = c_i|\vec{n}| + o(\vec{n}), \quad i \in \{1, \dots, d\}, \qquad |\vec{c}\,| := \sum_{i=1}^{d} c_i = 1, \tag{1.10}$$

was studied in [8] for $\vec{c} = (c_1, \dots, c_d) \in (0,1)^d$ (hereafter, $\lim_{\mathcal{N}_{\vec{c}}}$ stands for the limit as $|\vec{n}| \to \infty$, $\vec{n} \in \mathcal{N}_{\vec{c}}$). The following theorem was proved.

Theorem 1.1 ([8]). *Let $\vec{\mu}$ be an Angelesco system (1.9) such that for each $i \in \{1, \dots, d\}$ the measure μ_i is absolutely continuous with respect to the Lebesgue measure on Δ_i and the density $\mu_i'(x) := \mathrm{d}\mu_i(x)/\mathrm{d}x$ extends to a holomorphic and non-vanishing function in some neighborhood of Δ_i. Then the ray limits (1.10) of coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ from (1.7) exist for any $\vec{c} \in (0,1)^d$:*

$$\lim_{\mathcal{N}_{\vec{c}}} a_{\vec{n},i} = A_{\vec{c},i} \quad \text{and} \quad \lim_{\mathcal{N}_{\vec{c}}} b_{\vec{n},i} = B_{\vec{c},i}, \quad i \in \{1, \dots, d\}. \tag{1.11}$$

This result and expressions for $A_{\vec{c},i}$ and $B_{\vec{c},i}$ were obtained from the strong asymptotics of the MOPs also established in [8] (along the ray $\vec{c} = (1/d, \dots, 1/d)$ the limits in (1.11) can be deduced from the results in [10]). As it happens, the numbers $A_{\vec{c},i}$ and $B_{\vec{c},i}$ depend only on the vector $\vec{c}$ and the intervals $\{\Delta_i\}_{i=1}^d$ (see (2.5) for the case $d = 2$ where $\vec{c} = (c, 1-c)$ and $A_{\vec{c},i} = A_{c,i}$, $B_{\vec{c},i} = B_{c,i}$).

1.4. **Results and structure of the paper.** In this paper, we restrict ourselves to the case $d = 2$. Our main technical achievement is an extension of the results in [8] on the strong asymptotics of the Angelesco MOPs to the full range of $\vec{c}$: $\vec{c} \in [0,1]^2$. As a corollary of this extension, we get the following result.

Theorem 1.2. *Let $\vec{\mu}$ be as in Theorem 1.1 with $d = 2$. Then the ray limits*

$$\lim_{\mathcal{N}_c} a_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim_{\mathcal{N}_c} b_{\vec{n},i} = B_{c,i} \tag{1.12}$$

exist for any $c \in [0,1]$ and $i \in \{1,2\}$, where $\mathcal{N}_c := \mathcal{N}_{(c,1-c)}$ is any sequence satisfying (1.10).

Theorem 1.2 can be used to characterize the essential spectrum of the Jacobi operator $\mathcal{J}_{\vec{\kappa}}$, defined in (1.3), generated by an Angelesco system.

Definition. *Let $\mathsf{P} := \{\widehat{a}_{\vec{n},i}, \widehat{b}_{\vec{n},i}\}_{\vec{n} \in \mathbb{Z}^2_{\geqslant 0},\, i=1,2}$ be a set of real numbers that satisfy (1.2) for $d = 2$ and the constants $\{A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}\}_{c \in [0,1]}$ be limits from (1.12) (notice that P does not have to be a set of the recurrence coefficients of any Angelesco system, but the limits $\{A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}\}_{c \in [0,1]}$ are generated by some Δ_1 and Δ_2). We say that $\mathsf{P} \in \mathsf{P}_{Ang}(\Delta_1, \Delta_2)$ if P satisfies*

$$\lim_{\mathcal{N}_c} \widehat{a}_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim_{\mathcal{N}_c} \widehat{b}_{\vec{n},i} = B_{c,i} \tag{1.13}$$

for any $c \in [0,1]$ and $i \in \{1,2\}$, where, again, $\mathcal{N}_c := \mathcal{N}_{(c,1-c)}$ is any sequence satisfying (1.10).

By Theorem 1.2, the class $\mathsf{P}_{Ang}(\Delta_1, \Delta_2)$ is not empty since the recurrence coefficients of any Angelesco system with analytic weights supported on Δ_1 and Δ_2 belong in $\mathsf{P}_{Ang}(\Delta_1, \Delta_2)$. Consider Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$ defined in (1.3) with coefficients in $\mathsf{P}_{Ang}(\Delta_1, \Delta_2)$. The following result gives characterization of its essential spectrum.

Theorem 1.3. *Let $\mathcal{J}_{\vec{\kappa}}$ be the Jacobi operator defined by (1.3) and corresponding to a collection of parameters $\mathsf{P} \in \mathsf{P}_{Ang}(\Delta_1, \Delta_2)$, then $\sigma_{\mathrm{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$. In particular, the essential spectrum of the Jacobi matrix generated by an Angelesco system with analytic weights supported on Δ_1 and Δ_2 is $\Delta_1 \cup \Delta_2$.*

We prove this theorem in Section 2. The necessary definitions and statements of the main results on strong asymptotics of MOPs are adduced in Section 3. Auxiliary results and their proofs are relegated to Sections 4 and 5. Proofs of the main results can be found in Sections 6 and 7.

2. Expressions for the ray limits and proof of Theorem 1.3

2.1. **Expressions for the ray limits.** In this subsection we give formulas for the limits in (1.12).

Let $\Delta_1 = [\alpha_1, \beta_1]$ and $\Delta_2 = [\alpha_2, \beta_2]$ be two intervals on the real line such that $\beta_1 < \alpha_2$. Denote by ω_1 and ω_2 the arcsine distributions on Δ_1 and Δ_2, respectively. It is known [40] that

$$E(\omega_i, \omega_i) \leqslant E(\nu, \nu), \quad E(\mu, \nu) := -\int \log|x-y| \mathrm{d}\mu(x)\mathrm{d}\nu(y),$$

for any probability Borel ν measure on Δ_i. The logarithmic potentials of these measures satisfy

$$\ell_i - V^{\omega_i} \equiv 0 \quad \text{on} \quad \Delta_i,$$

for some constants ℓ_1 and ℓ_2, where $V^{\nu}(z) := -\int \log|z-x| \mathrm{d}\nu(x)$. Now, given $c \in (0,1)$, define

$$M_c := \big\{(\nu_1, \nu_2) : \ \mathrm{supp}(\nu_i) \subseteq \Delta_i, \ \|\nu_1\| = c, \ \|\nu_2\| = 1-c\big\}. \tag{2.1}$$

It is known [26] that there exists the unique pair of measures $(\omega_{c,1}, \omega_{c,2}) \in M_c$ such that

$$I(\omega_{c,1}, \omega_{c,2}) \leqslant I(\nu_1, \nu_2), \quad I(\nu_1, \nu_2) := 2E(\nu_1, \nu_1) + 2E(\nu_2, \nu_2) + E(\nu_1, \nu_2) + E(\nu_2, \nu_1), \tag{2.2}$$

for all pairs $(\nu_1, \nu_2) \in M_c$. Moreover, $\mathrm{supp}(\omega_{c,1}) = [\alpha_1, \beta_{c,1}] =: \Delta_{c,1}$ and $\mathrm{supp}(\omega_{c,2}) = [\alpha_{c,2}, \beta_2] =: \Delta_{c,2}$. Similarly to the case of a single interval, there exist constants $\ell_{c,i}$, $i \in \{1,2\}$, such that

$$\begin{cases} \ell_{c,1} - V^{2\omega_{c,1}+\omega_{c,2}} \equiv 0 & \text{on} \quad \mathrm{supp}(\omega_{c,1}), \\ \ell_{c,2} - V^{\omega_{c,1}+2\omega_{c,2}} \equiv 0 & \text{on} \quad \mathrm{supp}(\omega_{c,2}). \end{cases} \tag{2.3}$$

The dependence of the intervals $\Delta_{c,i}$ on the parameter c is described in greater detail in Section 4.

Let $\mathfrak{R}_c$, $c \in (0,1)$, be a 3-sheeted Riemann surface realized as follows: cut a copy of $\overline{\mathbb{C}}$ along $\Delta_{c,1} \cup \Delta_{c,2}$, which henceforth is denoted by $\mathfrak{R}_c^{(0)}$, the second copy of $\overline{\mathbb{C}}$ is cut along $\Delta_{c,1}$ and is denoted by $\mathfrak{R}_c^{(1)}$, while the third copy is cut along $\Delta_{c,2}$ and is denoted by $\mathfrak{R}_c^{(2)}$. These copies are then glued to each other crosswise along the corresponding cuts, see Figure 2. It can be easily verified that thus constructed Riemann surface has genus

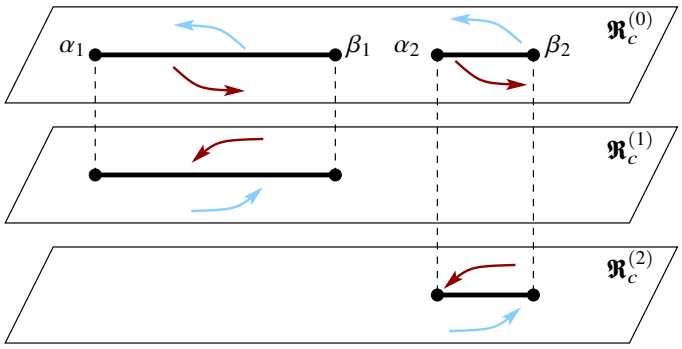

Figure 2. Surface $\mathfrak{R}_c$ when $\beta_{c,1} = \beta_1$ and $\alpha_{c,2} = \alpha_2$.

0. We denote by π the natural projection from $\mathfrak{R}_c$ to $\overline{\mathbb{C}}$ and employ the notation z for a generic point on $\mathfrak{R}_c$ with $\pi(z) = z$ as well as $z^{(i)}$ for a point on $\mathfrak{R}_c^{(i)}$ with $\pi(z^{(i)}) = z$. We call a linear combination $\sum n_i z_i$, $n_i \in \mathbb{Z}$, a *divisor*. The degree of $\sum n_i z_i$ is defined as $\sum n_i$. We say that $\sum n_i z_i$ is a zero/pole divisor of a rational function on $\mathfrak{R}_c$ if this function has a zero at z_i of multiplicity n_i when $n_i > 0$, a pole at z_i of order $-n_i$ when $n_i < 0$, and has no other zeros of poles. Zero/pole divisors necessarily have degree zero. Since $\mathfrak{R}_c$ has genus zero, one can arbitrarily prescribe zero/pole divisors of rational functions on $\mathfrak{R}_c$ as long as the degree of the divisor is zero. A rational function with a given divisor is unique up to multiplication by a constant.

Proposition 2.1. *Let $\mathfrak{R}_c$, $c \in (0,1)$, be as above and $\chi_c(z)$ be the conformal map of $\mathfrak{R}_c$ onto $\overline{\mathbb{C}}$ such that*

$$\chi_c\big(z^{(0)}\big) = z + O\big(z^{-1}\big) \quad \text{as} \quad z \to \infty. \tag{2.4}$$

Further, let the numbers $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ be defined by

$$\chi_c\big(z^{(i)}\big) =: B_{c,i} + A_{c,i} z^{-1} + O\big(z^{-2}\big) \quad \text{as} \quad z \to \infty, \ i \in \{1,2\}. \tag{2.5}$$

Finally, let $w_i(z) := \sqrt{(z-\alpha_i)(z-\beta_i)}$ *be the branch holomorphic outside of* Δ_i *and normalized so that* $w_i(z)/z \to 1$ *as* $z \to \infty$*, in which case*

$$\varphi_i(z) := \frac{1}{2}\left(z - \frac{\beta_i+\alpha_i}{2} + w_i(z)\right) \tag{2.6}$$

is the conformal map of $\overline{\mathbb{C}}\backslash\Delta_i$ *onto the complement of the disk* $B_{(\beta_i-\alpha_i)/4}(0)$ *satisfying* $\varphi_i(z) = z + O(1)$ *as* $z \to \infty$*. Then it holds that*

$$\lim_{c\to 0}\begin{cases} A_{c,2} &= \left[(\beta_2-\alpha_2)/4\right]^2 &=: A_{0,2},\\ B_{c,2} &= (\beta_2+\alpha_2)/2 &=: B_{0,2},\\ A_{c,1} &= 0 &=: A_{0,1},\\ B_{c,1} &= B_{0,2}+\varphi_2(\alpha_1) &=: B_{0,1},\end{cases} \tag{2.7}$$

and analogous limits hold when $c \to 1$*. Moreover, all the constants* $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ *are continuous functions of the parameter* $c \in [0,1]$*.*

Let us stress that the numbers $A_{c,i}$ and $B_{c,i}$ defined in (2.5) are precisely the ones appearing in (1.12). Even though the expression for $B_{0,1}$ might seem strange, it has a meaning from the point of view of spectral theory of Jacobi matrices, see (A.8).

We prove Proposition 2.1 in Section 5. It is worth noting that the constants $A_{c,1}$ and $A_{c,2}$ are always positive (except for $A_{0,1}$ and $A_{1,2}$, of course). Indeed, denote by $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_{c,1}, \boldsymbol{\alpha}_{c,2}, \boldsymbol{\beta}_2$ the ramification points of $\mathfrak{R}_c$ with natural projections $\alpha_1, \beta_{c,1}, \alpha_{c,2}, \beta_2$, respectively. Then the symmetries of $\mathfrak{R}_c$ and $\chi_c(z)$ yield that $\chi_c(z)$ is real and changes from $-\infty$ to ∞ when z moves along the cycle

$$\infty^{(0)} \to \boldsymbol{\alpha}_1 \to \infty^{(1)} \to \boldsymbol{\beta}_{c,1} \to \boldsymbol{\alpha}_{c,2} \to \infty^{(2)} \to \boldsymbol{\beta}_2 \to \infty^{(0)}$$

whose natural projection is the extended real line. Thus, $\chi_c(z)$ is increasing when it moves past $\infty^{(1)}$ and $\infty^{(2)}$, which yields the claim (this argument also shows that $B_{c,1} < B_{c,2}$).

2.2. **Proof of Theorem 1.3.** Our proof will be based on a characterization of the essential support of a Jacobi matrix on a tree obtained in [13, Theorem 4]. We need some preliminaries to formulate this result. Suppose $\mathcal{T}$ is a 3-homogeneous rooted tree with root at O (a binary tree), which means that O has two neighbors and any other vertex has three neighbors. Later in the text, we will use the notation $Z \sim Y$ to indicate that vertices Z and Y are neighbors and the symbol $\mathcal{V}$ will denote the set of all vertices of $\mathcal{T}$. Given a real function V defined on $\mathcal{V}$ and a real positive function W defined on all edges, we make an assumption

$$\sup_{Y\in\mathcal{V}} |V_Y| < \infty, \;\; 0 < W_{Z,Y}, \quad \sup_{Z\sim Y, Y\in\mathcal{V}} W_{Z,Y} < \infty\,, \tag{2.8}$$

to introduce $\mathcal{J}$, a bounded self-adjoint Jacobi matrix

$$(\mathcal{J}f)_Y := V_Y f_Y + \sum_{Z\sim Y} W_{Z,Y}^{1/2} f_Z, \quad Y \in \mathcal{V}\,, \tag{2.9}$$

defined on $\ell^2(\mathcal{V})$. One example one can think of is $\mathcal{J}_{\vec{k}}$ introduced in (1.3). Consider a set of distinct vertices (a path) $\{Y_n\}$, $n \in \mathbb{N}$, in $\mathcal{V}$ such that $Y_n \sim Y_{n+1}$ for every n. Clearly, every such path on the tree escapes to infinity, i.e., $\operatorname{dist}(O, Y_n) \to \infty, n \to \infty$. We want to define $\mathcal{R}$-limit (or "right limit") of $\mathcal{J}$ along this path. To do that, suppose $\mathcal{G}$ is a 3-homogeneous tree (without a root), O' is a fixed vertex on $\mathcal{G}$, and $\mathcal{J}'$ is a bounded self-adjoint operator on $\mathcal{G}$. Recall that $B_r(Y)$ stands for the ball of radius r centered at Y and denote the restriction operator to this ball by $P_{B_r(Y)}$. Consider two finite matrices: $P_{B_r(Y_{n_j})}\mathcal{J}P_{B_r(Y_{n_j})}$ and $P_{B_r(O')}\mathcal{J}'P_{B_r(O')}$. If we identify $\ell^2(B_r(O'))$ and $\ell^2(B_r(Y_{n_j}))$ by following the structure of the tree (and there are many ways to do that), then these matrices are defined on the same finite dimensional Euclidean space. If this identification can be done so that all sections of $\mathcal{J}'$ appear as the limits, we call $\mathcal{J}'$ an $\mathcal{R}$-limit or right limit:

Definition. *We say that* $\mathcal{J}'$ *is an* $\mathcal{R}$*-limit of* $\mathcal{J}$ *along* $\{Y_n\}$ *if there is a subsequence* $\{n_j\}$ *such that*

$$P_{B_r(Y_{n_j})}\mathcal{J}P_{B_r(Y_{n_j})} \to P_{B_r(O')}\mathcal{J}'P_{B_r(O')} \quad \text{as} \quad j \to \infty$$

for every fixed $r \in \mathbb{N}$*. Matrix* $\mathcal{J}'$ *is called simply an* $\mathcal{R}$*-limit of* $\mathcal{J}$ *if there exists a path along which* $\mathcal{J}'$ *is an* $\mathcal{R}$*-limit of* $\mathcal{J}$*.*

Remark. For the rigorous definition of $\mathcal{R}$-limit on more general graphs, see [13].

Theorem 2.1. ([13, Theorem 4]) *We have*

$$\sigma_{\mathrm{ess}}(\mathcal{J}) = \overline{\bigcup_{\mathcal{J}' \text{ is an } \mathcal{R}-\text{ limit of } \mathcal{J}} \sigma(\mathcal{J}')}\,.$$

Remark. [13, Theorem 4] was stated for the regular trees only, but the proof is valid for rooted trees as well.

Auxiliary operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$. Recall that $\mathcal{T}$ denotes the 3-homogeneous rooted tree with the root denoted by O and $\mathcal{V}$ stands for the set of all its vertices. There are two edges meeting at the root O. We label one of them type 1 and the other one – type 2. Now, consider two vertices that are at distance 1 from O. Each of them is coincident with exactly three edges. One of the edges for each vertex was labelled already, and we label the remaining two as an edge type 1 and an edge of type 2. We continue inductively by considering all edges that are at distance $2, 3$, etc. from O and calling one of the unlabelled edges type 1 and the other one type 2. Now that all edges of $\mathcal{T}$ have types assigned to them, we continue by labeling the vertices. If a vertex Y meets two edges of type 1 and one edge of type 2, we call it a vertex of type 1; otherwise, if it is incident with two edges of type 2 and one edge of type 1, we call it type 2. We do not need to assign any type to the root O. At a vertex $Y \neq O$ of type ι_Y, see (1.1), we define both operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ by the same formula:

$$(\mathcal{L}_c^{(l)}\psi)_Y = \sum_{j\in\{1,2\}, Y'\sim Y,\, \text{type of edge } (Y,Y')=j} \sqrt{A_{c,j}}\psi_{Y'} + B_{c,\iota_Y}\psi_Y, \quad l \in \{1,2\}; \tag{2.10}$$

and at the root O we define the operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ differently from each other by

$$(\mathcal{L}_c^{(l)}\psi)_O = \sum_{j\in\{1,2\}, Y'\sim O,\, \text{type of edge } (O,Y')=j} \sqrt{A_{c,j}}\psi_{Y'} + B_{c,l}\psi_O\,, \quad l \in \{1,2\}.$$

Notice that these operators represent Jacobi matrices on $\mathcal{T}$ when $c \in (0,1)$. However, if $c \in \{0,1\}$ either $A_{c,1}$ or $A_{c,2}$ becomes zero and $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$ are no longer Jacobi matrices, strictly speaking.

Remark. The operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ already appeared in [8] as the strong limits of Jacobi matrices on finite trees that correspond to $\{P_{\vec{n}}\}$, the polynomials of the second type (see formula (3.3) and Subsection 4.5 in [8]). We defined $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ by assigning the "types" to vertices of the tree and then defining the Jacobi matrix accordingly. This is an example of more general construction that generates trees satisfying a finite cone type condition. The Laplacian defined on trees with finite cone type and its perturbations were studied in, e.g., [31, 32, 33].

Lemma 2.1. *If $\mathcal{J}$ has coefficients in $\mathsf{P}_{Ang}(\Delta_1,\Delta_2)$, then the $\mathcal{R}$-limits of $\mathcal{J}$ and the $\mathcal{R}$-limits of $\mathcal{L}_c^{(l)}$, $l \in \{1,2\}$, are related by the following identity*

$$\bigcup_{c\in[0,1]} \left\{\mathcal{J}' : \mathcal{J}' \text{ is an } \mathcal{R}-\text{limit of } \mathcal{L}_c^{(l)}\right\} = \{\mathcal{J}'' : \mathcal{J}'' \text{ is an } \mathcal{R}-\text{limit of } \mathcal{J}\}\,. \tag{2.11}$$

Proof. This follows from the definition of the $\mathcal{R}$-limit, construction of $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$, and from the assumption (1.13). □

We further study auxiliary operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ in Appendix A.

Proof of Theorem 1.3. Assumptions (1.13) characterize the behavior of the coefficients at infinity. Thus, Weyl's theorem on the essential spectrum [41] implies that any two Jacobi matrices with parameters in $\mathsf{P}_{Ang}(\Delta_1,\Delta_2)$ have the same essential spectra. Moreover, by the same Weyl's theorem, this essential spectrum is independent of the choice of parameter $\vec{\kappa}$ in (1.3). Hence, it is enough to prove the theorem for the Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$ generated by some Angelesco system with analytic weights and with $\vec{\kappa} = \vec{e}_2$. In [8, Section 4] we established that $\Delta_1 \cup \Delta_2 \subseteq \sigma(\mathcal{J}_{\vec{e}_2})$. Thus, $\Delta_1 \cup \Delta_2 \subseteq \sigma_{\mathrm{ess}}(\mathcal{J}_{\vec{e}_2})$ as follows from the definition of the essential spectrum.

To prove the opposite inclusion, take any $\mathcal{J}$ for which the coefficients belong to $\mathsf{P}_{Ang}(\Delta_1,\Delta_2)$. The application of Theorem 2.1 and Theorem A.1 to $\mathcal{L}_c^{(1)}$ gives

$$\overline{\bigcup_{\mathcal{J}' \text{ is an } \mathcal{R}-\text{ limit of } \mathcal{L}_c^{(1)}} \sigma(\mathcal{J}')} = \sigma_{\mathrm{ess}}(\mathcal{L}_c^{(1)}) = \Delta_{c,1} \cup \Delta_{c,2},$$

which yields an inclusion

$$\bigcup_{c\in[0,1]} \bigcup_{\mathcal{J}' \text{ is an } \mathcal{R}-\text{ limit of } \mathcal{L}_c^{(1)}} \sigma(\mathcal{J}') \subseteq \bigcup_{c\in[0,1]} \left(\Delta_{c,1} \cup \Delta_{c,2}\right) = \Delta_1 \cup \Delta_2\,, \tag{2.12}$$

where the last equality follows from the properties of $\Delta_{c,1}$ and $\Delta_{c,2}$ (which we also discuss later in Proposition 4.1). Moreover, since

$$\sigma_{\rm ess}(\mathcal{J}) = \overline{\bigcup_{c\in[0,1]} \bigcup_{\mathcal{J}' \text{ is an } \mathcal{R}-\text{ limit of } \mathcal{L}_c^{(1)}} \sigma(\mathcal{J}')}$$

by Theorem 2.1 and (2.11), we get from (2.12) that $\sigma_{\rm ess}(\mathcal{J}) \subseteq \Delta_1 \cup \Delta_2$, which proves the theorem. □

3. Multiple Orthogonal Polynomials for Angelesco Systems

In this section we state the results on asymptotic behavior of the forms $Q_{\vec{n}}(x)$ and polynomials $P_{\vec{n}}(x)$ defined in (1.4) and (1.5), respectively, along ray sequences $\mathcal{N}_c = \mathcal{N}_{(c,1-c)}$ defined in (1.10) under the assumption that the measures of orthogonality are as in Theorem 1.1. The study of strong asymptotics of multiple orthogonal polynomials has a long history, see for example [29, 36, 6, 45]. Below, we follow the Riemann-Hilbert approach used in [45], where the strong asymptotics of MOPs was derived for Angelesco systems with analytic weights for non-marginal ray sequences. Here, we extend the results of [45] to marginal sequences, which is a non-trivial problem requiring new ideas.

As before, we assume that the intervals $\Delta_1 = [\alpha_1, \beta_1]$ and $\Delta_2 = [\alpha_2, \beta_2]$ are disjoint and $\beta_1 < \alpha_2$. In accordance with the definition of the intervals $\Delta_{c,1}, \Delta_{c,2}$ after (2.2), we shall also set $\Delta_{0,1} := \{\alpha_1\}, \Delta_{0,2} := \Delta_2$ and $\Delta_{1,1} := \Delta_1, \Delta_{1,2} := \{\beta_2\}$.

Throughout the paper, we use the following notation: given a system of Jordan arcs and curves Σ, we denote by Σ° the subset of Σ consisting of points that possess a neighborhood that is separated by Σ into exactly two connected components. In particular, $\Delta_i^\circ = (\alpha_i, \beta_i)$, $i \in \{1, 2\}$.

3.1. Fully Marginal Ray Sequences.

In this subsection we consider solely infinite ray sequences of the form

$$\mathcal{N}_{i-1} = \{\vec{n} : \text{ there exists } C > 0 \text{ such that } n_i \leqslant C\}, \quad i \in \{1, 2\}. \tag{3.1}$$

To describe the asymptotics we need to introduce the so-called Szegő functions of the measures μ_1, μ_2. To this end, let us set

$$\rho_i(x) := -2\pi \mathrm{i}\mu_i'(x), \quad x \in \Delta_i. \tag{3.2}$$

Observe that $(\rho_i w_{i+})(x) > 0$ for $x \in \Delta_i^\circ := (\alpha_i, \beta_i)$, where $w_i(z)$ was introduced in Proposition 2.1. Put

$$S_{\rho_i}(z) := \exp\left\{\frac{w_i(z)}{2\pi\mathrm{i}} \int_{\Delta_i} \frac{\log(\rho_i w_{i+})(x)}{z-x} \frac{\mathrm{d}x}{w_{i+}(x)}\right\}, \quad i \in \{1, 2\}. \tag{3.3}$$

Then each $S_{\rho_i}(z)$ is a holomorphic and non-vanishing function in $\overline{\mathbb{C}}\backslash\Delta_i$ that is uniquely (up to a sign) characterized by the properties[1]

$$\begin{cases} (S_{\rho_i+} S_{\rho_i-})(x)(\rho_i w_{i+})(x) \equiv 1, & x \in \Delta_i^\circ, \\ |S_{\rho_i}(z)| \sim |z - x_*|^{-1/4}, & z \to x_* \in \{\alpha_i, \beta_i\}. \end{cases} \tag{3.4}$$

Notice also that if $\rho_i(x)$ is replaced by $\rho_i(x)/w_{i+}(x)$ in (3.3), then $S_{\rho_i/w_{i+}}(z)$ retains all the described properties except it is actually bounded around β_i and α_i. The following theorem holds.

Theorem 3.1. *Under the conditions of Theorem 1.2, it holds that*

$$P_{\vec{n}}(z) = (1 + o(1))\big(S_{\rho_2}(z)/S_{\rho_2}(\infty)\big) S^{n_1}(z; \alpha_1)(z - \alpha_1)^{n_1} \varphi_2^{n_2}(z)$$

uniformly on bounded subsets of $\overline{\mathbb{C}}\backslash(\Delta_{0,1} \cup \Delta_2)$ *along any* $\mathcal{N}_0$ *satisfying* (3.1)*, where* $\varphi_2(z)$ *was introduced in* (2.6) *and*

$$S(z; x_0) := \left(\frac{\varphi_2(z) - \varphi_2(x_0)}{\varphi_2(x_0)\varphi_2(z) - A_{0,2}} \frac{\varphi_2(x_0)\varphi_2(z)}{z - x_0}\right)^{1/2}, \quad z \in \overline{\mathbb{C}}\backslash\Delta_2, \tag{3.5}$$

$x_0 \in (-\infty, \infty)\backslash\Delta_2$ *and the root is chosen so that* $S(\infty; x_0) = 1$*. An analogous asymptotic formula holds along* $\mathcal{N}_1$ *satisfying* (3.1).

Since $\varphi_{2+}(x)\varphi_{2-}(x) \equiv A_{0,2}$ for $x \in \Delta_2$, an explicit computation shows that

$$S(x; x_0)_+ S(x; x_0)_- = |S(x; x_0)_\pm|^2 \equiv -\varphi_2(x_0)(x - x_0)^{-1}, \quad x \in \Delta_2^\circ.$$

[1] $A(z) \sim B(z)$ as $z \to z_0$ means that the ratio $A(z)/B(z)$ is uniformly bounded away from zero and infinity as $z \to z_0$.

As $S(z;x_0)$ is non-vanishing and holomorphic in $\overline{\mathbb{C}}\backslash\Delta_2$ as well as bounded around α_2,β_2 a standard argument shows that

$$S(z;x_0)=S_\varrho(z)/S_\varrho(\infty),\quad S_\varrho^{-1}(\infty)=\sqrt{-\varphi_2(x_0)},\quad \varrho(x):=(x-x_0)/w_{2+}(x),$$

where $S_\varrho(\infty)>0$ when $x_0<\alpha_2$ while $S_\varrho(\infty)\in\mathrm{i}\mathbb{R}$ when $x_0>\beta_2$ with the choice of the square root depending on the determination of $\log(x-x_0)$ used. We prove Theorem 3.1 in Section 6.

3.2. **Szegő Functions.** Let us set $\mathbf{\Delta}_{c,i}:=\pi^{-1}(\Delta_{c,i})$, $i\in\{1,2\}$, and orient it so that $\mathfrak{R}_c^{(0)}$ remains on the left when the cycle is traversed in the positive direction. Put

$$w_{c,i}(z):=\sqrt{(z-\alpha_{c,i})(z-\beta_{c,i})}=z+O(1),\quad z\to\infty, \tag{3.6}$$

to be the branch holomorphic outside of $\Delta_{c,i}$. In what follows, it will be convenient to introduce the following notation

$$F^{(k)}(z):=F\big(z^{(k)}\big),\quad k\in\{0,1,2\},$$

for a function $F(z)$ defined on $\mathfrak{R}_c\backslash(\mathbf{\Delta}_{c,1}\cup\mathbf{\Delta}_{c,2})$. Then the following proposition holds.

Proposition 3.1. *Given $c\in(0,1)$ and functions $\rho_1(x),\rho_2(x)$ as in (3.2) and Theorem 1.1, there exists a function $S_c(z)$ non-vanishing and holomorphic in $\mathfrak{R}_c\backslash(\mathbf{\Delta}_{c,1}\cup\mathbf{\Delta}_{c,2})$ such that*

$$\begin{cases} S_{c\pm}^{(i)}(x)=S_{c\mp}^{(0)}(x)(\rho_i w_{c,i+})(x),\quad x\in\Delta_{c,i},\\ \big(S_c^{(0)}S_c^{(1)}S_c^{(2)}\big)(z)\equiv 1,\quad z\in\overline{\mathbb{C}},\\ \big|S_c^{(0)}(z)\big|\sim|z-x_0|^{-1/4}\quad as\quad z\to x_0\in\big\{\alpha_1,\beta_{c,1},\alpha_{c,2},\beta_2\big\}. \end{cases} \tag{3.7}$$

Properties (3.7) determine $S_c(z)$ uniquely up to a multiplication by a cubic root of unity. Moreover, if $c\to c_\star\in(0,1)$, then

$$S_c^{(k)}(z)=\big[1+o(1)\big]S_{c_\star}^{(k)}(z), \tag{3.8}$$

locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{c_\star,k}$ when $k\in\{1,2\}$, and in $\overline{\mathbb{C}}\backslash(\Delta_{c_\star,1}\cup\Delta_{c_\star,2})$ when $k=0$. Furthermore, it holds that

$$\frac{S_c^{(k)}(z)}{S_c^{(k)}(\infty)}=\big(1+o(1)\big)\begin{cases} S_{\rho_2}(z)/S_{\rho_2}(\infty), & k=0,\\ 1, & k=1,\\ S_{\rho_2}(\infty)/S_{\rho_2}(z), & k=2, \end{cases} \tag{3.9}$$

as $c\to 0$, where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ when $k\in\{0,1\}$ and uniformly in $\overline{\mathbb{C}}$ when $k=2$ (that is, including the traces on Δ_2), while it also holds that

$$\lim_{c\to 0}S_c^{(0)}(\infty)c^{1/3}=VS_{\rho_2}(\infty),\quad \lim_{c\to 0}S_c^{(1)}(\infty)c^{-2/3}=V^{-2},\quad and\quad \lim_{c\to 0}S_c^{(2)}(\infty)c^{1/3}=V/S_{\rho_2}(\infty), \tag{3.10}$$

where $V:=\big(2\pi\mu_1'(\alpha_1)|w_2(\alpha_1)|S_{\rho_2}(\alpha_1)\big)^{-1/3}$. Limits analogous to (3.9) and (3.10) also hold as $c\to 1$.

The construction leading to Proposition 3.1 is not new. As soon as strong asymptotics of MOPs became a question of interest, it was well understood that classical Szegő functions need to be replaced by solutions to a boundary value problem (3.7). The original approach reformulated (3.7) as a certain extremal problem, see [6]. Another approach using discontinuous Cauchy kernels on the corresponding Riemann surface was developed in [11]. The latter construction is exactly the one we adopt in Section 5 to prove Proposition 3.1. Even though out of necessity, but unlike previous works, we do examine here what happens to the Szegő functions $S_c(z)$ when one of the intervals $\Delta_{c,1}$, $\Delta_{c,2}$ is collapsing.

3.3. **Non-Fully Marginal and Non-Marginal Ray Sequences.** In this section we assume that sequences $\mathcal{N}_c$, $c\in[0,1]$, satisfy

$$\varepsilon_{\vec{n}}:=1/\min\{n_1,n_2\}\to 0\quad as\quad |\vec{n}|\to\infty,\quad \vec{n}\in\mathcal{N}_c. \tag{3.11}$$

We start by introducing an analog of the functions $\varphi_1(z),\varphi_2(z)$ in the non-fully marginal and non-marginal cases. Given a multi-index $\vec{n}$, let

$$c_{\vec{n}}:=n_1/|\vec{n}|. \tag{3.12}$$

To alleviate the notation, in what follows we shall use the subindex $\vec{n}$ instead of $c_{\vec{n}}$ for quantities depending on $c_{\vec{n}}$ such that $\mathfrak{R}_{\vec{n}}=\mathfrak{R}_{c_{\vec{n}}}$, $S_{\vec{n}}(z)=S_{c_{\vec{n}}}(z)$, etc. We shall denote by $\Phi_{\vec{n}}(z)$ a rational function on $\mathfrak{R}_{\vec{n}}$ which is

non-zero and finite everywhere except at the points on top of infinity, has a pole of order $|\vec{n}|$ at $\infty^{(0)}$, a zero of multiplicity n_i at $\infty^{(i)}$ for each $i \in \{1,2\}$, and satisfies

$$\left(\Phi_{\vec{n}}^{(0)}\Phi_{\vec{n}}^{(1)}\Phi_{\vec{n}}^{(2)}\right)(z) \equiv 1, \quad z \in \overline{\mathbb{C}}. \tag{3.13}$$

Equality in (3.13) is a simple matter of a normalization since the logarithm of the absolute value of the left-hand side of (3.13) extends to a harmonic function on $\mathbb{C}$ which has a well defined limit at infinity and therefore is a constant.

Theorem 3.2. *Under the conditions of Theorem 1.2, let $P_{\vec{n}}(z)$ be the polynomials satisfying (1.5). Given $c \in [0,1]$, let $\mathcal{N}_c = \{\vec{n}\}$ be a sequence for which (3.11) holds. Then $\vec{n} \in \mathcal{N}_c$ we have that*

$$\begin{cases} P_{\vec{n}}(z) &= (1+o(1))\gamma_{\vec{n}}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)^{(0)}(z), \\ P_{\vec{n}}(x) &= (1+o(1))\gamma_{\vec{n}}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)_+^{(0)}(x) + (1+o(1))\gamma_{\vec{n}}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)_-^{(0)}(x), \end{cases}$$

where the relations holds uniformly on closed subsets of $\overline{\mathbb{C}}\setminus(\Delta_{c,1}\cup\Delta_{c,2})$ and compact subsets $\Delta_{c,1}^\circ \cup \Delta_{c,2}^\circ$, respectively, and $\gamma_{\vec{n}}$ is the constant such that

$$\lim_{z\to\infty} \gamma_{\vec{n}} z^{|\vec{n}|}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)^{(0)}(z) = 1.$$

When $c \neq c^*, c^{**}$, see Proposition 4.1 further below, the error rate $o(1)$ can be replaced by $O_c(\varepsilon_{\vec{n}})$, where the dependence of $O_c(\varepsilon_{\vec{n}})$ on c is uniform for c on compact subsets $[0,1]\setminus\{c^*, c^{**}\}$.

In the above theorem the functions $S_{\vec{n}}^{(0)}(z)$ could be replaced by their limits as discussed in Proposition 3.1. However, we can do this only at the expense of the error rate $O_c(\varepsilon_{\vec{n}})$.

To describe asymptotic behavior of the forms $Q_{\vec{n}}(x)$, we need to introduce one additional function. Let $\Pi_{\vec{n}}(z)$ be a rational function on $\mathfrak{R}_{\vec{n}}$ with the zero/pole divisor and the normalization given by

$$2\left(\infty^{(1)}+\infty^{(2)}\right) - \boldsymbol{\alpha}_1 - \boldsymbol{\beta}_{\vec{n},1} - \boldsymbol{\alpha}_{\vec{n},2} - \boldsymbol{\beta}_2 \quad \text{and} \quad \Pi_{\vec{n}}^{(0)}(\infty) = 1,$$

where $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_{\vec{n},1}, \boldsymbol{\alpha}_{\vec{n},2}, \boldsymbol{\beta}_2$ are the ramification points of $\mathfrak{R}_{\vec{n}}$. Then the following theorem holds.

Theorem 3.3. *Under the conditions of Theorem 1.2, let $A_{\vec{n}}^{(i)}(z)$ be the polynomials defined in (1.4), $i \in \{1,2\}$. Given $c \in [0,1]$, let $\mathcal{N}_c = \{\vec{n}\}$ be a sequence for which (3.11) holds. Then for $\vec{n} \in \mathcal{N}_c$ we have that*

$$A_{\vec{n}}^{(i)}(z) = -(1+o(1))\frac{(\Pi_{\vec{n}}^{(i)} w_{\vec{n},i})(z)}{\gamma_{\vec{n}}(S_{\vec{n}}\Phi_{\vec{n}})^{(i)}(z)},$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus\Delta_{c,i}$ for $i \in \{1,2\}$ when $c \in (0,1)$, $i = 2$ when $c = 0$, and $i = 1$ when $c = 1$, while

$$A_{\vec{n}}^{(i)}(z) = o(1)\left(\tau_{\vec{n}}\left(w_{\vec{n},i}\Phi_{\vec{n}}^{(i)}\right)(z)\right)^{-1},$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus\Delta_{0,1}$ for $i = 1$ when $c = 0$ and of $\overline{\mathbb{C}}\setminus\Delta_{1,2}$ for $i = 2$ when $c = 1$, where $\tau_{\vec{n}} := \gamma_{\vec{n}} S_{\vec{n}}^{(0)}(\infty)$, i.e., it is a constant such that $\lim_{z\to\infty}\tau_{\vec{n}}|z|^{|\vec{n}|}\Phi_{\vec{n}}^{(0)}(z) = 1$. Moreover,

$$A_{\vec{n}}^{(i)}(x) = -(1+o(1))\frac{(\Pi_{\vec{n}}^{(i)} w_{\vec{n},i})_+(x)}{\gamma_{\vec{n}}(S_{\vec{n}}\Phi_{\vec{n}})_+^{(i)}(x)} - (1+o(1))\frac{(\Pi_{\vec{n}}^{(i)} w_{\vec{n},i})_-(x)}{\gamma_{\vec{n}}(S_{\vec{n}}\Phi_{\vec{n}})_-^{(i)}(x)},$$

uniformly on compact subsets of $\Delta_{c,i}^\circ$, $i \in \{1,2\}$. As in the case of Theorem 3.2, the error rate can be improved to $O_c(\varepsilon_{\vec{n}})$ when $c \in [0,1]\setminus\{c^*, c^{**}\}$ with dependence on c being locally uniform.

Let $\left(\widehat{\mu}_1, \widehat{\mu}_2\right)$ be a vector of Markov functions of the measures μ_i, that is,

$$\widehat{\mu}_i(z) := \int \frac{\mathrm{d}\mu_i(x)}{z-x} = \frac{1}{2\pi\mathrm{i}}\int_{\Delta_i}\frac{\rho_i(x)}{x-z}\,\mathrm{d}x, \qquad z \in \overline{\mathbb{C}}\setminus\Delta_i,\ i \in \{1,2\}.$$

Observe also that $(\widehat{\mu}_{i+} - \widehat{\mu}_{i-})(x) = \rho_i(x)$, $x \in \Delta_i^\circ$, by Sokhotski-Plemelj formulae. Then one can deduce from orthogonality relations (1.5) that there exist polynomials $P_{\vec{n}}^{(i)}(z)$ such that

$$R_{\vec{n}}^{(i)}(z) := \left(P_{\vec{n}}\widehat{\mu}_i - P_{\vec{n}}^{(i)}\right)(z) = O\left(z^{-(n_i+1)}\right) \quad \text{as} \quad z \to \infty,$$

$i \in \{1,2\}$. The vector of rational functions $\big(P^{(1)}_{\vec{n}}/P_{\vec{n}}, P^{(2)}_{\vec{n}}/P_{\vec{n}}\big)$ is called the *Hermite-Padé approximant* for $\big(\widehat{\mu}_1, \widehat{\mu}_2\big)$ corresponding to the multi-index $\vec{n}$. It further can be shown that

$$R^{(i)}_{\vec{n}}(z) = \frac{1}{2\pi \mathrm{i}} \int \frac{(P_{\vec{n}}\rho_i)(x)}{x-z}\mathrm{d}x, \quad z \in \overline{\mathbb{C}}\backslash\Delta_i,\ i \in \{1,2\}. \tag{3.14}$$

It also follows from (1.4) that there exists polynomial $A_{\vec{n}}(x)$ such that

$$O\big(z^{-|\vec{n}|}\big) = \sum_{i=1}^{2} \big(A^{(i)}_{\vec{n}}\widehat{\mu}_i\big)(z) - A_{\vec{n}}(z) =: L_{\vec{n}}(z) = \int \frac{Q_{\vec{n}}(x)}{z-x}, \tag{3.15}$$

where the asymptotic formula is valid for $z \to \infty$. Then the following result holds.

Theorem 3.4. *Under the conditions of Theorems 3.2–3.2, it holds for $\vec{n} \in \mathcal{N}_c$ that*

$$R^{(i)}_{\vec{n}}(z) = (1+o(1))\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}(z)w^{-1}_{\vec{n},i}(z),$$

uniformly on closed subsets of $\overline{\mathbb{C}}\backslash\Delta_{c,i}$, that is, including the traces on $\Delta_i\backslash\Delta_{c,i}$ for $i \in \{1,2\}$ when $c \in (0,1)$, for $i = 2$ when $c = 0$, and for $i = 1$ when $c = 1$, while

$$R^{(i)}_{\vec{n}}(z) = o(1)\tau_{\vec{n}}\Phi^{(i)}_{\vec{n}}(z)w^{-1}_{\vec{n},i}(z)$$

uniformly on closed subsets of $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ for $i = 1$ when $c = 0$ and of $\overline{\mathbb{C}}\backslash\Delta_{1,2}$ for $i = 2$ when $c = 1$. Moreover,

$$L_{\vec{n}}(z) = (1+o(1))\frac{\Pi^{(0)}_{\vec{n}}(z)}{\gamma_{\vec{n}}(S_{\vec{n}}\Phi_{\vec{n}})^{(0)}(z)},$$

uniformly on closed subsets of $\overline{\mathbb{C}}\backslash(\Delta_{c,1}\cup\Delta_{c,2})$. As in Theorems 3.2 and 3.3 the error rate can be improved to $O_c(\varepsilon_{\vec{n}})$ when $c \in [0,1]\backslash\{c^*, c^{**}\}$ with dependence on c being locally uniform.

Theorems 3.2–3.4 are proven in Chapter 7.

4. On the Supports of the Equilibrium measures

In this section we discuss further properties of the vector equilibrium problem (2.2)–(2.3) as well as prove some auxiliary lemmas needed later.

With the notation introduced in the beginning of Section 2.1, the following proposition holds.

Proposition 4.1. *There exist constants $0 < c^* < c^{**} < 1$ such that*

$$\begin{cases} \beta_{c,1} < \beta_1, \quad \alpha_{c,2} = \alpha_2, \quad 0 < c < c^*, \\ \beta_{c,1} = \beta_1, \quad \alpha_{c,2} = \alpha_2, \quad c^* \leqslant c \leqslant c^{**}, \\ \beta_{c,1} = \beta_1, \quad \alpha_{c,2} > \alpha_2, \quad 1 > c > c^{**}. \end{cases}$$

Moreover, it holds that[2]

$$\omega_{c,i} \xrightarrow{*} \omega_{c_\star,i}, \quad \alpha_{c,2} \to \alpha_{c_\star,2}, \quad \beta_{c,1} \to \beta_{c_\star,1}, \quad \ell_{c,i} \to \ell_{c_\star,i}, \quad V^{\omega_{c,i}} \to V^{\omega_{c_\star,i}} \quad as \quad c \to c_\star \in (0,1)$$

for $i \in \{1,2\}$, where the convergence of potentials is uniform on compact subsets of $\mathbb{C}$. Furthermore,

$$\begin{cases} \omega_{c,2} \xrightarrow{*} \omega_2, \quad \beta_{c,1} \to \alpha_1, \quad \ell_{c,2} \to 2\ell_2, \quad \ell_{c,1} \to V^{\omega_2}(\alpha_1) \quad as \quad c \to 0, \\ \omega_{c,1} \xrightarrow{*} \omega_1, \quad \alpha_{c,2} \to \beta_2, \quad \ell_{c,1} \to 2\ell_1, \quad \ell_{c,2} \to V^{\omega_1}(\beta_2) \quad as \quad c \to 1, \end{cases}$$

and $V^{\omega_{c,i}} \to V^{\omega_i}$ uniformly on compact subsets of $\mathbb{C}$ as $c \to 2-i$, $i \in \{1,2\}$.

Further, recall the surface $\mathfrak{R}_c$ constructed just before Proposition 2.1. Given a rational function $F(z)$ on $\mathfrak{R}_c$, we denote its divisor of zeros and poles by (F) and write

$$(F) = m_1\boldsymbol{z}_1 + \cdots + m_l\boldsymbol{z}_l - k_1\boldsymbol{p}_1 - \cdots - k_t\boldsymbol{p}_t$$

to mean that $F(z)$ has a zero of order m_i at $\boldsymbol{z}_i$ for each $i \in \{1,\ldots,l\}$, a pole of order k_i at $\boldsymbol{p}_i$ for each $i \in \{1,\ldots,t\}$, and otherwise it is non-vanishing and finite, where necessarily $\sum_{i=1}^{l} m_i = \sum_{i=1}^{t} k_i$.

[2]Given compactly supported measures ν_n, $n \in \mathbb{Z}_{\geqslant 0}$, $\nu_n \xrightarrow{*} \nu_0$ as $n \to \infty$ means that $\int f\,\mathrm{d}\nu_n \to \int f\,\mathrm{d}\nu_0$ as $n \to \infty$ for any compactly supported continuous function f.

It can be easily checked using Schwarz reflection principle, as it was done in [45, Proposition 2.1] for c rational, that the function

$$H_c(z) := \begin{cases} -V^{\omega_{c,1}+\omega_{c,2}}(z) + \frac{\ell_{c,1}+\ell_{c,2}}{3}, & z \in \mathfrak{R}_c^{(0)}, \\ V^{\omega_{c,i}}(z) - \ell_{c,i} + \frac{\ell_{c,1}+\ell_{c,2}}{3}, & z \in \mathfrak{R}_c^{(i)}, \quad i \in \{1,2\}, \end{cases} \tag{4.1}$$

is harmonic on $\mathfrak{R}_c \setminus \{\infty^{(0)}, \infty^{(1)}, \infty^{(2)}\}$. Therefore, the function $h_c(z) := 2\partial_z H_c(z)$, where $2\partial_z := \partial_x - \mathrm{i}\partial_y$, is rational on $\mathfrak{R}_c$. In fact, it holds that

$$\begin{cases} h_c^{(0)}(z) = \displaystyle\int \frac{\mathrm{d}(\omega_{c,1}+\omega_{c,2})(x)}{z-x}, & z \in \mathbb{C}\setminus\big(\Delta_{c,1}\cup\Delta_{c,2}\big), \\ h_c^{(i)}(z) = \displaystyle\int \frac{\mathrm{d}\omega_{c,i}(x)}{x-z}, & z \in \mathbb{C}\setminus\Delta_{c,i}, \quad i \in \{1,2\}. \end{cases} \tag{4.2}$$

The importance of this function lies in the following: it was shown in [45, Propositions 2.1 and 2.3] that

$$\Phi_{\vec{n}}(z) = C_{\vec{n}} \exp\left\{ |\vec{n}| \int_{\beta_2}^{z} h_{c_{\vec{n}}}(x)\mathrm{d}x \right\} \quad \text{and} \quad \frac{1}{|\vec{n}|}\log\left|\Phi_{\vec{n}}(z)\right| = H_{c_{\vec{n}}}(z) \tag{4.3}$$

for $z \in \mathfrak{R}_{\vec{n}}$, where the constant $C_{\vec{n}}$ should be chosen so that (3.13) is satisfied.

Proposition 4.2. *Let $\mathcal{D}_c := \alpha_1 + \beta_{c,1} + \alpha_{c,2} + \beta_2$ be the divisor of the ramification points of $\mathfrak{R}_c$. It holds that*

$$(h_c) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_c - \mathcal{D}_c \tag{4.4}$$

for some $z_c \in \mathfrak{R}_c^{(0)}$ such that $z_c \in [\beta_{c,1}, \alpha_{c,2}]$. Moreover, z_c is a continuous increasing function of c and

$$\begin{cases} z_c = \beta_{c,1}, & c \leqslant c^*, \\ z_c = \alpha_{c,2}, & c \geqslant c^{**}. \end{cases}$$

This proposition has the following implication: *point z_c uniquely determines the vector equilibrium measure* $(\omega_{c,1}, \omega_{c,2})$. Indeed, choose $z_\star \in (\alpha_1, \beta_2)$. Set $\beta_{\star,1} = \min\{\beta_1, z_\star\}$ and $\alpha_{\star,2} = \max\{\alpha_2, z_\star\}$. Construct Riemann surface $\mathfrak{R}_\star$ with respect to the cuts $[\alpha_1, \beta_{\star,1}]$ and $[\alpha_{\star,2}, \beta_2]$ as before. Let $h_\star(z)$ be a rational function on $\mathfrak{R}_\star$ with the zero/pole divisor

$$(h_\star) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_\star - \alpha_1 - \beta_{\star,1} - \alpha_{\star,2} - \beta_2,$$

where $\alpha_1, \beta_{\star,1}, \alpha_{\star,2}, \beta_2$ are the ramification points of $\mathfrak{R}_\star$ and $z_\star \in \mathfrak{R}_\star^{(0)}$. Clearly, $h_\star(z^{(0)}) + h_\star(z^{(1)}) + h_\star(z^{(2)}) \equiv 0$ as this sum must be an entire function that vanishes at infinity. Normalize $h_\star(z)$ so that $h_\star(z^{(0)}) = 1/z + O(1/z^2)$ as $z \to \infty$. Set $c_\star := -\lim_{z\to\infty} z h_\star(z^{(1)})$. Then $\mathfrak{R}_\star = \mathfrak{R}_{c_\star}$, $z_\star = z_{c_\star}$, and respectively $h_\star(z) = h_{c_\star}(z)$. It further follows from Privalov's lemma [39, Section III.2] that

$$\mathrm{d}\omega_{c_\star,i}(x) = \left(h_{\star+}^{(i)}(x) - h_{\star-}^{(i)}(x)\right)\frac{\mathrm{d}x}{2\pi\mathrm{i}}, \quad i \in \{1,2\},$$

and thus, we have recovered the vector equilibrium measure from $z_\star$.

Proof of Propositions 4.1 and 4.2. Besides relations (2.3), it also holds that the left hand sides of (2.3) are strictly less than zero on $\Delta_1\setminus\Delta_{c,1}$ and $\Delta_2\setminus\Delta_{c,2}$, respectively, see [26]. In particular, we can write

$$V^{\frac{1}{c}\omega_{c,1}}(x) + \frac{1}{2c}V^{\omega_{c,2}}(x) - \frac{\ell_{c,1}}{2c} \begin{cases} \equiv 0 & \text{on} \quad \mathrm{supp}(\omega_{c,1}), \\ \geqslant 0 & \text{on} \quad \Delta_1\setminus\mathrm{supp}(\omega_{c,1}), \end{cases}$$

which, in view of [42, Theorem I.3.3], can be interpreted in the following way: the measure $\frac{1}{c}\omega_{c,1}$ is the weighted logarithmic equilibrium distribution on Δ_1 in the presence of the external field $\frac{1}{2c}V^{\omega_{c,2}}(x)$. Hence, its support maximizes the Mhaskar-Saff functional [42, Chapter IV]:

$$F_c(K) := \log \mathrm{cap}(K) - \frac{1}{2c}\int V^{\omega_{c,2}}\mathrm{d}\omega_K,$$

where $K \subseteq [\alpha_1, \beta_1]$ is compact, $\mathrm{cap}(K)$ is the logarithmic capacity of K, and ω_K is the logarithmic equilibrium distribution on K (when K is an interval, ω_K is the arcsine distribution on K). As mentioned before (2.3), the maximizer of this functional is an interval containing α_1 (this was proven in [26]). Therefore, it is enough to consider compact sets K only of the form $[\alpha_1, \beta]$. Thus, the functional $F(K)$ reduces to the function

$$F_c(\beta) := \log\frac{\beta-\alpha_1}{4} - \frac{1}{2c}\int_{\alpha_1}^{\beta} V^{\omega_{c,2}}(x)\frac{\mathrm{d}x}{\pi\sqrt{(\beta-x)(x-\alpha_1)}},$$

where we used explicit expressions for the logarithmic capacity and the equilibrium measure of an interval. To find the maximum of $F_c(\beta)$ on Δ_1, let us compute its derivative. To this end, it can be readily checked that

$$\frac{1}{h}\left(\int_{\alpha_1}^{\beta+h} f(x)\frac{\mathrm{d}x}{\pi\sqrt{(\beta+h-x)(x-\alpha_1)}} - \int_{\alpha_1}^{\beta} f(x)\frac{\mathrm{d}x}{\pi\sqrt{(\beta-x)(x-\alpha_1)}}\right) = \int_{\alpha_1}^{\beta}\frac{1}{h}\left(f\left(x+h\frac{x-\alpha_1}{\beta-\alpha_1}\right)-f(x)\right)\frac{\mathrm{d}x}{\pi\sqrt{(\beta-x)(x-\alpha_1)}}$$

for every differentiable function $f(x)$ on Δ_1. Observe also that $V^{\omega_{c,2}}(x)$ is harmonic off Δ_2 and therefore $f_c(x) := V^{\omega_{c,2}}(x) = -\int \log|x-y|\mathrm{d}\omega_{c,2}(y)$ is a smooth function on Δ_1. Hence, by taking the limit as $h \to 0$ in the above equality, we get

$$F_c'(\beta) = \frac{1}{\beta-\alpha_1} - \frac{1}{4\pi c}\int_{-1}^{1} f_c'\left(\frac{\beta-\alpha_1}{2}x+\frac{\beta+\alpha_1}{2}\right)\sqrt{\frac{1+x}{1-x}}\mathrm{d}x. \tag{4.5}$$

It is also obvious that $f_c'(x) = \int (y-x)^{-1}\mathrm{d}\omega_{c,2}(y)$, which is an increasing positive function on Δ_1. Thus, $F_c'(\beta)$ is a decreasing function of β and therefore has at most one zero. Moreover, it holds that

$$\frac{1-c}{\beta_2-\alpha_1} < f_c'(x) < \frac{1-c}{\alpha_2-\beta_1}, \quad x \in \Delta_1. \tag{4.6}$$

Hence, $F_c'(\beta_1) < 0$ for all c small. As $\lim_{\beta\to\alpha_1^+} F_c'(\beta) = +\infty$, we get that $\beta_{c,1} \in (\alpha_1, \beta_1)$ for all c small. Using $F_c'(\beta_{c,1}) = 0$ and the above estimates, we get from (4.5) that

$$\frac{4c}{1-c}(\alpha_2-\beta_1) < \beta_{c,1}-\alpha_1 < \frac{4c}{1-c}(\beta_2-\alpha_1) \tag{4.7}$$

for all small c. This, in particular, implies that $\beta_{c,1} \to \alpha_1$ as $c \to 0$. An analogous argument shows that $\alpha_{c,2}$ approaches β_2 when $c \to 1$. It further follows from (4.6) that $f_c'(x)$ uniformly converges to zero on Δ_1 as $c \to 1$. Thus, $F_c'(\beta) > 0$ for all $\beta \in \Delta_1$ and all c close to 1. That is, $\Delta_{c,1} = \Delta_1$ in this case. Similarly, we also get that $\Delta_{c,2} = \Delta_2$ for all c small.

Let us now describe what happens to the components of the vector equilibrium measure and their potentials as $c \to 0$. Clearly, $V^{\omega_{c,1}}(z) \to 0$ uniformly on compact subsets of $\mathbb{C}\backslash\Delta_{0,1}$ in this case. To show that $\omega_{c,2} \xrightarrow{*} \omega_2$ as $c \to 0$, notice that

$$\|\sigma\|\ell_2 = \int V^{\omega_2}\mathrm{d}\sigma = \int V^{\sigma}\mathrm{d}\omega_2 \begin{cases} \geqslant \inf_{\Delta_2} V^{\sigma}, \\ \leqslant \sup_{\Delta_2} V^{\sigma}, \end{cases}$$

for any Borel measure σ supported on Δ_2 since ω_2 is a probability measure. It follows from (2.3) that $V^{\omega_{c,2}}(x)$ is continuous on $\Delta_2 = \Delta_{c,2}$. Therefore,

$$\begin{cases} 2\ell_2(1-c) \geqslant \min\limits_{\Delta_2} V^{2\omega_{c,2}} = V^{2\omega_{c,2}}(x_{\min}) = \ell_{c,2} - V^{\omega_{c,1}}(x_{\min}) = \ell_{c,2} + o(1), \\ 2\ell_2(1-c) \leqslant \max\limits_{\Delta_2} V^{2\omega_{c,2}} = V^{2\omega_{c,2}}(x_{\max}) = \ell_{c,2} - V^{\omega_{c,1}}(x_{\max}) = \ell_{c,2} + o(1), \end{cases}$$

which implies that $\ell_{c,2} = 2\ell_2 + o(1)$ as $c \to 0$. Let ω be a weak* limit point of $\omega_{c,2}$ as $c \to 0$. Then ω is a probability measure and

$$V^{\omega}(x) \leqslant \liminf_{c\to 0} V^{\omega_{c,2}}(x) = \liminf_{c\to 0}\left(\ell_{c,2} - V^{\omega_{c,1}}(x)\right)/2 = \ell_2, \quad x \in \Delta_2,$$

where the first inequality follows from the Principle of Descent [42, Theorem I.6.8]. Therefore, $E(\omega,\omega) \leqslant \ell_2 = E(\omega_2,\omega_2)$, which implies that $\omega = \omega_2$ by the uniqueness of the equilibrium measure. To deduce the behavior of the constants $\ell_{c,1}$ as $c \to 0$, observe that

$$\begin{cases} V^{2\omega_{c,1}+\omega_{c,2}}(x) \leqslant \ell_{c,1}, & x \in (-\infty, \alpha_1], \\ V^{2\omega_{c,1}+\omega_{c,2}}(x) \geqslant \ell_{c,1}, & x \in [\beta_{c,1}, \beta_1], \end{cases}$$

where the first claim can be easily obtained from (2.3) and the second one was already mentioned at the beginning of the proof. Then

$$V^{2\omega_{c,1}+\omega_{c,2}}(\alpha_1-\epsilon) \leqslant \ell_{c,1} \leqslant V^{2\omega_{c,1}+\omega_{c,2}}(\alpha_1+\epsilon)$$

for any $\epsilon > 0$ since $\beta_{c,1} < \alpha_1 + \epsilon$ for all c small enough. Hence, we get that

$$V^{\omega_2}(\alpha_1-\epsilon) \leqslant \liminf_{c\to 0} \ell_{c,1} \leqslant \limsup_{c\to 0} \ell_{c,1} \leqslant V^{\omega_2}(\alpha_1+\epsilon).$$

Since $V^{\omega_2}(x)$ is continuous on the real line and ϵ is arbitrary, we get that $\ell_{c,1} \to V^{\omega_2}(\alpha_1)$ as $c \to 0$. The respective claims for the limits as $c \to 1$ can be shown in a similar fashion.

Let us point out one consequence of the fact that $\omega_{c,2} \xrightarrow{*} \omega_2$ as $c \to 0$ that will be useful to us later. It holds that

$$f_c'(z) := \int \frac{\mathrm{d}\omega_{c,2}(y)}{y-z} \to \int \frac{\mathrm{d}\omega_2(y)}{y-z} = \frac{1}{\pi}\int_{\alpha_2}^{\beta_2} \frac{1}{y-z}\frac{\mathrm{d}y}{\sqrt{(y-\alpha_2)(\beta_2-y)}} = -\frac{1}{w_2(z)},$$

locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_2$, where, as before, $w_2(z) := \sqrt{(z-\alpha_2)(z-\beta_2)}$. Therefore, we can improve (4.7) to

$$\frac{4c}{\beta_{c,1}-\alpha_1} = \frac{1}{|w_2(\alpha_1)|} + o(1) \tag{4.8}$$

as $c \to 0$, where we again used (4.5).

The facts that $\omega_{c,i} \xrightarrow{*} \omega_{c_\star,i}$ and $\ell_{c,i} \to \ell_{c_\star,i}$ as $c \to c_\star \in (0,1)$, $i \in \{1,2\}$, were shown in the proof of [45, Proposition 2.1]. Let us now show that $\beta_{c,1} \to \beta_{c_\star,1}$ in this case (that is, that $\beta_{c,1}$ is a continuous function of c). Weak* convergence of measures necessitates that $\liminf_{c\to c_\star} \beta_{c,1} \geqslant \beta_{c_\star,1}$. Assume to the contrary that there exists a subsequence $c_n \to c_\star$ such that $\beta_{c_\star,1} < \beta_* := \liminf_{n\to\infty} \beta_{c_n,1}$. Then

$$\liminf_{n\to\infty} \ell_{c_n,1} = \liminf_{n\to\infty} V^{2\omega_{c_n,1}+\omega_{c_n,2}}(x) \geqslant V^{2\omega_{c_\star,1}+\omega_{c_\star,2}}(x) > \ell_{c_\star,1}$$

for $x \in (\beta_{c_\star,1}, \beta_*)$ due to the Principle of Descent [42, Theorem I.6.8]. However, the above conclusion clearly contradicts the claim $\ell_{c,1} \to \ell_{c_\star,1}$ as $c \to c_\star$. The convergence $\alpha_{c,2} \to \alpha_{c_\star,2}$ as $c \to c_\star$ can be shown analogously (unfortunately, this convergence of the endpoints was asserted without justification in the proof [45, Proposition 2.1]). Given the convergence of the endpoint, the uniform convergence of the potentials as $c \to c_\star \in (0,1)$ was established in the proof of [45, Proposition 2.1] using harmonicity of $H_c(z)$. The same arguments can be applied to show that $V^{\omega_{c,i}} \to V^{\omega_i}$ uniformly on compact subsets of $\mathbb{C}$ as $c \to 2-i$, $i \in \{1,2\}$.

Let us now establish the existence of the constants $0 < c^* < c^{**} < 1$ and the monotonicity properties of $\beta_{c,1}$ and $\alpha_{c,2}$. Claim (4.4) was obtained in [45, Proposition 2.3]. There it was further shown that

$$\beta_{c,1} < \beta_1 \quad \Rightarrow \quad z_c = \beta_{c,1} \quad \text{and} \quad \alpha_{c,2} > \alpha_2 \quad \Rightarrow \quad z_c = \alpha_{c,2}. \tag{4.9}$$

Assume now that $\beta_{c_1,1} = \beta_{c_2,1} < \beta_1$. Then the functions $h_{c_1}(z)$ and $h_{c_2}(z)$ are defined on the same Riemann surface. Their difference has at least four zeros (double zero at $\infty^{(0)}$ and simple zeros at $\infty^{(1)}$ and $\infty^{(2)}$) and at most three poles $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2$. This is possible only if the function is identically zero and therefore $c_1 = c_2$ as $h_c^{(1)}(z) = cz^{-1} + O(z^{-2})$ by (4.2). Since $\beta_{c,1} \to \alpha_1$ as $c \to 0$, this shows the existence of c^* and proves monotonicity of $\beta_{c,1}$ as a function of c (it is a continuous and injective function of c). The existence of c^{**} and monotonicity of $\alpha_{c,2}$ are proven analogously. It also follows from (4.9) that $c^* \leqslant c^{**}$. As it was shown in [45, Proposition 2.3] that $z_{c^*} = \beta_{c^*,1} (= \beta_1)$ and $z_{c^{**}} = \alpha_{c^{**},2} (= \alpha_2)$, we in fact get that $c^* < c^{**}$.

It only remains to prove that z_c is a continuous increasing function of c on $[c^*, c^{**}]$. To show monotonicity, take $c^* \leqslant c_1 < c_2 \leqslant c^{**}$. It follows easily from (4.2) that each $h_c(x^{(0)})$ is a decreasing function of $x \in (\beta_1, \alpha_2)$. Thus, to prove that $z_{c_1} < z_{c_2}$, it is enough to show that $h(x^{(0)}) > 0$ in (β_1, α_2), where $h(z) := (h_{c_2} - h_{c_1})(z)$. Notice that $h(x^{(0)}) = -h(x^{(1)}) - h(x^{(2)})$ by (4.2) and therefore it is sufficient to argue that $h(x^{(1)}) < 0$ on (β_1, ∞) and $h(x^{(2)}) < 0$ on $(-\infty, \alpha_2)$. These claims are obvious for all $|x|$ large enough since

$$h(z^{(1)}) = -\frac{c_2 - c_1}{z} + O\big(z^{-2}\big) \quad \text{and} \quad h(z^{(2)}) = \frac{c_2 - c_1}{z} + O\big(z^{-2}\big)$$

as $z \to \infty$ according to (4.2). As explained after (4.9), $h(z)$ vanishes only at $\infty^{(0)}$, $\infty^{(1)}$, and $\infty^{(2)}$. Therefore, $h(z^{(1)})$ and $h(z^{(2)})$ cannot change sign on (β_1, ∞) and $(-\infty, \alpha_2)$, respectively. Hence, these functions are negative everywhere on the considered rays by continuity.

To show continuity of z_c as a function of $c \in [c^*, c^{**}]$, we shall once again use the fact that $h_c(x^{(0)})$ is a decreasing function on (β_1, α_2). When $c \in (c^*, c^{**})$, $h_c(x^{(0)})$ is unbounded on both ends of (β_1, α_2) and therefore changes sign from $+$ to $-$ when passing through z_c (recall that $h_c(z)$ has poles at $\boldsymbol{\beta}_1$ and $\boldsymbol{\alpha}_2$ in this case). When $c = c^*$, $h_c(x^{(0)})$ is unbounded only at α_2 and, since it is non-vanishing, is negative on $[\beta_1, \alpha_2)$. Similarly, when $c = c^{**}$, it is unbounded at β_1 only and therefore is positive on $(\beta_1, \alpha_2]$. In any case, z_c is the point where the potential $V^{\omega_{c,1}+\omega_{c,2}}(x)$ achieves its minimum on $[\beta_1, \alpha_2]$. Thus, if $z_{c_n} \to z_\star$ as $c_n \to c_\star$ when $n \to \infty$, $c_n, c_\star \in (c^*, c^{**})$, then

$$V^{\omega_{c_0,1}+\omega_{c_0,2}}(z_\star) \leqslant \liminf_{n\to\infty} V^{\omega_{c_n,1}+\omega_{c_n,2}}(z_{c_n}) \leqslant \liminf_{n\to\infty} V^{\omega_{c_n,1}+\omega_{c_n,2}}(z_{c_\star}) = V^{\omega_{c_\star,1}+\omega_{c_\star,2}}(z_{c_\star}),$$

where the first inequality follows from the weak* convergence of measures and the Principle of Descent [42, Theorem I.6.8], the second one from the just discussed extremal property of z_{c_n}, and the last equality holds due to the weak* convergence of measures and the fact that $z_{c_\star}$ does not belong to the supports of the measures in

question. Since $V^{\omega_{c_\star,1}+\omega_{c_\star,2}}(x)$ is smallest at $z_{c_\star}$, we get that $z_\star = z_{c_\star}$. When $c_\star = c^*$, essentially the same argument works. One just needs to replace $z_{c_\star} = \beta_1$ with $\beta_1 + \epsilon$ for any $\epsilon > 0$. Since $V^{\omega_{c_\star,1}+\omega_{c_\star,2}}(x)$ is increasing on $[\beta_1, \alpha_2]$, this shows that $z_\star \leqslant z_{c_\star} + \epsilon$ for any $\epsilon > 0$ and therefore $z_\star = z_{c_\star}$. Clearly, an analogous modification works when $c_\star = c^{**}$. □

5. Proof of Propositions 2.1 and 3.1

On several occasions, we shall refer to the following consequences of Koebe's 1/4-theorem, [38, Theorem 1.3]. Given $r > 0$, let

$$a(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad b(z) = \sum_{k=0}^{\infty} b_k z^{-k}, \quad \text{and} \quad d(z) = \sum_{k=-\infty}^{1} d_k z^k$$

be univalent in $D_a = \{|z - z_0| < r\}$, $D_b = \{|z| > 1/r\}$, and $D_d = \{|z| > r\}$, respectively. Then,

$$\{|z - a_0| < ra_1/4\} \subseteq a(D_a), \quad \{|z - b_0| < rb_1/4\} \subseteq b(D_b), \quad \text{and} \quad \{|z| > 4rd_1\} \subseteq d(D_d), \tag{5.1}$$

where $f(D)$ stands for image of a domain D under the function $f(z)$.

5.1. **Proof of Proposition 2.1.** Recall that $\chi_c(z)$ is univalent on $\mathfrak{R}_c$ and $\chi_c^{(0)}(z) = z + O(z^{-1})$ as $z \to \infty$, see (2.4). Hence, it follows from (5.1) that there exists a finite constant R independent of c such that $\{|z| > R\} \subset \chi_c\big(\mathfrak{R}_c^{(0)}\big)$ for all $c \in (0,1)$. In particular, it holds that $|\chi_c(\boldsymbol{x})| \leqslant R$, $\boldsymbol{x} \in \boldsymbol{\Delta}_{c,1}$, as well as $|B_{c,i}| \leqslant R$, $i \in \{1,2\}$, see (2.5), for all $c \in (0,1)$. For all $c \leqslant c^{**}$ (in which case $\Delta_{c,2} = \Delta_2$), define

$$\varphi(z) := \frac{1}{2} \begin{cases} z - (\beta_2 + \alpha_2)/2 + w_2(z), & z \in \mathfrak{R}_c^{(0)} \backslash \boldsymbol{\Delta}_{c,1}, \\ z - (\beta_2 + \alpha_2)/2 - w_2(z), & z \in \mathfrak{R}_c^{(2)}. \end{cases}$$

This is a meromorphic function in $\big(\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}\big)\backslash \boldsymbol{\Delta}_{c,1}$ with a simple pole at $\infty^{(0)}$, a simple zero at $\infty^{(2)}$, and otherwise non-vanishing and finite. It is normalized so that $\varphi(z^{(0)}) = z + O(1)$ as $z \to \infty$. Observe that $\varphi(\boldsymbol{z})$ continuously extends to the closed set $\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}$. It can be readily checked that the image of $\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}$ under $\varphi(\boldsymbol{z})$ is equal to $\overline{\mathbb{C}}$ and $\varphi(\boldsymbol{z})$ is one-to-one everywhere except on $\boldsymbol{\Delta}_{c,1}$ that is mapped into an interval

$$\varphi(\boldsymbol{\Delta}_{c,1}) =: I_{c,1} = \big[\varphi(\boldsymbol{\alpha}_1), \varphi(\boldsymbol{\beta}_{c,1})\big] \to \big\{\varphi(\boldsymbol{\alpha}_1)\big\} \quad \text{as} \quad c \to 0.$$

Notice also that $\varphi^{(0)}(z) = \varphi_2(z)$ for $z \in \overline{\mathbb{C}}\backslash\Delta_2$, see (2.6).

Define $f_c(z) := \big(\chi_c\big(\varphi^{-1}(z)\big) - B_{c,2}\big)/z$. Then $f_c(z)$ is a holomorphic function in $\overline{\mathbb{C}}\backslash I_{c,1}$ (there is no pole at the origin as $\varphi^{-1}(0) = \infty^{(2)}$ and $\chi_c(\boldsymbol{z}) - B_{c,2}$ vanishes there) with bounded traces on $I_{c,1}$ that assumes value 1 at infinity. Hence, it follows from Cauchy's integral formula that

$$f_c(z) = 1 + \int_{I_{c,1}} \frac{(f_{c+} - f_{c-})(x)}{x - z} \frac{\mathrm{d}x}{2\pi \mathrm{i}}, \quad z \in \overline{\mathbb{C}}\backslash I_{c,1}.$$

Since the traces $f_{c\pm}(z)$ are bounded above in absolute value on $I_{c,1}$ independently of c and $|I_{c,1}| \to 0$ as $c \to 0$, we see that $f_c(z) \to 1$ as $c \to 0$ locally uniformly in $\overline{\mathbb{C}}\backslash\{\varphi(\boldsymbol{\alpha}_1)\}$. Hence, it holds that

$$\chi_c(\boldsymbol{z}) = B_{c,2} + \big(1 + o(1)\big)\varphi(\boldsymbol{z})$$

locally uniformly on $\big(\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}\big)\backslash \boldsymbol{\Delta}_{c,1}$. Since the image of $\big(\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}\big)\backslash \boldsymbol{\Delta}_{c,1}$ under $\varphi(\boldsymbol{z})$ is $\overline{\mathbb{C}}\backslash I_{c,1}$ and $|I_{c,1}| \to 0$ as $c \to 0$, for any $\epsilon > 0$ there exists $\delta > 0$ such that the image of $\big(\mathfrak{R}_c^{(0)} \backslash \pi^{-1}(\{|z - \alpha_1| < \epsilon\})\big) \cup \mathfrak{R}_c^{(2)}$ under $\chi_c(\boldsymbol{z})$ contains $\overline{\mathbb{C}}\backslash\{|z - B_{c,2} - \varphi(\boldsymbol{\alpha}_1)| < \delta\}$. Due to univalency of $\chi_c(\boldsymbol{z})$ on $\mathfrak{R}_c$, this means that the image of $\big(\mathfrak{R}_c^{(0)} \cap \pi^{-1}(\{|z - \alpha_1| < \epsilon\})\big) \cup \mathfrak{R}_c^{(1)}$ is contained in $\{|z - B_{c,2} - \varphi(\boldsymbol{\alpha}_1)| < \delta\}$. Altogether, we get that

$$\chi_c(\boldsymbol{z}) = B_{c,2} + \big(1 + o(1)\big) \begin{cases} \varphi(\boldsymbol{z}), & z \in \mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)}, \\ \varphi(\boldsymbol{\alpha}_1), & z \in \mathfrak{R}_c^{(1)}, \end{cases} \tag{5.2}$$

where $o(1)$ holds uniformly on the entire surface $\mathfrak{R}_c$. Since

$$\varphi^{(0)}(z) = z - \frac{\beta_2 + \alpha_2}{2} + O\left(\frac{1}{z}\right) \quad \text{and} \quad \varphi^{(2)}(z) = \frac{(\beta_2 - \alpha_2)^2}{16} \frac{1}{z} + O\left(\frac{1}{z^2}\right),$$

the desired limits (2.7) easily follow.

Continuity of $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ as functions of c comes from the continuous dependence of $\alpha_{c,2}$ and $\beta_{c,1}$ on c, see Proposition 4.2, and therefore the continuous dependence $\chi_c(\boldsymbol{z})$ on c.

5.2. **Auxiliary Estimates, I.** In the forthcoming analysis, the following functions will play an important role:

$$\Upsilon_{c,i}(z) := A_{c,i}\big(\chi_c(z) - B_{c,i}\big)^{-1}, \quad i \in \{1,2\}. \tag{5.3}$$

It follows from the properties of $\chi_c(z)$, see (2.4) and (2.5), that $\Upsilon_{c,i}(z)$ is a conformal map of $\mathfrak{R}_c$ onto $\overline{\mathbb{C}}$ that maps $\infty^{(i)}$ into ∞ and $\infty^{(0)}$ into 0. Moreover, it holds that

$$\Upsilon^{(1)}_{c,1}(z) = z + O(1) \quad \text{and} \quad \Upsilon^{(0)}_{c,1}(z) = A_{c,1}z^{-1} + O\big(z^{-2}\big) \quad \text{as} \quad z \to \infty. \tag{5.4}$$

It was explained in [45, Section 7], see [45, Equation (7.2)], that

$$\Upsilon_{c,i}(z) \to \Upsilon_{c_\star,i}(z) \quad \text{as} \quad c \to c_\star \in (0,1), \tag{5.5}$$

uniformly on $\mathfrak{R}_{c_\star}\backslash\mathfrak{U}$ for each $i \in \{1,2\}$, where $\mathfrak{U}$ is any open set the containing ramification points of $\mathfrak{R}_{c_\star}$ (if $\mathfrak{U}_c \subset \mathfrak{R}_c$ is an open set such that $\pi\big(\mathfrak{R}^{(k)}_{c_\star}\backslash\mathfrak{U}\big) = \pi\big(\mathfrak{R}^{(k)}_{c}\backslash\mathfrak{U}_c\big)$ for each $k \in \{0,1,2\}$, then the bordered Riemann surfaces $\mathfrak{R}_{c_\star}\backslash\mathfrak{U}$ and $\mathfrak{R}_c\backslash\mathfrak{U}_c$ are identical for all c sufficiently close to $c_\star$ and we can think of $\Upsilon_{c,i}(z)$ as a function on $\mathfrak{R}_{c_\star}\backslash\mathfrak{U}$). On the other hand, when $c \to 0$, the following is true.

Lemma 5.1. *It holds that*

$$\Upsilon_{c,2}(z) = (1+o(1))\begin{cases} \psi(z), & z \in \mathfrak{R}^{(0)}_c \cup \mathfrak{R}^{(2)}_c, \\ \psi(\alpha_1), & z \in \mathfrak{R}^{(1)}_c, \end{cases} \tag{5.6}$$

as $c \to 0$, where $o(1)$ holds uniformly on the entire surface $\mathfrak{R}_c$ and

$$\psi(z) := \frac{A_{0,2}}{\varphi(z)} = \frac{1}{2}\begin{cases} z - (\beta_2+\alpha_2)/2 - w_2(z), & z \in \mathfrak{R}^{(0)}_c\backslash\mathbf{\Delta}_{c,1}, \\ z - (\beta_2+\alpha_2)/2 + w_2(z), & z \in \mathfrak{R}^{(2)}_c, \end{cases}$$

that is $\psi^{(2)}(z)$ maps $\mathfrak{R}^{(2)}_c$ conformally onto $\{|z| > (\beta_2-\alpha_2)/4\}$ and $\psi^{(0)}(z)\psi^{(2)}(z) \equiv A_{0,2}$. Moreover, it holds that[3]

$$\big|\Upsilon^{(0)}_{c,1}(z)\big| \sim c|\phi_c^{-1}(z)|, \quad \big|\Upsilon^{(1)}_{c,1}(z)\big| \sim c|\phi_c(z)|, \quad \text{and} \quad \big|\Upsilon^{(2)}_{c,1}(z)\big| \sim c^2 \tag{5.7}$$

on $\overline{\mathbb{C}}$ (including the traces on $\Delta_{c,1} \cup \Delta_2$, $\Delta_{c,1}$, and Δ_2, respectively) as $c \to 0$, where

$$\phi_c(z) := \frac{2}{\beta_{c,1}-\alpha_1}\left(z - \frac{\beta_{c,1}+\alpha_1}{2} + w_{c,1}(z)\right) \tag{5.8}$$

is the conformal map of $\overline{\mathbb{C}}\backslash\Delta_{c,1}$ onto $\{|z| > 1\}$ that fixes the point at infinity and has positive derivative there. In addition, it holds that $\Upsilon^{(1)}_{c,1}(z) = z - \alpha_1 + O(c)$ uniformly in $\overline{\mathbb{C}}$ as $c \to 0$.

Proof. Formula (5.6) follows immediately from (5.2), the very definition (5.3), and the first limit in (2.7). It also is immediate from (5.3) and (5.2) that

$$\big|\Upsilon^{(2)}_{c,1}(z)\big| = \left|\frac{A_{c,1}}{(1+o(1))\varphi(\alpha_1) + (1+o(1))\varphi^{(2)}(z)}\right| \sim A_{c,1} \tag{5.9}$$

in $\overline{\mathbb{C}}$ (including the traces on Δ_2) as $c \to 0$ since $|\varphi^{(2)}(z)| \leqslant (\beta_2-\alpha_2)/4 < |\varphi(\alpha_1)|$, see (2.6). It can be readily verified that the symmetric functions of the branches of a rational function on $\mathfrak{R}_c$ must be rational functions on $\overline{\mathbb{C}}$. Since $\Upsilon^{(1)}_{c,1}(z)$ has a simple pole at infinity, $\Upsilon^{(0)}_{c,1}(z)$ has a simple zero there, and $\Upsilon^{(k)}_{c,1}(z)$, $k \in \{0,1,2\}$, are otherwise non-vanishing and finite, the product of three branches of $\Upsilon_{c,1}(z)$ must be a constant. Thus, similarly to (5.9), it holds that

$$\big|\Upsilon^{(0)}_{c,1}(z)\Upsilon^{(1)}_{c,1}(z)\Upsilon^{(2)}_{c,1}(z)\big| = \frac{A^2_{c,1}}{B_{c,2}-B_{c,1}} = -\frac{A^2_{c,1}}{(1+o(1))\varphi(\alpha_1)} \sim A^2_{c,1} \tag{5.10}$$

in $\overline{\mathbb{C}}$ as $c \to 0$ (recall that $\varphi(\alpha_1) < 0$). For each $z \notin \mathbf{\Delta}_{c,1} \cup \mathbf{\Delta}_{c,2}$, let $\overline{z}$ be the point on the same sheet of $\mathfrak{R}_c$ as z with $\pi(\overline{z}) = \overline{z}$ and then extend this definition by continuity to $\mathbf{\Delta}_{c,1} \cup \mathbf{\Delta}_{c,2}$. The function $\overline{\Upsilon_{c,1}(\overline{z})}$ is meromorphic on $\mathfrak{R}_c$ and has the same zero/pole divisor and normalization as $\Upsilon_{c,1}(z)$. Therefore, $\overline{\Upsilon_{c,1}(\overline{z})} = \Upsilon_{c,1}(z)$. In particular, $\Upsilon^{(2)}_{c,1}(x)$ is real on $\Delta_{c,1}$ and the traces of $\Upsilon^{(k)}_{c,1}(z)$ on $\Delta_{c,1}$, $k \in \{0,1\}$, are conjugate-symmetric. Hence, we get from (5.9) and (5.10) that

$$A_{c,1} \sim A^{-1}_{c,1}\big|\Upsilon^{(2)}_{c,1}(x)\Upsilon^{(1)}_{c,1+}(x)\Upsilon^{(0)}_{c,1+}(x)\big| \sim \big|\Upsilon^{(1)}_{c,1+}(x)\big|^2 = \big|\Upsilon^{(0)}_{c,1+}(x)\big|^2, \quad x \in \Delta_{c,1}, \tag{5.11}$$

[3]Given non-negative functions $A_c(z)$ and $B_c(z)$, we write $A_c(z) \lesssim B_c(z)$ (resp. $A_c(z) \sim B_c(z)$) as $c \to 0$ on K_c for some family of closed sets $\{K_c\}$, if there exists $\epsilon > 0$ such that $A_c(z) \leqslant CB_c(z)$ (resp. $C^{-1}A_c(z) \leqslant B_c(z) \leqslant CA_c(z)$) for all $z \in K_c$ and each $c \in [0, \epsilon]$, where C depends only on ϵ.

as $c \to 0$. Thus, (5.9), (5.11), and the maximum modulus principle applied to $\Upsilon^{(0)}_{c,1}(z)\phi_c(z)$ and $\Upsilon^{(1)}_{c,1}(z)/\phi_c(z)$ yield (5.7) with c^2 replaced by $A_{c,1}$. That is, we need to show that $A_{c,1} \sim c^2$ as $c \to 0$.

As is mentioned above, the sum $\Upsilon^{(0)}_{c,1}(z) + \Upsilon^{(1)}_{c,1}(z) + \Upsilon^{(2)}_{c,1}(z)$ is a rational function on $\overline{\mathbb{C}}$. Since it has only one pole, which is simple and located at infinity, it is a monic (see (5.4)) polynomial of degree 1. In particular, it holds that

$$\beta_{c,1} - \alpha_1 = 2\Upsilon^{(0)}_{c,1}(\beta_{c,1}) + \Upsilon^{(2)}_{c,1}(\beta_{c,1}) - 2\Upsilon^{(0)}_{c,1}(\alpha_1) - \Upsilon^{(2)}_{c,1}(\alpha_1), \tag{5.12}$$

where we used the fact that $\Upsilon^{(0)}_{c,1}(\gamma) = \Upsilon^{(1)}_{c,1}(\gamma) = \Upsilon_{c,1}(\gamma)$ for $\gamma \in \{\alpha_1, \beta_{c,1}\}$. Thus, it follows from (4.7) and (5.12) (lower bound) together with (5.9) and (5.11) (upper bound) that

$$c \lesssim 2\big|\Upsilon^{(0)}_{c,1}(\beta_{c,1})\big| + \big|\Upsilon^{(2)}_{c,1}(\beta_{c,1})\big| + 2\big|\Upsilon^{(0)}_{c,1}(\alpha_1)\big| + \big|\Upsilon^{(2)}_{c,1}(\alpha_1)\big| \lesssim A^{1/2}_{c,1} + A_{c,1} \lesssim A^{1/2}_{c,1}$$

as $c \to 0$, where we also used the fact that $A_{c,1} \to 0$ as $c \to 0$ for the last inequality. On the other hand, it holds that

$$\Upsilon^{(1)}_{c,2}(z) = -\frac{A_{c,2}}{B_{c,2} - B_{c,1}} + \frac{A_{c,1}A_{c,2}}{B_{c,2} - B_{c,1}}\frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

as $z \to \infty$ by the very definitions (5.3) and (2.4). Therefore, we can deduce from Cauchy's integral formula that

$$\frac{A_{c,1}A_{c,2}}{B_{c,2} - B_{c,1}} = \left|\frac{1}{2\pi \mathrm{i}}\int_{\Delta_{c,1}} \left(\Upsilon^{(1)}_{c,2+}(x) - \Upsilon^{(1)}_{c,2-}(x)\right) \mathrm{d}x\right| \leqslant \frac{\beta_{c,1} - \alpha_1}{\pi} \max_{\boldsymbol{x} \in \boldsymbol{\Delta}_{c,1}} \left|\Upsilon^{(1)}_{c,2}(\boldsymbol{x}) + Z\right| \tag{5.13}$$

for any complex number Z. Now, if we show that

$$\max_{\boldsymbol{x} \in \boldsymbol{\Delta}_{c,1}} \left|\Upsilon^{(1)}_{c,2}(\boldsymbol{x}) + \frac{A_{c,2}}{B_{c,2} - B_{c,1}}\right| \lesssim A^{1/2}_{c,1} \tag{5.14}$$

as $c \to 0$, inequalities (4.7) and (5.13) together with limits (2.7) will allow us to conclude that $A^{1/2}_{c,1} \lesssim c$ as $c \to 0$, which will finish the proof of (5.7). To prove (5.14), observe that

$$\Upsilon_{c,2}(\boldsymbol{z}) = \frac{A_{c,2}}{\chi_c(\boldsymbol{z}) - B_{c,2}} = \frac{A_{c,2}}{B_{c,1} - B_{c,2} + A_{c,1}\Upsilon^{-1}_{c,1}(\boldsymbol{z})} = \frac{A_{c,2}}{B_{c,2} - B_{c,1}} \frac{\Upsilon_{c,1}(\boldsymbol{z})}{\frac{A_{c,1}}{B_{c,2} - B_{c,1}} - \Upsilon_{c,1}(\boldsymbol{z})}$$

according to their very definition (5.3). Thus,

$$\Upsilon_{c,2}(\boldsymbol{z}) + \frac{A_{c,2}}{B_{c,2} - B_{c,1}} = \frac{A_{c,2}}{B_{c,2} - B_{c,1}} \frac{A_{c,1}}{A_{c,1} - (B_{c,2} - B_{c,1})\Upsilon_{c,1}(\boldsymbol{z})}.$$

The desired estimate (5.14) now follows from (5.11) and (2.7).

To prove the last claim of the lemma, observe that $\Upsilon^{(1)}_{c,1}(z) - (z - \alpha_1)$ is holomorphic in $\overline{\mathbb{C}}\setminus\Delta_{c,1}$ and

$$\big|\Upsilon^{(1)}_{c,1\pm}(x) - (x - \alpha_1)\big| \leqslant \max_{x \in \Delta_{c,1}} \big|\Upsilon^{(1)}_{c,1\pm}(x)\big| + \beta_{c,1} - \alpha_1 \lesssim c, \quad x \in \Delta_{c,1},$$

as $c \to 0$ by (4.7) and (5.11). The desired claim now follows from the maximum modulus principle. □

In our analysis, it will be convenient to apply Lemma 5.1 in the following form.

Lemma 5.2. *For each $0 < \delta \leqslant (\alpha_2 - \beta_1)/2$ fixed, it holds that*

$$\begin{cases} c^{-1}\big|\Upsilon^{(0)}_{c,1}(z)\big|,\ c^{-1}\big|\Upsilon^{(1)}_{c,1}(z)\big|,\ c^{-2}\big|\Upsilon^{(2)}_{c,1}(z)\big| \sim 1, \\ (1-c)^{-2}\big|\Upsilon^{(0)}_{c,2}(z)\big|,\ (1-c)^{-2}\big|\Upsilon^{(1)}_{c,2}(z)\big|,\ \big|\Upsilon^{(2)}_{c,2}(z)\big| \sim 1, \end{cases} \tag{5.15}$$

on $K_{c,\delta,1} := \{z : \operatorname{dist}(z, \Delta_{c,1}) \leqslant c\delta\}$ for all $c \in (0,1)$ and that

$$\begin{cases} c^{-2}\big|\Upsilon^{(0)}_{c,1}(z)\big|,\ \big|\Upsilon^{(1)}_{c,1}(z)\big|,\ c^{-2}\big|\Upsilon^{(2)}_{c,1}(z)\big| \sim 1, \\ (1-c)^{-1}\big|\Upsilon^{(0)}_{c,2}(z)\big|,\ (1-c)^{-2}\big|\Upsilon^{(1)}_{c,2}(z)\big|,\ (1-c)^{-1}\big|\Upsilon^{(2)}_{c,2}(z)\big| \sim 1, \end{cases} \tag{5.16}$$

on $K_{c,\delta,2} := \{z : \operatorname{dist}(z, \Delta_{c,2}) \leqslant (1-c)\delta\}$ for all $c \in (0,1)$, where the constants of proportionality depend only on δ.

Proof. We provide the proofs only for $\Upsilon_{c,1}(z)$, understanding that the arguments for $\Upsilon_{c,2}(z)$ are essentially identical. Recall that $\Upsilon_{c,1}(z)$ is a conformal map of $\mathfrak{R}_c$ onto $\overline{\mathbb{C}}$ that maps $\infty^{(0)}$ into 0 and $\infty^{(1)}$ into ∞. Let $r := \max\{|\alpha_1|, |\beta_2|\}$. Then it follows from (5.1) and (5.4) that

$$\big\{|z| < A_{c,1}/4(r+\delta)\big\} \subset \Upsilon_{c,1}^{(0)}\big(\{|z| > r+\delta\}\big) \quad \text{and} \quad \big\{|z| > 4(r+\delta)\big\} \subset \Upsilon_{c,1}^{(1)}\big(\{|z| > r+\delta\}\big).$$

Thus, it holds that

$$\frac{A_{c,1}}{4(r+\delta)} \leqslant |\Upsilon_{c,1}(z)| \leqslant 4(r+\delta) \quad \text{for all} \quad z \in K_{c,\delta,1} \cup K_{c,\delta,2}.$$

Since $A_{c,1} \to ((\beta_1 - \alpha_1)/4)^2$ by the limit analogous to the one for $A_{c,2}$ in (2.7), this establishes the desired bounds in (5.15) and (5.16) for all $c \in [\epsilon, 1)$ and any $\epsilon > 0$ fixed with the constants of proportionality dependent on ϵ and δ. On the other hand, the bounds for $c \in (0, \epsilon]$ readily follow from (5.7) and (5.8) as

$$1 \leqslant |\phi_c(z)| \leqslant 4\frac{c\delta + \beta_{c,1} - \alpha_1}{\beta_{c,1} - \alpha_1} < 4 + \frac{\delta}{\beta_2 - \alpha_1} \quad \text{and} \quad c|\phi_c(z)| \sim |z - \alpha_1| \tag{5.17}$$

on $K_{c,\delta,1}$ and $K_{c,\delta,2}$, respectively, as $c \to 0$ by elementary estimates and (4.7). The estimates of $\Upsilon_{c,2}^{(k)}(z)$ can be verified similarly. □

Let a function $\Pi_c(z)$ be defined on $\mathfrak{R}_c$ analogously to the way $\Pi_{\vec{n}}(z)$ was defined on $\mathfrak{R}_{\vec{n}}$ just before Theorem 3.3. Further, let $\Pi_{c,i}(z)$, $i \in \{1,2\}$, be rational functions on $\mathfrak{R}_c$ with the divisors and normalization given by

$$(\Pi_{c,i}) = \infty^{(0)} + \infty^{(i)} + 2\infty^{(3-i)} - \mathcal{D}_c \quad \text{and} \quad \Pi_{c,i}^{(i)}(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right), \tag{5.18}$$

where $\mathcal{D}_c$ is the divisor of the ramification points of $\mathfrak{R}_c$, see Proposition 4.2.

Lemma 5.3. *It holds that*

$$(-1)^{3-i}(w_{c,1}w_{c,2})(z)\Pi_{c,3-i}(z) = \begin{cases} \left(\Upsilon_{c,i}^{(2)} - \Upsilon_{c,i}^{(1)}\right)(z), & z \in \mathfrak{R}_c^{(0)}, \\ \left(\Upsilon_{c,i}^{(0)} - \Upsilon_{c,i}^{(2)}\right)(z), & z \in \mathfrak{R}_c^{(1)}, \\ \left(\Upsilon_{c,i}^{(1)} - \Upsilon_{c,i}^{(0)}\right)(z), & z \in \mathfrak{R}_c^{(2)}, \end{cases} \tag{5.19}$$

for $i \in \{1,2\}$ and

$$(w_{c,1}w_{c,2})(z)\Pi_c(z) = \begin{cases} \left(\Upsilon_{c,2}^{(2)}\Upsilon_{c,1}^{(1)} - \Upsilon_{c,2}^{(1)}\Upsilon_{c,1}^{(2)}\right)(z), & z \in \mathfrak{R}_c^{(0)}, \\ \left(\Upsilon_{c,2}^{(0)}\Upsilon_{c,1}^{(2)} - \Upsilon_{c,2}^{(2)}\Upsilon_{c,1}^{(0)}\right)(z), & z \in \mathfrak{R}_c^{(1)}, \\ \left(\Upsilon_{c,2}^{(1)}\Upsilon_{c,1}^{(0)} - \Upsilon_{c,2}^{(0)}\Upsilon_{c,1}^{(1)}\right)(z), & z \in \mathfrak{R}_c^{(2)}. \end{cases} \tag{5.20}$$

Moreover, it holds that

$$\Pi_c^{(0)}(z) = (1 + o(1))\frac{\psi^{(2)}(z)}{w_2(z)}\frac{z - \alpha_1 + O(c)}{w_{c,1}(z)} = (1 + o(1))\frac{\psi^{(2)}(z)}{w_2(z)} \tag{5.21}$$

as $c \to 0$, where the first relation holds uniformly in $\overline{\mathbb{C}}$ (that is, including the traces on $\Delta_{c,1} \cup \Delta_2$) and the second one locally uniformly in $\overline{\mathbb{C}}\setminus\Delta_{0,1}$.

Proof. Representations (5.19) and (5.20) can be easily verified by observing that the right-hand sides are continuous across $\mathbf{\Delta}_{c,1}$ and $\mathbf{\Delta}_{c,2}$ and by comparing the zero/pole divisors and the normalizations of the left-hand and right-hand sides, see (2.5), (5.3), and (5.18). Asymptotic formula (5.21) follows immediately from the first relation in (5.20), asymptotic formulae (5.6) and (5.7), and the last claim of Lemma 5.1. □

5.3. **Proof of Proposition 3.1.** It was shown in [45, Section 6] that the Szegő function $S_c(z)$ satisfying (3.7) is given by

$$S_c(z) := \exp\left\{\frac{1}{6\pi\mathrm{i}}\sum_{i=1}^{2}\int_{\mathbf{\Delta}_{c,i}} \log(\rho_i w_{c,i+})(s)C_z(s)\right\},$$

where $C_z(s)$ is the third kind differential on $\mathfrak{R}_c$ with three simple poles at z, z_1, z_2 that have the same natural projection z and respective residues $-2, 1, 1$. Limit (3.8) was in fact proven in [45, Section 7]. Thus, it only

remains to show the validity of (3.9) and (3.10). In order to do that we shall use an alternative construction of $S_c(z)$ that is more amenable to asymptotic analysis.

Since we are interested in what happens when $c \to 0$, we shall assume that $c \leqslant \min\{1/2, c^{**}\}$ (the choice of $1/2$ is rather arbitrary, but convenient to use in (4.7)). Set

$$D_{c,1}(z) := \left(\frac{z - (\beta_{c,1} + \alpha_{c,1})/2 + w_{c,1}(z)}{2w_{c,1}(z)}\right)^{1/2}, \quad z \in \overline{\mathbb{C}}\backslash\Delta_{c,1},$$

where we take the branch of the square root such that $D_{c,1}(z)$ is holomorphic and non-vanishing in the domain of the definition and has value 1 at infinity. The traces of $D_{c,1}(z)$ on $\Delta_{c,1}$ satisfy

$$|D_{c,1\pm}(x)|^2 = \big(D_{c,1+}D_{c,1-}\big)(x) = \frac{\beta_{c,1} - \alpha_1}{4|w_{c,1}(x)|} = \frac{\mathrm{i}}{4}\frac{\beta_{c,1} - \alpha_1}{w_{c,1+}(x)}, \quad x \in \Delta_{c,1}. \tag{5.22}$$

Let $\delta > 0$ be as in Lemma 5.2, that is, $\delta \leqslant (\alpha_2 - \beta_1)/2$. Then it follows from (4.7) that $\delta c \leqslant |\Delta_{c,1}|/8$. Using (4.7) once more together with our assumption that $c \leqslant 1/2$, we get that

$$\begin{cases} \sqrt{3(\alpha_2 - \beta_1)} < |w_{c,1}(s)|/(c\sqrt{\delta}) < 3\sqrt{\beta_2 - \alpha_1}, & |s - \alpha_1| = \delta c,\ |s - \beta_{c,1}| = \delta c, \\ \sqrt{\delta(\alpha_2 - \beta_1)} < |w_{c,1\pm}(x)|/c < 8(\beta_2 - \alpha_1), & \alpha_1 + \delta c \leqslant x \leqslant \beta_{c,1} - \delta c, \end{cases} \tag{5.23}$$

(the constants in the above inequalities are in no way sharp, but sufficient for our purposes). Therefore, (5.23) and similar straightforward estimates of $|2z - \alpha_1 - \beta_{c,1}|$ using (4.7) as well as (5.22) and the maximum modulus principle for holomorphic functions applied to both $D_{c,1}(z)$ and $D_{c,1}^{-1}(z)$ yield that

$$\begin{cases} |D_{c,1}(s)| \sim \delta^{-1/4}, & |s - \alpha_1| = \delta c,\ |s - \beta_{c,1}| = \delta c, \\ 1 \lesssim |D_{c,1}(z)| \lesssim \delta^{-1/4}, & 0 < \delta c \leqslant \operatorname{dist}(z, \{\alpha_1, \beta_{c,1}\}), \end{cases} \tag{5.24}$$

uniformly on the respective sets, where the constants of proportionality do not depend on c, δ. Additionally, since $\beta_{c,1} \to \alpha_1$ as $c \to 0$ and therefore $w_{c,1}(z) = z - \alpha_1 + o(1)$ locally uniformly in $\mathbb{C}\backslash\Delta_{0,1}$ as $c \to 0$, it holds locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ that

$$D_{c,1}(z) = 1 + o(1) \quad \text{as} \quad c \to 0. \tag{5.25}$$

Now, let $D_{c,\rho_1}(z)$ be the Szegő function of the restriction of $\rho_1(x)$ to $\Delta_{c,1}$ normalized to have value 1 at infinity. That is,

$$D_{c,\rho_1}(z) = \exp\left\{\frac{w_{c,1}(z)}{2\pi\mathrm{i}}\int_{\Delta_{c,1}} \frac{\log\rho_1(x)}{z - x}\frac{\mathrm{d}x}{w_{c,1+}(x)} - \int_{\Delta_{c,1}} \frac{\log\rho_1(x)}{w_{c,1+}(x)}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\right\}, \tag{5.26}$$

$z \in \overline{\mathbb{C}}\backslash\Delta_{c,1}$, where we set $\log\rho_1(x) := \log\mu_1'(x) + \log(2\pi) - \pi\mathrm{i}/2$, see (3.2) and recall that $\mu_1'(x)$ is positive on Δ_1. Observe that

$$-\int_{\Delta_{c,1}} \frac{1}{w_{c,1+}(x)}\frac{\mathrm{d}x}{\pi\mathrm{i}} = 1 \quad \text{and} \quad \frac{1}{\pi\mathrm{i}}\int_{\Delta_{c,1}} \frac{1}{z - x}\frac{\mathrm{d}x}{w_{c,1+}(x)} = -\frac{1}{w_{c,1}(z)}, \tag{5.27}$$

by Cauchy's theorem and integral formula. Hence, $D_{c,\rho_1}(z) = D_{c,\mu_1'}(z)$ is a holomorphic and non-vanishing function in $\overline{\mathbb{C}}\backslash\Delta_{c,1}$ with continuous and conjugate-symmetric traces on $\Delta_{c,1}$ that satisfy

$$\rho_1(x)|D_{c,\rho_1\pm}(x)|^2 = \big(\rho_1 D_{c,\rho_1+}D_{c,\rho_1-}\big)(x) = G_{c,\rho_1} := \exp\left\{-\int_{\Delta_{c,1}} \frac{\log\rho_1(x)}{w_{c,1+}(x)}\frac{\mathrm{d}x}{\pi\mathrm{i}}\right\}, \tag{5.28}$$

according to Plemelj-Sokhotski formulae. Now, analyticity of $\rho_1(x)$ in a neighborhood of Δ_1 implies that $\max_{x\in\Delta_{c,1}} |\rho_1(x)/\rho_1(\alpha_1) - 1| \to 0$ as $c \to 0$. Combining this estimate with (5.27) yields that

$$-\int_{\Delta_{c,1}} \frac{\log\rho_1(x)}{w_{c,1+}(x)}\frac{\mathrm{d}x}{\pi\mathrm{i}} = \log\rho_1(\alpha_1) - \int_{\Delta_{c,1}} \frac{\log(\rho_1(x)/\rho_1(\alpha_1))}{w_{c,1+}(x)}\frac{\mathrm{d}x}{\pi\mathrm{i}} = \log\rho_1(\alpha_1) + o(1)$$

when $c \to 0$ as well as that

$$\begin{aligned} \frac{w_{c,1}(z)}{\pi\mathrm{i}}\int_{\Delta_{c,1}} \frac{\log\rho_1(x)}{z - x}\frac{\mathrm{d}x}{w_{c,1+}(x)} &= \frac{w_{c,1}(z)}{\pi\mathrm{i}}\int_{\Delta_{c,1}} \frac{\log(\rho_1(x)/\rho_1(\alpha_1))}{z - x}\frac{\mathrm{d}x}{w_{c,1+}(x)} - \log\rho_1(\alpha_1) \\ &= o(1) - \log\rho_1(\alpha_1) \end{aligned}$$

uniformly on compact subsets of $\mathbb{C}\backslash\Delta_{0,1}$ when $c \to 0$. Thus, it follows from the maximum modulus principle that

$$D_{c,\rho_1}(z) = 1 + o(1) \quad \text{and} \quad G_{c,\rho_1} = \big(1 + o(1)\big)\rho_1(\alpha_1) \tag{5.29}$$

locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ as $c \to 0$. One can also see from its very definition in (5.28) combined with the second formula of (5.29) that G_{c,ρ_1} extends to a non-vanishing continuous function of $c \in [0,1]$ (it is constant for all $c \geqslant c^*$). This observation as well as (5.28) combined with positivity of $\rho_1(x)$ on Δ_1 show that $|D_{c,\rho_1\pm}(x)| \sim 1$ uniformly on $\Delta_{c,1}$ for all $c \in (0,1)$. Then the maximum modulus principle for holomorphic functions applied to $D_{c,\rho_1}(z)$ and $D_{c,\rho_1}^{-1}(z)$ yields that

$$G_{c,\rho_1}, |D_{c,\rho_1}(z)| \sim 1, \tag{5.30}$$

uniformly in $\overline{\mathbb{C}}$ for all $c \in (0,1)$ (notice that $|D_{c,\rho_1}(z)|$ is a continuous function on the entire sphere $\overline{\mathbb{C}}$ independent of c when $c \geqslant c^*$).

Let $\Gamma_{c,2} := \chi_c(\mathbf{\Delta}_{c,2})$, which are clockwise oriented analytic Jordan curves (recall that $\mathbf{\Delta}_{c,2}$ is oriented so that $\mathfrak{R}_c^{(0)}$ remains on the left when $\mathbf{\Delta}_{c,2}$ is traversed in the positive direction and that $\chi_c(z)$ is conformal on $\mathfrak{R}_c$ and maps $\infty^{(0)}$ into ∞). The function

$$S_{c,2}(z) := \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{c,2}} \frac{\log(D_{c,1}D_{c,\rho_1})\big(\pi\big(\chi_c^{-1}(s)\big)\big)}{s-\chi_c(z)}\mathrm{d}s\right\} \tag{5.31}$$

is holomorphic and bounded in $\mathfrak{R}_c\backslash\mathbf{\Delta}_{c,2}$ and has value 1 at $\infty^{(0)}$. It follows from Plemelj-Sokhotski formulae that

$$S_{c,2-}(\boldsymbol{x}) = S_{c,2+}(\boldsymbol{x})(D_{c,1}D_{c,\rho_1})(x), \quad \boldsymbol{x} \in \mathbf{\Delta}_{c,2}. \tag{5.32}$$

Observe also that $(D_{c,1}D_{c,\rho_1})(\pi(z))$ is holomorphic in a neighborhood of $\mathbf{\Delta}_{c,2}$. Therefore, $S_{c,2}(z)$ can be continued analytically across each side of $\mathbf{\Delta}_{c,2}$. In fact, this continuation has an integral representation similar to (5.31), where one simply needs to homologously deform $\Gamma_{c,2}$ within the domain of holomorphy of $(D_{c,1}D_{c,\rho_1})\big(\pi\big(\chi_c^{-1}(s)\big)\big)$. Moreover, it holds that

$$S_{c,2}(z) = 1 + o(1) \quad \text{as} \quad c \to 0 \quad \text{and} \quad |S_{c,2}(z)| \sim 1, \quad c \in (0, c^{**}], \tag{5.33}$$

uniformly on $\mathfrak{R}_c$ (again, this means including the traces on $\mathbf{\Delta}_{c,2}$). Indeed, observe that the analytic curves $\Gamma_{c,2}$ approach the circle $\big\{|z - B_{0,2}| = (\beta_2 - \alpha_2)/4\big\}$ by (2.7) and (5.2). Let $\delta > 0$ be small enough so that the integrand in (5.31) is analytic in a neighborhood of the closure of the annular domain bounded by $\Gamma_{c,2}$ and $C_\delta := \{|z - B_{0,2}| = 2\delta + (\beta_2 - \alpha_2)/4\}$. Assuming that C_δ is clockwise oriented, it follows from Cauchy's theorem that $\Gamma_{c,2}$ can be replaced by C_δ whenever $z \in \mathfrak{R}_c^{(2)}$, i.e., whenever $\chi_c(z)$ is interior or on $\Gamma_{c,2}$. Then it trivially holds that

$$|S_{c,2}(z)| \leqslant \exp\left\{\frac{|C_\delta|}{2\pi\delta}\max_{s\in C_\delta}\big|\log(D_{c,1}D_{c,\rho_1})\big(\pi\big(\chi_c^{-1}(s)\big)\big)\big|\right\},$$

for $z \in \mathfrak{R}_c^{(2)}$, where $|C_\delta|$ is the arclength of C_δ. The desired limit in $\mathfrak{R}_c^{(2)}$ now follows from (5.25) and (5.29) while the uniform boundedness follows from (5.24) and (5.30). Clearly, the estimates in the remaining part of $\mathfrak{R}_c$ can be obtained analogously by deforming $\Gamma_{c,2}$ into the circles $\{|z - B_{0,2}| = -2\delta + (\beta_2 - \alpha_2)/4\}$.

As a part of the final piece of our construction, let $\Gamma_{c,1} := \chi_c(\mathbf{\Delta}_{c,1})$. Similarly to $\Gamma_{c,2}$, these are clockwise oriented analytic Jordan curves that collapse into a point $B_{0,1}$ by (2.7) and (5.2). Let

$$S_{c,1}(z) := \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{c,1}} \frac{\log\big[S_{\rho_2}\big(\pi\big(\chi_c^{-1}(s)\big)\big)/S_{\rho_2}(\infty)\big]}{s-\chi_c(z)}\mathrm{d}s\right\}, \tag{5.34}$$

which is a holomorphic and bounded function on $\mathfrak{R}_c$ that has value 1 at $\infty^{(1)}$ and whose traces on $\mathbf{\Delta}_{c,1}$ are continuous and satisfy

$$S_{c,1-}(\boldsymbol{x}) = S_{c,1+}(\boldsymbol{x})S_{\rho_2}(x)/S_{\rho_2}(\infty), \quad \boldsymbol{x} \in \mathbf{\Delta}_{c,1}, \tag{5.35}$$

by Plemelj-Sokhotski formulae. Notice that all the observation about analytic continuations (contour deformation) made for $S_{c,2}(z)$ apply to $S_{c,1}(z)$ as well. Since the Cauchy kernel is integrated against the pullback of a fixed function $S_{\rho_2}(z)/S_{\rho_2}(\infty)$ from $\Delta_{c,1}$ while the curves $\Gamma_{c,1}$ collapse into a point, straightforward estimates of Cauchy integrals as well as analytic continuation (deformation of a contour) technique yield that

$$S_{c,1}(z) = 1 + o(1) \quad \text{as} \quad c \to 0 \quad \text{and} \quad |S_{c,1}(z)| \sim 1, \quad c \in (0, c^{**}], \tag{5.36}$$

locally uniformly on $(\mathfrak{R}_c^{(0)} \cup \mathfrak{R}_c^{(2)})\backslash\Delta_{c,1}$ and uniformly on $\mathfrak{R}_c$, respectively. To examine what happens to $S_{c,1}(z)$ on $\mathfrak{R}_c^{(1)}$, given $\epsilon > 0$, let $C_\epsilon := \{|z - B_{c,1}| = \epsilon\}$ be clockwise oriented circle. It follows from (5.2) that the

Jordan curve $\chi_c^{-1}(C_\epsilon)$ belongs to $\mathfrak{R}_c^{(0)}$ and is homologous to $\mathbf{\Delta}_{c,1}$ for all c sufficiently small. A straightforward computation shows that

$$\int_{C_\epsilon} \frac{\log\left[S_{\rho_2}\left(\pi\left(\chi_c^{-1}(s)\right)\right)/S_{\rho_2}(\infty)\right]}{s-B_{c,1}}\frac{\mathrm{d}s}{2\pi\mathrm{i}} = \log\frac{S_{\rho_2}(\alpha_1)}{S_{\rho_2}(\infty)} + O\left(\max_{z\in\pi\left(\chi_c^{-1}(C_\epsilon)\right)}\left|\log\frac{S_{\rho_2}(z)}{S_{\rho_2}(\alpha_1)}\right|\right). \tag{5.37}$$

It further follows from (2.7) and (5.2) that Jordan curves $\pi\left(\chi_c^{-1}(C_\epsilon)\right)$ converge to the analytic Jordan curve $(\varphi_2+B_{0,2})^{-1}(C_\epsilon)$ (recall that $\varphi_2(z)=\varphi^{(0)}(z)$) and the latter curves collapse into a point α_1 as $\epsilon\to 0$. Hence, by taking the limit as $c\to 0$ and then the limit as $\epsilon\to 0$ of the $O(\cdot)$ in (5.37) gives 0. Therefore, analytic continuation (deformation of a contour) technique and (5.34) imply that

$$\lim_{c\to 0} S_{c,1}\left(\infty^{(1)}\right) = \lim_{c\to 0}\exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{C_\epsilon}\frac{\log\left[S_{\rho_2}\left(\pi\left(\chi_c^{-1}(s)\right)\right)/S_{\rho_2}(\infty)\right]}{s-B_{c,1}}\mathrm{d}s\right\} = \frac{S_{\rho_2}(\alpha_1)}{S_{\rho_2}(\infty)}. \tag{5.38}$$

Finally, we are ready to state an alternative formula for the functions $S_c(z)$ when $c\leqslant c^{**}$. Since relations (3.7) characterize $S_c(z)$ up to multiplication by a cubic root of unity, it follows from the normalization of $D_{c,1}(z)$ and $D_{c,\rho_1}(z)$ at infinity, the normalization of $S_{c,1}(z)$ and $S_{c,2}(z)$ at $\infty^{(0)}$, and relations (3.4), (5.22), (5.28), (5.32), and (5.35) that

$$\frac{S_c(z)}{S_c\left(\infty^{(0)}\right)} = \left(S_{c,1}S_{c,2}\right)(z)\begin{cases} S_{\rho_2}^{-1}(\infty)(D_{c,1}D_{c,\rho_1}S_{\rho_2})(z), & z\in\mathfrak{R}_c^{(0)}\backslash\left(\mathbf{\Delta}_{c,1}\cup\mathbf{\Delta}_{c,2}\right),\\ \mathrm{i}\frac{\beta_{c,1}-\alpha_1}{4}G_{c,\rho_1}(D_{c,1}D_{c,\rho_1})^{-1}(z), & z\in\mathfrak{R}_c^{(1)}\backslash\mathbf{\Delta}_{c,1},\\ (S_{\rho_2}(\infty)S_{\rho_2}(z))^{-1}, & z\in\mathfrak{R}_c^{(2)}\backslash\mathbf{\Delta}_{c,2}.\end{cases} \tag{5.39}$$

Now, it follows from (5.33) and (5.36) that

$$S_c^{(2)}(z)/S_c^{(0)}(\infty) = (1+o(1))(S_{\rho_2}(\infty)S_{\rho_2}(z))^{-1} \tag{5.40}$$

uniformly in $\overline{\mathbb{C}}$ (that is, including the traces on Δ_2) as $c\to 0$. Similarly, it follows from (5.25), (5.29), (5.33), and (5.36) that

$$S_c^{(0)}(z)/S_c^{(0)}(\infty) = (1+o(1))S_{\rho_2}(z)/S_{\rho_2}(\infty) \tag{5.41}$$

locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ as $c\to 0$. Further, it follows from the middle relation in (3.7) and the last two asymptotic formulae that

$$\frac{S_c^{(1)}(z)}{S_c^{(0)}(\infty)} = \frac{1}{S_c^{(0)}(\infty)^3}\frac{S_c^{(0)}(\infty)}{S_c^{(0)}(z)}\frac{S_c^{(0)}(\infty)}{S_c^{(2)}(z)} = (1+o(1))\frac{S_{\rho_2}(\infty)^2}{S_c^{(0)}(\infty)^3} \tag{5.42}$$

locally uniformly in $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ as $c\to 0$. Since relations (5.40)–(5.42) also provide asymptotics for the ratios of $S_c^{(k)}(\infty)/S_c^{(0)}(\infty)$, the limits in (3.9) easily follow. In fact, we deduce from (5.40) and (5.42) that

$$S_c^{(2)}(\infty) = (1+o(1))\frac{S_c^{(0)}(\infty)}{S_{\rho_2}^2(\infty)} \quad\text{and}\quad S_c^{(1)}(\infty) = (1+o(1))\left(\frac{S_{\rho_2}(\infty)}{S_c^{(0)}(\infty)}\right)^2. \tag{5.43}$$

On the other hand, it follows from the normalization $D_{c,1}(z)$ and $D_{c,\rho_1}(z)$ at infinity, (3.2), (4.8), (5.29), (5.33), and (5.38) that

$$\lim_{c\to 0}\frac{1}{c}\frac{S_c^{(1)}(\infty)}{S_c^{(0)}(\infty)} = \frac{2\pi\mu_1'(\alpha_1)|w_2(\alpha_1)|S_{\rho_2}(\alpha_1)}{S_{\rho_2}(\infty)}. \tag{5.44}$$

Plugging in the second asymptotic formula of (5.43) into (5.44) yields the first limit in (3.10). The other two now follow from (5.43).

5.4. **Auxiliary Estimates, II.** The sole purpose of this subsection is to state the following lemma that follows from (5.24), (5.30), (5.33), (5.36), (5.39), as well as the analogous results for $c\in[c^*,1)$ and $c\to 1$.

Lemma 5.4. *It holds uniformly on $\mathfrak{R}_c$ for all $c\in(0,1)$ that*

$$c\left|\frac{S_c^{(0)}(\infty)}{S_c^{(1)}(\infty)}\right|,\ (1-c)\left|\frac{S_c^{(0)}(\infty)}{S_c^{(2)}(\infty)}\right| \sim 1.$$

Moreover, let $\delta > 0$ be such that $0 < \delta c \leqslant |\Delta_{c,1}|/8$ and $0 < \delta(1-c) \leqslant |\Delta_{c,2}|/8$ for all $c \in (0,1)$. Then it holds for all $c \in (0,1)$ that

$$\left|\frac{S_c^{(0)}(z)}{S_c^{(0)}(\infty)}\right| \sim \delta^{-1/4}$$

uniformly on each circle $\{|z-\alpha_1| = \delta c\}$, $\{|z-\beta_{c,1}| = \delta c\}$, $\{|z-\alpha_{c,2}| = \delta(1-c)\}$, and $\{|z-\beta_2| = \delta(1-c)\}$; and

$$1 \lesssim \left|\frac{S_c^{(0)}(z)}{S_c^{(0)}(\infty)}\right| \lesssim \delta^{-1/4}$$

uniformly on $\{\delta c \leqslant \operatorname{dist}(z, \{\alpha_1, \beta_{c,1}\})\}$ and $\{\delta(1-c) \leqslant \operatorname{dist}(z, \{\alpha_{c,2}, \beta_2\})\}$. In addition, it holds for all $c \in (0,1)$ and each $i \in \{1,2\}$ that

$$\left|\frac{S_c^{(i)}(z)}{S_c^{(i)}(\infty)}\right| \sim \delta^{1/4}$$

uniformly on circles $\{|z-\alpha_{c,i}| = \delta(i-1-(-1)^i c)\}$ and $\{|z-\beta_{c,i}| = \delta(i-1-(-1)^i c)\}$; and

$$\delta^{1/4} \lesssim \left|\frac{S_c^{(i)}(z)}{S_c^{(i)}(\infty)}\right| \lesssim 1$$

uniformly on $\{\delta(i-1-(-1)^i c) \leqslant \operatorname{dist}(z, \{\alpha_{c,i}, \beta_{c,i}\})\}$.

6. Proof of Theorem 3.1

Let $\alpha_1 \leqslant x_{\vec{n},1} < x_{\vec{n},2} < \ldots < x_{\vec{n},n_1} \leqslant \beta_1$ be the zeros of $P_{\vec{n}}(x)$ on Δ_1. Then we can write

$$P_{\vec{n}}(x) =: P_{\vec{n},1}(x)P_{\vec{n},2}(x), \quad P_{\vec{n},1}(x) := \prod_{i=1}^{n_1}(x - x_{\vec{n},i}).$$

Observe that the polynomials $\{P_{\vec{n},1}(x)\}_{\vec{n}\in\mathcal{N}_0}$ form a normal family in a neighborhood of Δ_2. As $\deg(P_{\vec{n},2}) = n_2$ and it holds that

$$\int x^l P_{\vec{n},2}(x)P_{\vec{n},1}(x)\mathrm{d}\mu_2(x) = 0, \quad l \in \{0,\ldots,n_2-1\},$$

by (1.5), the asymptotics of $P_{\vec{n},2}(z)$ follows from [12, Theorem 2.7]. Namely, it holds that

$$P_{\vec{n},2}(z) = (1+o(1))\big(S_{\rho_2}(z)/S_{\rho_2}(\infty)\big)\left(\prod_{i=1}^{n_1} S(z;x_{\vec{n},i})\right)\varphi_2^{n_2}(z) \tag{6.1}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\backslash\Delta_2$. Thus, to obtain the asymptotic formula for $P_{\vec{n}}(z)$, we only need to show that all the zeros $\{x_{\vec{n},i}\}_{i=1}^{n_1}$ approach α_1. We shall do it in a slightly more general setting.

Lemma 6.1. *Suppose that μ_2 is an absolutely continuous Szegő measure, i.e., $\int_{\Delta_2}\log\mu_2'(x)\mathrm{d}x > -\infty$, and that $\mathcal{N}_0$ is any marginal sequence, that is, $n_1/n_2 \to 0$ as $|\vec{n}| \to \infty$ for $\vec{n} \in \mathcal{N}_0$. Assuming formula (6.1) remains valid, it holds that $x_{\vec{n},n_1} \to \alpha_1$ as $|\vec{n}| \to \infty$ for $\vec{n} \in \mathcal{N}_0$. Moreover,*

$$\lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0}\ \lim_{z\to\infty}\left(\frac{P_{\vec{n}+\vec{e}_i}(z)}{P_{\vec{n}}(z)} - z\right) = -B_{0,i}, \quad i \in \{1,2\}. \tag{6.2}$$

Proof. Assume to the contrary that there exists $\epsilon > 0$ such that $\alpha_1 + \epsilon \leqslant x_{\vec{n},n_1}$ along some subsequence $\mathcal{N}' \subset \mathcal{N}_0$. Let $\rho_{\vec{n},1}(x) := P_{\vec{n},1}(x)/(x - x_{\vec{n},n_1})$. Then it follows from (1.5) that

$$\int_{\alpha_1}^{x_{\vec{n},n_1}} \rho_{\vec{n},1}^2(x)|P_{\vec{n},2}(x)|(x_{\vec{n},n_1} - x)\mathrm{d}\mu_1(x) = \int_{x_{\vec{n},n_1}}^{\beta_1} \rho_{\vec{n},1}^2(x)|P_{\vec{n},2}(x)|(x - x_{\vec{n},n_1})\mathrm{d}\mu_1(x), \tag{6.3}$$

(since all the zeros of $P_{\vec{n},2}(x)$ belong to Δ_2, it has a constant sign on Δ_1). As the zeros of the monic polynomial $P_{\vec{n},1}(z)$ belong to Δ_1, we have that $|P_{\vec{n},1}(x)| \leqslant |\beta_1 - \alpha_1|^{n_1}$, $x \in \Delta_1$. Moreover, since each $S(z;x_0)$ is a non-vanishing function in $\overline{\mathbb{C}}\backslash\Delta_2$, compactness of Δ_1 implies that there exists a constant $C_1 > 1$ such that $C_1^{-1} \leqslant |S(x;x_0)| \leqslant C_1$ for any $x, x_0 \in \Delta_1$. Therefore, we can deduce from (6.1) that

$$\int_{x_{\vec{n},n_1}}^{\beta_1} \rho_{\vec{n},1}^2(x)|P_{\vec{n},2}(x)|(x - x_{\vec{n},n_1})\mathrm{d}\mu_1(x) \leqslant C_2^{n_1}|\varphi_2(\alpha_1+\epsilon)|^{n_2} \tag{6.4}$$

for some absolute constant $C_2 > 0$. On the other hand, by restricting the interval of integration from $[\alpha_1, x_{\vec{n},n_1}]$ to $[\alpha_1, \alpha_1 + \epsilon/2]$ and then using (6.1), the lower estimate of the Szegő functions $S(z;x_0)$, the facts that $\mu_1'(x)$ is non-vanishing and $|\varphi_2(x)|$ is decreasing for $x < \alpha_2$ we get that

$$\begin{aligned}(6.5)\quad \int_{\alpha_1}^{x_{\vec{n},n_1}} \rho_{\vec{n},1}^2(x)|P_{\vec{n},2}(x)|(x_{\vec{n},n_1}-x)\mathrm{d}\mu_1(x) &\geqslant C_3^{n_1}|\varphi_2(\alpha_1+\epsilon/2)|^{n_2}\int_{\alpha_1}^{\alpha_1+\epsilon/2}\rho_{\vec{n},1}^2(x)\mathrm{d}x\\ &\geqslant C_3^{n_1}\min_{\vec{n}\in\mathcal{N}'}\left(\int_{\alpha_1}^{\alpha_1+\epsilon/2}L_{n_1-1}^2(x)\mathrm{d}x\right)|\varphi_2(\alpha_1+\epsilon/2)|^{n_2} \geqslant C_4^{n_1}|\varphi_2(\alpha_1+\epsilon/2)|^{n_2}\end{aligned}$$

for some constants $C_3, C_4 > 0$ that might depend on ϵ, but are independent of $\vec{n}$, where $L_n(x)$ is the n-th monic orthogonal polynomial with respect to $\mathrm{d}x$ on $[\alpha_1, \alpha_1+\epsilon/2]$ (rescaled Legendre polynomial) and the last estimate follows from [37, Table 18.3.1]. Since $n_1/n_2 \to 0$ and $|\varphi_2(x)|$ is decreasing on $(-\infty, \alpha_2)$, we have that

$$C_4^{n_1/n_2}|\varphi_2(\alpha_1+\epsilon/2)| > C_2^{n_1/n_2}|\varphi_2(\alpha_1+\epsilon)|$$

for all $|\vec{n}|$ large, $\vec{n}\in\mathcal{N}_0$. Hence, the above estimate shows that (6.4)–(6.5) are incompatible with (6.3). Thus, it indeed holds that $x_{\vec{n},n_1} \to \alpha_1$ as $|\vec{n}|\to\infty$, $\vec{n}\in\mathcal{N}_0$. Further, it holds that

$$\lim_{z\to\infty}\left(\frac{P_{\vec{n}+\vec{e}_1,1}(z)}{P_{\vec{n},1}(z)} - z\right) = -\sum_{i=1}^{n_1+1} x_{\vec{n}+\vec{e}_1,i} + \sum_{i=1}^{n_1} x_{\vec{n},i} = -\alpha_1 + o(1) - \sum_{i=1}^{n_1}\left(x_{\vec{n}+\vec{e}_1,i+1} - x_{\vec{n},i}\right).$$

It is known that the zeros of $P_{\vec{n}}(z)$ and $P_{\vec{n}+\vec{e}_1}(z)$ interlace, see for example [8, Lemma A.2]. Therefore,

$$0 \leqslant \sum_{i=1}^{n_1}\left(x_{\vec{n}+\vec{e}_1,i+1} - x_{\vec{n},i}\right) \leqslant x_{\vec{n}+\vec{e}_1,n_1} - x_{\vec{n},1} = o(1),$$

where the last conclusion follows from the fact that $x_{\vec{n},1}, x_{\vec{n}+\vec{e}_1,n_1} \to \alpha_1$ (observe that $\{\vec{n}+\vec{e}_1 : \vec{n}\in\mathcal{N}_0\}$ is also a marginal sequence). Thus,

$$(6.6)\qquad \lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0}\lim_{z\to\infty}\left(\frac{P_{\vec{n}+\vec{e}_1,1}(z)}{P_{\vec{n},1}(z)} - z\right) = -\alpha_1.$$

Furthermore, it follows from the explicit definition (3.5) that

$$S^2(z;x_0) = \frac{1 - \frac{B_{0,2}+\varphi_2(x_0)}{z} + O(z^{-2})}{1 - \frac{B_{0,2}+A_{0,2}\varphi_2^{-1}(x_0)}{z} + O(z^{-2})}\,\frac{1 - \frac{B_{0,2}}{z} + O(z^{-2})}{1 - \frac{x_0}{z}},$$

where we used (2.6) to get that $\varphi_2(z) = z - B_{0,2} + O(z^{-1})$ as $z\to\infty$. Since

$$(6.7)\qquad B_{0,2} + \varphi_2(x_0) - x_0 - A_{0,2}\varphi_2^{-1}(x_0) = 2(B_{0,2} + \varphi_2(x_0) - x_0),$$

we have that $S(z;x_0) = 1 - (B_{0,2}+\varphi_2(x_0)-x_0)z^{-1} + O(z^{-2})$ as $z\to\infty$. Now, interlacing of the zeros $\{x_{\vec{n}+\vec{e}_1,i}\}_{i=1}^{n_1+1}$ and $\{x_{\vec{n},i}\}_{i=1}^{n_1}$, their convergence to α_1, and monotonicity of $\varphi_2(z)$ yield similarly to (6.6) that

$$(6.8)\qquad \lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0}\lim_{z\to\infty} z\left(\frac{\prod_{i=1}^{n_1+1}S(z;x_{\vec{n}+\vec{e}_1,i})}{\prod_{i=1}^{n_1}S(z;x_{\vec{n},i})} - 1\right) = -(B_{0,2}+\varphi_2(\alpha_1)-\alpha_1).$$

Hence, it follows from (6.1), (6.6), (6.8), and (2.7) that the limit in (6.2) when $i=1$ is equal to

$$\lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0}\lim_{z\to\infty}\left(\frac{P_{\vec{n}+\vec{e}_1,1}(z)}{P_{\vec{n},1}(z)}\frac{\prod_{i=1}^{n_1+1}S(z;x_{\vec{n}+\vec{e}_1,i})}{\prod_{i=1}^{n_1}S(z;x_{\vec{n},i})} - z\right) = -B_{0,1}.$$

Since $\{\vec{n}+\vec{e}_2 : \vec{n}\in\mathcal{N}_0\}$ is a marginal sequence as well and the zeros of $P_{\vec{n}}(z)$ and $P_{\vec{n}+\vec{e}_2}(z)$ also interlace, the limit in (6.2) for $i=2$ follows similarly to the case $i=1$. □

7. Proof of Theorems 3.2–3.4

To prove Theorems 3.2–3.4 we use the extension to multiple orthogonal polynomials [24] of by now classical approach of Fokas, Its, and Kitaev [20, 21] connecting orthogonal polynomials to matrix Riemann-Hilbert problems. The RH problem is then analyzed via the non-linear steepest descent method of Deift and Zhou [17].

As was agreed in Section 3.3, we label quantities dependent on $c_{\vec{n}}$ only by the subindex $\vec{n}$ as in $\beta_{\vec{n},1} := \beta_{c_{\vec{n}},1}$, $\Delta_{\vec{n},i} := \Delta_{c_{\vec{n}},i}$, etc. If Δ is a closed interval, we denote by Δ° the open interval with the same endpoints. Moreover, when convenient, we write $\alpha_{\vec{n},1}(=\alpha_1)$ and $\beta_{\vec{n},2}(=\beta_2)$ even though they do not depend on the index $\vec{n}$.

Throughout this section, the reader must keep in mind the definition of constants c^* and c^{**} in Proposition 4.1. Moreover, we would like to use the symbol c as a free parameter from the interval $[0,1]$, as was done in the previous sections. Thus, we slightly modify the notation from the statement of Theorems 3.2–3.4 and assume that we deal with a sequence of multi-indices $\mathcal{N}_{c_\star}$ such that

$$c_{\vec{n}} = n_1/|\vec{n}| \to c_\star \in [0,1] \quad \text{and} \quad n_1, n_2 \to \infty \quad \text{as} \quad |\vec{n}| \to \infty, \quad \vec{n} \in \mathcal{N}_{c_\star}.$$

We let $[\boldsymbol{A}]_{i,j}$ to stand for (i,j)-th entry of a matrix $\boldsymbol{A}$ and $\boldsymbol{E}_{i,j}$ to be the matrix whose entries are all zero except for $[\boldsymbol{E}_{i,j}]_{i,j} = 1$. We set $\boldsymbol{I}$ to be the identity matrix, $\sigma_3 := \operatorname{diag}(1,-1)$ to be the third Pauli matrix, and $\sigma(\vec{n}) := \operatorname{diag}(|\vec{n}|, -n_1, -n_2)$. Finally, for compactness of notation, we introduce transformations T_i, $i \in \{1,2\}$, that act on 2×2 matrices in the following way:

$$\mathsf{T}_1\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22}\end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad \mathsf{T}_2\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22}\end{pmatrix} = \begin{pmatrix} e_{11} & 0 & e_{12} \\ 0 & 1 & 0 \\ e_{21} & 0 & e_{22}\end{pmatrix}.$$

7.1. **Initial RH Problem.** Let the measures μ_1, μ_2 be as in Theorem 1.2 and the functions $\rho_1(x), \rho_2(x)$ be given by (3.2). Consider the following Riemann-Hilbert problem (RHP-$\boldsymbol{Y}$): find a 2×2 matrix function $\boldsymbol{Y}(z)$ such that

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C}\setminus(\Delta_1 \cup \Delta_2)$ and $\lim_{z\to\infty} \boldsymbol{Y}(z) z^{-\sigma(\vec{n})} = \boldsymbol{I}$;
(b) $\boldsymbol{Y}(z)$ has continuous traces on Δ_i° that satisfy $\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x)(\boldsymbol{I} + \rho_i(x)\boldsymbol{E}_{1,i+1})$, $i \in \{1,2\}$;
(c) the entries of the $(i+1)$-st column of $\boldsymbol{Y}(z)$ behave like $O(\log|z-\xi|)$ as $z \to \xi \in \{\alpha_i, \beta_i\}$, while the remaining entries stay bounded, $i \in \{1,2\}$.

Lemma 7.1 (Proposition 3.1 of [45]). *Solution of* RHP-$\boldsymbol{Y}$ *is unique and given by*

$$\boldsymbol{Y}(z) := \begin{pmatrix} P_{\vec{n}}(z) & R^{(1)}_{\vec{n}}(z) & R^{(2)}_{\vec{n}}(z) \\ m_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(z) & m_{\vec{n},1}R^{(1)}_{\vec{n}-\vec{e}_1}(z) & m_{\vec{n},1}R^{(2)}_{\vec{n}-\vec{e}_1}(z) \\ m_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(z) & m_{\vec{n},2}R^{(1)}_{\vec{n}-\vec{e}_2}(z) & m_{\vec{n},2}R^{(2)}_{\vec{n}-\vec{e}_2}(z)\end{pmatrix}, \tag{7.1}$$

where $P_{\vec{n}}(z)$ is the polynomial satisfying (1.5), $R^{(i)}_{\vec{n}}(z)$, $i \in \{1,2\}$, are its functions of the second kind, see (3.14), $m_{\vec{n},i}$ are constants such that $\lim_{z\to\infty} m_{\vec{n},i}R^{(i)}_{\vec{n}-\vec{e}_i}(z)z^{n_i} = 1$ and $\vec{e}_1 := (1,0)$, $\vec{e}_2 := (0,1)$.

7.2. **Opening of the Lenses.** Given $c \in (0,1)$ and $\delta > 0$, denote by $U_{c,\delta,e}$ an open square with vertices $e \pm c\delta, e \pm \mathrm{i}c\delta$ when $e \in \{\alpha_1, \beta_{c,1}\}$ and $e \pm (1-c)\delta, e \pm \mathrm{i}(1-c)\delta$ when $e \in \{\alpha_{c,2}, \beta_2\}$. Define $\delta_i(c)$, $i \in \{1,2\}$, via

$$\begin{cases} \delta_1(c) := \dfrac{1}{8c}\begin{cases} \min\{\beta_{c,1}-\alpha_1, \beta_1 - \beta_{c,1}\}, & c < c^*, \\ \min\{\beta_1 - \alpha_1, \alpha_2 - \beta_1\}, & c^* \leqslant c, \end{cases} \\ \delta_2(c) := \dfrac{1}{8(1-c)}\begin{cases} \min\{\beta_2 - \alpha_2, \alpha_2 - \beta_1\}, & c \leqslant c^{**}, \\ \min\{\beta_2 - \alpha_{c,2}, \alpha_{c,2} - \alpha_2\}, & c^{**} < c. \end{cases} \end{cases}$$

Of course, it holds that $c\delta_1(c)$ (resp. $(1-c)\delta_2(c)$) is constant for $c \geqslant c^*$ (resp. $c \leqslant c^{**}$). Moreover, $\delta_1(c)$ (resp. $\delta_2(c)$) approaches a non-zero constant as $c \to 0^+$ (resp. $c \to 1^-$) by (4.8) and it approaches 0 as $c \to c^{*-}$ (resp. $c \to c^{**+}$). Set $\delta(c) := \min\{\delta_1(c), \delta_2(c)\}$. For brevity, we write

$$U_e := U_{c_{\vec{n}},\delta,e}, \quad \vec{n} \in \mathcal{N}_{c_\star}, \quad e \in E_{\vec{n}} := E_{c_{\vec{n}}},\ E_c := \{\alpha_1, \beta_{c,1}, \alpha_{c,2}, \beta_2\},$$

assuming that $\delta \in (0, \delta(c_\star))$. In particular, all the domains U_e are disjoint and $\beta_1 \notin \overline{U}_{\beta_{c,1}}$ when $c_\star < c^*$ while $\alpha_2 \notin \overline{U}_{\alpha_{c,2}}$ when $c_\star > c^{**}$, again, for all $|\vec{n}|$ large enough, $\vec{n} \in \mathcal{N}_{c_\star}$.

Section 7.4 contains a construction of maps $\zeta_e(z)$, conformal in U_e, $e \in E_c$, such that $\zeta_e(z)$ is real on the real line, vanishes at e, and maps $(\Delta_{c,1} \cup \Delta_{c,2}) \cap U_e$ into the negative reals (these subsets of $\Delta_{c,1} \cup \Delta_{c,2}$ are covered by the darker shading on Figure 3). Using these conformal maps corresponding to $c_{\vec{n}}$ for $\vec{n} \in \mathcal{N}_{c_\star}$, we can select piecewise smooth open Jordan arcs $\Gamma^{\pm}_{\vec{n},i}$, connecting $\alpha_{\vec{n},i}$ to $\beta_{\vec{n},i}$, defined by the following properties:

$$\zeta_{\beta_{\vec{n},i}}\big(\Gamma^{\pm}_{\vec{n},i} \cap U_{\beta_{\vec{n},i}}\big) \subset I_\pm := \big\{z : \arg(z) = \pm 2\pi/3\big\}, \quad \zeta_{\alpha_{\vec{n},i}}\big(\Gamma^{\pm}_{\vec{n},i} \cap U_{\alpha_{\vec{n},i}}\big) \subset I_\mp, \tag{7.2}$$

and $\Gamma^{\pm}_{\vec{n},i}$ consist of straight line segments outside of $U_{\alpha_{\vec{n},i}}$ and $U_{\beta_{\vec{n},i}}$, see Figure 3. When $c_\star = c^*$, we slightly modify (7.2) and require that

$$\widetilde{\zeta}_{\beta_{\vec{n},1}}\big(\Gamma^{\pm}_{\vec{n},1} \cap U_{\beta_{\vec{n},1}}\big) \subset I_\pm, \quad \widetilde{\zeta}_{\beta_{\vec{n},1}}(z) := \zeta_{\beta_{\vec{n},1}}(z) - \zeta_{\beta_{\vec{n},1}}(\beta_1), \tag{7.3}$$

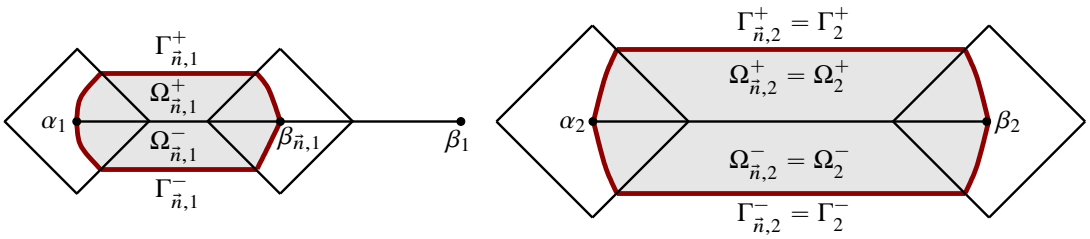

FIGURE 3. The squares $U_{\alpha_{\vec{n},i}}$, $U_{\beta_{\vec{n},i}}$, and U_{β_1}, arcs $\Gamma^{\pm}_{\vec{n},i}$, and domains $\Omega^{\pm}_{\vec{n},i}$ (shaded).

with an analogous modification holding for $c_\star = c^{**}$ at $\alpha_{\vec{n},2}$. We denote by $\Omega^{\pm}_{\vec{n},i}$ the domains delimited by $\Gamma^{\pm}_{\vec{n},i}$ and $\Delta_{\vec{n},i}$, see Figure 3.

Given $\boldsymbol{Y}(z)$, the solution of RHP-$\boldsymbol{Y}$, set

$$\boldsymbol{X}(z) := \boldsymbol{Y}(z) \begin{cases} \mathsf{T}_i \begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i(z) & 1 \end{pmatrix}, & z \in \Omega^{\pm}_{\vec{n},i},\ i \in \{1,2\}, \\ \boldsymbol{I}, & \text{otherwise.} \end{cases} \tag{7.4}$$

It can be readily verified that $\boldsymbol{X}(z)$ solves the following Riemann-Hilbert problem (RHP-$\boldsymbol{X}$):

(a) $\boldsymbol{X}(z)$ is analytic in $\mathbb{C}\setminus\bigcup_{i=1}^{2}\left(\Delta_i \cup \Gamma^+_{\vec{n},i} \cup \Gamma^-_{\vec{n},i}\right)$ and $\lim_{z\to\infty} \boldsymbol{X}(z)z^{-\sigma(\vec{n})} = \boldsymbol{I}$;

(b) $\boldsymbol{X}(z)$ has continuous traces on $\bigcup_{i=1}^{2}\left(\Delta_i^\circ \cup \Gamma^+_{\vec{n},i} \cup \Gamma^-_{\vec{n},i}\right)$ that satisfy

$$\boldsymbol{X}_+(s) = \boldsymbol{X}_-(s) \begin{cases} \mathsf{T}_i \begin{pmatrix} 0 & \rho_i(s) \\ -1/\rho_i(s) & 0 \end{pmatrix}, & s \in \Delta_{\vec{n},i}, \\ \mathsf{T}_i \begin{pmatrix} 1 & 0 \\ 1/\rho_i(s) & 1 \end{pmatrix}, & s \in \Gamma^+_{\vec{n},i} \cup \Gamma^-_{\vec{n},i}, \\ \mathsf{T}_i \begin{pmatrix} 1 & \rho_i(s) \\ 0 & 1 \end{pmatrix}, & s \in \Delta_i^\circ \setminus \Delta_{\vec{n},i}, \end{cases}$$

for each $i \in \{1,2\}$;

(c) the entries of the first and $(i+1)$-st columns of $\boldsymbol{X}(z)$ behave like $O(\log|z-\xi|)$ as $z \to \xi \in \{\alpha_i, \beta_i\}$, while the remaining entries stay bounded, $i \in \{1,2\}$.

The following lemma is contained in [45, Lemma 8.1].

Lemma 7.2. *RHP-$\boldsymbol{X}$ is solvable if and only if RHP-$\boldsymbol{Y}$ is solvable. When solutions of RHP-$\boldsymbol{X}$ and RHP-$\boldsymbol{Y}$ exist, they are unique and connected by (7.4).*

7.3. **Auxiliary Parametrices.** The following Riemann-Hilbert problem (RHP-$\boldsymbol{N}$) is essentially obtained by discarding the jumps of $\boldsymbol{X}(z)$ outside of $\Delta_{\vec{n},1} \cup \Delta_{\vec{n},2}$:

(a) $\boldsymbol{N}(z)$ is analytic in $\mathbb{C}\setminus(\Delta_{\vec{n},1} \cup \Delta_{\vec{n},2})$ and $\lim_{z\to\infty} \boldsymbol{N}(z)z^{-\sigma(\vec{n})} = \boldsymbol{I}$;

(b) $\boldsymbol{N}(z)$ has continuous traces on $\Delta^\circ_{\vec{n},i}$ that satisfy $\boldsymbol{N}_+(s) = \boldsymbol{N}_-(s)\mathsf{T}_i \begin{pmatrix} 0 & \rho_i(s) \\ -1/\rho_i(s) & 0 \end{pmatrix}$;

(c) it holds that $\boldsymbol{N}(z) = \boldsymbol{O}\left(|z-e|^{-1/4}\right)$ as $z \to e \in E_{\vec{n}}$.

Let $S_{\vec{n}}(z) := S_{c_{\vec{n}}}(z)$ be the one granted by Proposition 3.1. Put

$$\boldsymbol{S}(z) := \operatorname{diag}\left(S^{(0)}_{\vec{n}}(z), S^{(1)}_{\vec{n}}(z), S^{(2)}_{\vec{n}}(z)\right)$$

for $z \in \overline{\mathbb{C}}\setminus(\Delta_{\vec{n},1} \cup \Delta_{\vec{n},2})$. Further, let $\Phi_{\vec{n}}(z)$, $w_{\vec{n},i}(z) := w_{c_{\vec{n}},i}(z)$, and $\Upsilon_{\vec{n},i}(z) := \Upsilon_{c_{\vec{n}},i}(z)$ be the functions given by (3.13), (3.6), and (5.3), respectively. Define

$$\boldsymbol{M}(z) := \boldsymbol{S}^{-1}(\infty) \begin{pmatrix} 1 & 1/w_{\vec{n},1}(z) & 1/w_{\vec{n},2}(z) \\ \Upsilon^{(0)}_{\vec{n},1}(z) & \Upsilon^{(1)}_{\vec{n},1}(z)/w_{\vec{n},1}(z) & \Upsilon^{(2)}_{\vec{n},1}(z)/w_{\vec{n},2}(z) \\ \Upsilon^{(0)}_{\vec{n},2}(z) & \Upsilon^{(1)}_{\vec{n},2}(z)/w_{\vec{n},1}(z) & \Upsilon^{(2)}_{\vec{n},2}(z)/w_{\vec{n},2}(z) \end{pmatrix} \boldsymbol{S}(z). \tag{7.5}$$

Then RHP-N is solved by $\boldsymbol{N}(z) := \boldsymbol{C}(\boldsymbol{MD})(z)$, see [45, Section 8.2], where $\boldsymbol{C}$ is a diagonal matrix of constants such that

$$\lim_{z\to\infty} \boldsymbol{CD}(z) z^{-\sigma(\vec{n})} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}(z) := \operatorname{diag}\left(\Phi_{\vec{n}}^{(0)}(z), \Phi_{\vec{n}}^{(1)}(z), \Phi_{\vec{n}}^{(2)}(z)\right). \tag{7.6}$$

Since the jump matrix in RHP-N(b) has determinant 1, it follows from RHP-N(a,b) that $\det(\boldsymbol{N})(z)$ is holomorphic in $\overline{\mathbb{C}}\setminus E_{\vec{n}}$ with at most square root singularities at the points of $E_{\vec{n}}$. Thus, $\det(\boldsymbol{N})(z)$ is a constant and $\det(\boldsymbol{N})(z) \equiv 1$ by RHP-N(a). Therefore, it holds that $\det(\boldsymbol{M})(z) \equiv \det(\boldsymbol{D})(z) \equiv \det(\boldsymbol{C}) = 1$ due to the second relation in (3.7) and (3.13). Moreover, it follows from (5.19) and (5.20) that

$$\boldsymbol{M}^{-1}(z) = \boldsymbol{S}^{-1}(z) \begin{pmatrix} \Pi_{\vec{n}}^{(0)}(z) & \Pi_{\vec{n},1}^{(0)}(z) & \Pi_{\vec{n},2}^{(0)}(z) \\ w_{\vec{n},1}(z)\Pi_{\vec{n}}^{(1)}(z) & w_{\vec{n},1}(z)\Pi_{\vec{n},1}^{(1)}(z) & w_{\vec{n},1}(z)\Pi_{\vec{n},2}^{(1)}(z) \\ w_{\vec{n},2}(z)\Pi_{\vec{n}}^{(2)}(z) & w_{\vec{n},2}(z)\Pi_{\vec{n},1}^{(2)}(z) & w_{\vec{n},2}(z)\Pi_{\vec{n},2}^{(2)}(z) \end{pmatrix} \boldsymbol{S}(\infty). \tag{7.7}$$

We use the following convention: $|\boldsymbol{A}(z)| \lesssim |\boldsymbol{B}(z)|$ (resp. $|\boldsymbol{A}(z)| \sim |\boldsymbol{B}(z)|$ if all the individual entries satisfy $|[\boldsymbol{A}]_{i,j}(z)| \lesssim |[\boldsymbol{B}]_{i,j}(z)|$ (resp. $|[\boldsymbol{A}]_{i,j}(z)| \sim |[\boldsymbol{B}]_{i,j}(z)|$). Moreover, if the constants appearing in inequalities $\lesssim$ and $\sim$ do depend on a certain parameter, say δ, we write $\lesssim_\delta$ and $\sim_\delta$. Furthermore, we shall write $\boldsymbol{A}(z) = \boldsymbol{O}_\delta(1)$ if all the individual entries satisfy $|[\boldsymbol{A}]_{i,j}(z)| \lesssim_\delta 1$.

Lemma 7.3. *It holds that $\boldsymbol{M}^{\pm 1}(z) = \boldsymbol{O}_\delta(1)$ uniformly for z such that $0 < \delta c_{\vec{n}} \leqslant \operatorname{dist}(z, \{\alpha_1, \beta_{\vec{n},1}\})$ and $0 < \delta(1-c_{\vec{n}}) \leqslant \operatorname{dist}(z, \{\alpha_{\vec{n},2}, \beta_2\})$, where the estimate is independent of the parameter $c_{\vec{n}}$. Moreover, it holds that $|\boldsymbol{M}(z)|$ is*

$$\sim \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} & 1-c_{\vec{n}} \\ \delta^{-1/4} & \delta^{-1/4} & c_{\vec{n}}(1-c_{\vec{n}}) \\ (1-c_{\vec{n}})\delta^{-1/4} & (1-c_{\vec{n}})\delta^{-1/4} & 1 \end{pmatrix} \quad \text{and} \quad \sim \begin{pmatrix} \delta^{-1/4} & c_{\vec{n}} & \delta^{-1/4} \\ c_{\vec{n}}\delta^{-1/4} & 1 & c_{\vec{n}}\delta^{-1/4} \\ \delta^{-1/4} & c_{\vec{n}}(1-c_{\vec{n}}) & \delta^{-1/4} \end{pmatrix}$$

uniformly on $|z-\alpha_1| = \delta c_{\vec{n}}$, $|z-\beta_{\vec{n},1}| = \delta c_{\vec{n}}$ and on $|z-\alpha_{\vec{n},2}| = \delta(1-c_{\vec{n}})$, $|z-\beta_2| = \delta(1-c_{\vec{n}})$, respectively, where the constants of proportionality are independent of $c_{\vec{n}}$ and δ. Finally, it holds that $\boldsymbol{M}^{-1}(z)$ is equal to

$$\boldsymbol{O}\left(\frac{1}{\delta^{1/4}}\begin{pmatrix} 1 & 1 & 1-c_{\vec{n}} \\ 1 & 1 & 1-c_{\vec{n}} \\ (1-c_{\vec{n}})\delta^{-1/4} & (1-c_{\vec{n}})\delta^{-1/4} & \delta^{-1/4} \end{pmatrix}\right) \quad \text{and} \quad \boldsymbol{O}\left(\frac{1}{\delta^{1/4}}\begin{pmatrix} 1 & c_{\vec{n}} & 1 \\ c_{\vec{n}}\delta^{-1/4} & \delta^{-1/4} & c_{\vec{n}}\delta^{-1/4} \\ 1 & c_{\vec{n}} & 1 \end{pmatrix}\right)$$

uniformly on $|z-\alpha_1| = \delta c_{\vec{n}}$, $|z-\beta_{\vec{n},1}| = \delta c_{\vec{n}}$ and on $|z-\alpha_{\vec{n},2}| = \delta(1-c_{\vec{n}})$, $|z-\beta_2| = \delta(1-c_{\vec{n}})$, respectively, with $\boldsymbol{O}(\cdot)$ holding independently of $c_{\vec{n}}$ and δ.

Proof. Consider first z on one of the circles from the statement of the lemma. It follows from (3.10) and Lemma 5.4 that

$$\boldsymbol{S}(\infty) \sim \operatorname{diag}\left(c_{\vec{n}}^{-1/3}(1-c_{\vec{n}})^{-1/3}, c_{\vec{n}}^{2/3}(1-c_{\vec{n}})^{-1/3}, c_{\vec{n}}^{-1/3}(1-c_{\vec{n}})^{2/3}\right),$$

where the constants of proportionality are independent of $c_{\vec{n}}$. It further follows from Lemma 5.4 that

$$|\boldsymbol{S}(z)| \sim \boldsymbol{S}(\infty)\operatorname{diag}\left(\delta^{-1/4}, \delta^{1/4}, 1\right) \quad \text{and} \quad |\boldsymbol{S}(z)| \sim \boldsymbol{S}(\infty)\operatorname{diag}\left(\delta^{-1/4}, 1, \delta^{1/4}\right)$$

uniformly on $|z-\alpha_1| = \delta c_{\vec{n}}$, $|z-\beta_{\vec{n},1}| = \delta c_{\vec{n}}$ and on $|z-\alpha_{\vec{n},2}| = \delta(1-c_{\vec{n}})$, $|z-\beta_2| = \delta(1-c_{\vec{n}})$, respectively, where the constants of proportionality are independent of $c_{\vec{n}}$ and δ. Moreover, we deduce from Lemma 5.2 and (5.23) that $\boldsymbol{S}(\infty)\left|\left(\boldsymbol{MS}^{-1}\right)(z)\right|$ is

$$\sim \begin{pmatrix} 1 & c_{\vec{n}}^{-1}\delta^{-1/2} & 1 \\ c_{\vec{n}} & \delta^{-1/2} & c_{\vec{n}}^2 \\ (1-c_{\vec{n}})^2 & c_{\vec{n}}^{-1}(1-c_{\vec{n}})^2\delta^{-1/2} & 1 \end{pmatrix} \quad \text{and} \quad \sim \begin{pmatrix} 1 & 1 & (1-c_{\vec{n}})^{-1}\delta^{-1/2} \\ c_{\vec{n}}^2 & 1 & c_{\vec{n}}^2(1-c_{\vec{n}})^{-1}\delta^{-1/2} \\ 1-c_{\vec{n}} & (1-c_{\vec{n}})^2 & \delta^{-1/2} \end{pmatrix}$$

uniformly on $|z-\alpha_1| = \delta c_{\vec{n}}$, $|z-\beta_{\vec{n},1}| = \delta c_{\vec{n}}$ and on $|z-\alpha_{\vec{n},2}| = \delta(1-c_{\vec{n}})$, $|z-\beta_2| = \delta(1-c_{\vec{n}})$, respectively, where the constants of proportionality are independent of $c_{\vec{n}}$ and δ. The combination of the above three estimates yields the desired asymptotics of $\boldsymbol{M}(z)$ on the circles around $\alpha_1, \beta_{\vec{n},1}, \alpha_{\vec{n},2}, \beta_2$.

It further follows from Lemma 5.4 that

$$|\boldsymbol{S}_\pm(x)| \lesssim \boldsymbol{S}(\infty)\operatorname{diag}\left(\delta^{-1/4}, 1, 1\right)$$

uniformly for $x \in (\alpha_1 + \delta c_{\vec{n}}, \beta_{\vec{n},1} - \delta c_{\vec{n}}) \cup (\alpha_{\vec{n},2} + \delta(1-c_{\vec{n}}), \beta_2 - \delta(1-c_{\vec{n}}))$ where the constants of proportionality are independent of $c_{\vec{n}}$ and δ. Analogously, it follows from Lemma 5.2 and (5.23) that the

above estimate of $\boldsymbol{S}(\infty)\big(\boldsymbol{M}\boldsymbol{S}^{-1}\big)(z)$ on the circles remains valid as an upper estimate on $(\alpha_1+\delta c_{\vec{n}},\beta_{\vec{n},1}-\delta c_{\vec{n}})\cup(\alpha_{\vec{n},2}+\delta(1-c_{\vec{n}}),\beta_2-\delta(1-c_{\vec{n}}))$. The last two observations and the maximum modulus principle for holomorphic functions show that $\boldsymbol{M}(z)=\boldsymbol{O}_\delta(1)$ uniformly for z such that $0<\delta c_{\vec{n}}\leqslant \operatorname{dist}(z,\{\alpha_1,\beta_{\vec{n},1}\})$ and $0<\delta(1-c_{\vec{n}})\leqslant\operatorname{dist}(z,\{\alpha_{\vec{n},2},\beta_2\})$, where the estimate is independent of the parameter $c_{\vec{n}}$.

Finally, as $\det(\boldsymbol{M})(z)\equiv 1$, the estimates of $\boldsymbol{M}^{-1}(z)$ follow in a straightforward fashion from the ones for $\boldsymbol{M}(z)$. □

Besides $\boldsymbol{N}(z)$, we shall also need matrix functions that solve RHP-$\boldsymbol{X}$ within the domains U_e, introduced at the beginning of Section 7.2, with an additional matching condition on the boundary. More precisely, let $\varepsilon_{\vec{n}}$ be given by (3.11). For each $e\in\{\alpha_1,\beta_{\vec{n},1},\alpha_{\vec{n},2},\beta_2\}$ we are seeking a solution of the following RHP-$\boldsymbol{P}_e$:

(a,b,c) $\boldsymbol{P}_e(z)$ satisfies RHP-$\boldsymbol{X}$(a,b,c) within U_e;

(d) $\boldsymbol{P}_e(s)=\boldsymbol{M}(s)(\boldsymbol{I}+\boldsymbol{o}(1))\boldsymbol{D}(s)$ uniformly on $\partial U_e\setminus\bigcup_{i=1}^2\big(\Delta_i\cup\Gamma^+_{\vec{n},i}\cup\Gamma^-_{\vec{n},i}\big)$, where

$$|[\boldsymbol{o}(1)]_{j,k}|\leqslant C\varepsilon_{\vec{n}}\begin{cases}\delta^{-1/2}, & e=\alpha_1,\\ \delta^{-3/2}, & e=\beta_{\vec{n},1}\text{ when }c_\star<c^*,\\ \big(\delta(z_{c_\star}-\beta_1)\big)^{-1/2}, & e=\beta_1\text{ when }c_\star>c^*,\end{cases}$$

for some constant $C>0$ independent of $\vec{n}$ and δ, and analogous estimates hold around $\alpha_{\vec{n},2},\beta_2$ (in the cases $c_\star=c^*$ and $c_\star=c^{**}$ we cannot specify the exact rate of the error term), where the point z_c, or more precisely $\boldsymbol{z}_c$ was defined in Proposition 4.2.

We will solve RHP-$\boldsymbol{P}_e$ only for $e\in\{\alpha_1,\beta_{\vec{n},1}\}$ understanding that the solutions for $e\in\{\alpha_{\vec{n},2},\beta_2\}$ can be constructed similarly. Solution of each RHP-$\boldsymbol{P}_e$ will require a construction, carried out in the next subsection, of a local conformal map around α_1 and $\beta_{\vec{n},1}$. Recall that these maps were already used in (7.2).

7.4. **Conformal Maps.** In this subsection we construct local conformal maps needed to solve problems RHP-$\boldsymbol{P}_e$. To this end, recall the definition, given right after (4.1), and properties, described in Proposition 4.2, of the function $h_c(\boldsymbol{z})$ that is rational on the surface $\mathfrak{R}_c$.

7.4.1. *Local maps around α_1.* Given $c\in(0,1)$, define

$$\zeta_{c,\alpha_1}(z):=\left(\frac{1}{4}\int_{\alpha_1}^{z}\left(h_c^{(0)}-h_c^{(1)}\right)(s)\mathrm{d}s\right)^2,\quad \Re z<\beta_{c,1}. \tag{7.8}$$

Since $h^{(0)}_{c\pm}(x)=h^{(1)}_{c\mp}(x)$ on $\Delta^\circ_{c,1}$, the function $\zeta_{c,\alpha_1}(z)$ is holomorphic in the region of definition. When ω is a real measure on the real line, it trivially holds that

$$\int\frac{\mathrm{d}\omega(x)}{x-(x_0\pm\mathrm{i}y)}=\int\frac{(x-x_0)\mathrm{d}\omega(x)}{(x-x_0)^2+y^2}\mp\mathrm{i}y\int\frac{\mathrm{d}\omega(x)}{(x-x_0)^2+y^2}.$$

Therefore, if the traces of $\int(x-z)^{-1}\mathrm{d}\omega(x)$ exist at x_0, they are necessarily conjugate-symmetric. In particular, it follows from (4.2) that the integrand in (7.8) is purely imaginary on $\Delta^\circ_{c,1}$ and therefore $\zeta_{c,\alpha_1}(x)<0$ for $x\in\Delta^\circ_{c,1}$. It also clearly follows from (4.2) that $\zeta_{c,\alpha_1}(x)>0$ for $x<\alpha_1$. Moreover, since $h_c(\boldsymbol{z})$ has a pole at $\boldsymbol{\alpha}_1$, a ramification point of $\mathfrak{R}_c$ of order 2, $\zeta_{c,\alpha_1}(z)$ has a simple zero at α_1.

Lemma 7.4. *There exist $\delta_{\alpha_1}>0$, $A_{\alpha_1}>0$, and $D_{\alpha_1}>0$, independent of c, such that each $\zeta_{c,\alpha_1}(z)$ is conformal in $\{|z-\alpha_1|<\delta_{\alpha_1}c\}$, $4A_{\alpha_1}c\leqslant|\zeta'_{c,\alpha_1}(\alpha_1)|$, and $|\zeta'_{c,\alpha_1}(z)|\leqslant D_{\alpha_1}c$ when $\{|z-\alpha_1|<\delta_{\alpha_1}c\}$ for all $c\in(0,1)$.*

Proof. We start by proving the estimate on the size of $|\zeta'_{c,\alpha_1}(\alpha_1)|$. Assume first that $c\leqslant c^*$. Since $\boldsymbol{\alpha}_1$ is a simple pole of $h_c(\boldsymbol{z})$ and $h_c^{(0)}(x)<0$ for $x<\alpha_1$ by (4.2), it holds that $h_c^{(0)}(x)=u_c(\alpha_1-x)^{-1/2}+O(1)$ for $x<\alpha_1$ and sufficiently close to α_1, where the branch of the square root is principal and $u_c<0$. Since $h_c^{(1)}(x)=-u_c(\alpha_1-x)^{-1/2}+O(1)$ around α_1, it can be readily checked that $\zeta'_{c,\alpha_1}(\alpha_1)=-u_c^2$. It was shown in [7, Equation 2.7] that $h_c(\boldsymbol{z})$ solves

$$h^3-\big(1-c+c^2\big)\frac{z-d_c}{\Pi(z)}h-\frac{c-c^2}{\Pi(z)}=0, \tag{7.9}$$

where $\Pi(z):=(z-\alpha_1)(z-\alpha_2)(z-\beta_2)$ and d_c is the point such that the discriminant of (7.9), whose numerator is a cubic polynomial, vanishes at $\beta_{c,1}$ and has an additional double zero. By plugging the identity $h_c^{(0)}(x)=u_c(\alpha_1-x)^{-1/2}+O(1)$ into (7.9), it is easy to verify that

$$u_c^2=\big(1-c+c^2\big)\frac{d_c-\alpha_1}{(\alpha_2-\alpha_1)(\beta_2-\alpha_1)}. \tag{7.10}$$

The numerator of the discriminant of (7.9) is equal to

$$4\big(1-c+c^2\big)^3(z-d_c)^3-27\big(c-c^2\big)^2(z-\alpha_1)(z-\alpha_2)(z-\beta_2) \tag{7.11}$$

and must have a single sign change, which happens at $\beta_{c,1}$. If $d_c\leqslant\alpha_1$ were true, then the discriminant would have been positive at α_2,β_2 and non-negative at α_1, that is, it would have been positive on (α_1,β_2), which contradicts vanishing at $\beta_{c,1}$. On the other hand if, $d_c\geqslant\beta_{c,1}$ were to be true, then the discriminant would have been strictly negative at $\beta_{c,1}$, which, again, leads to a contradiction. Thus, $\alpha_1<d_c<\beta_{c,1}$. Now, (7.9) yields that

$$\frac{d_c-\alpha_1}{c}=\frac{1-c}{(\alpha_2-d_c)(\beta_2-d_c)}\frac{1}{h_c^3(d_c)}\geqslant\frac{(1-c^*)(\alpha_2-\beta_1)^3}{(\alpha_2-\alpha_1)(\beta_2-\alpha_1)}, \tag{7.12}$$

where we used (4.2) to observe that $h_c^{(2)}(d_c)\leqslant 1/(\alpha_2-\beta_1)$. The above inequality and (7.10) clearly yield the desired estimate for $|\zeta'_{c,\alpha_1}(\alpha_1)|=u_c^2$ when $c\leqslant c^*$. In fact, when $c\to 0$, it actually follows from the first equality in (7.12) that

$$\frac{c}{d_c-\alpha_1}=\frac{(\alpha_2-d_c)(\beta_2-d_c)}{1-c}\left(h^{(2)}(d_c)\right)^3\to(\alpha_2-\alpha_1)(\beta_2-\alpha_1)\left(\int\frac{\mathrm{d}\omega_2(x)}{x-\alpha_1}\right)^3=\frac{1}{|w_2(\alpha_1)|} \tag{7.13}$$

due to (4.2), the last conclusion of Proposition 4.1, and the formula before (4.8). In this case, (7.10) and (7.13) yield that

$$\zeta'_{c,\alpha_1}(\alpha_1)=-u_c^2=-c\frac{1+o(1)}{|w_2(\alpha_1)|}\quad\text{as}\quad c\to 0. \tag{7.14}$$

When $c\in\big[c^*,c^{**}\big]$ the surface $\mathfrak{R}_c$ is always the same. Hence, one can argue using local coordinates that the pull-backs of $h_c(z)$ from a fixed circular neighborhood of $\boldsymbol{\alpha}_1$ to a fixed neighborhood in $\mathbb{C}$ continuously depend on c. Since each $|\zeta'_{c,\alpha}(\alpha_1)|>0$ for $c\in(0,1)$, the desired estimate follows from compactness of $[c^*,c^{**}]$. When $c^{**}\leqslant c$, $h_c(z)$ satisfies an equation similar to (7.9). Using this equation, we again can argue that the estimate holds as $c\to 1$, thus, proving that it holds uniformly for all $c\in(0,1)$.

It remains to study conformality of $\zeta_{c,\alpha_1}(z)$. Denote by $\delta_{\alpha_1}(c)$ the supremum of δ such that $\zeta_{c,\alpha_1}(z)$ is conformal in $\{|z-\alpha_1|<2\delta c\}$. We take $\delta_{\alpha_1}:=\inf_{c\in(0,1)}\delta_{\alpha_1}(c)$. Since $\delta_{\alpha_1}(c)>0$ for $c\in(0,1)$ and continuously depends on c, we only need to study what happens as $c\to 0$ and $c\to 1$ to prove that $\delta_{\alpha_1}>0$. Assume first that $c\to 0$. Set $\hat\zeta_{c,\alpha_1}(s):=c^{-2}\zeta_{c,\alpha_1}(z(s))$, where $z(s):=\alpha_1+|\Delta_{c,1}|(1-s)/2$. Then it follows from (7.8) that

$$\hat\zeta_{c,\alpha_1}(s)=\left(\frac{|\Delta_{c,1}|}{8c}\int_1^s\big(\hat h_c^{(0)}-\hat h_c^{(1)}\big)(t)\mathrm{d}t\right)^2,$$

where $\hat h_c^{(k)}(s):=h_c^{(k)}(z(s))$. By using (7.9), (7.13), and (4.8), we see that $\hat h_c$ solves an algebraic equation of the form

$$\hat h^3-(1+o(1))\frac{2(1-s)-1-o(1)}{2(1-s)(|w_2(\alpha_1)|^2+o(1))}\hat h-\frac{1+o(1)}{2(1-s)(|w_2(\alpha_1)|^3+o(1))}=0,$$

where $o(1)$ holds uniformly on compact subsets of the plane as $c\to 0$. The above equation converges to

$$\left(\hat h^2+\frac{\hat h}{|w_2(\alpha_1)|}+\frac{1}{2(1-s)|w_2(\alpha_1)|^2}\right)\left(\hat h-\frac{1}{|w_2(\alpha_1)|}\right)=0. \tag{7.15}$$

Since the branches $\hat h_c^{(0)}(s)$ and $\hat h_c^{(1)}(s)$ have 1 as a branch point, their limits come from the quadratic factor in (7.15). This observation together with with (4.8) readily yield that

$$\hat\zeta_{c,\alpha_1}(s)\to\hat\zeta_{\alpha_1}(s):=\left(\frac{1}{2}\int_1^s\sqrt{\frac{t+1}{t-1}}\mathrm{d}t\right)^2=\frac{1}{4}\left(\sqrt{s^2-1}+\log\big(s+\sqrt{s^2-1}\big)\right)^2 \tag{7.16}$$

locally uniformly in $\{|1-s|<2\}$, where the branches of the square roots and the logarithm are principal and therefore $\hat\zeta_{\alpha_1}(s)$ is holomorphic in $\mathbb{C}\backslash(-\infty,-1]$ and is positive for $s\in(1,\infty)$. Using the explicit expression for $\hat\zeta_{\alpha_1}(s)$, we can conclude that it is conformal in $\{|1-s|<2\}$ and therefore $\liminf_{c\to 0}\delta_{\alpha_1}(c)\geqslant 4|w_2(\alpha_1)|$ by (4.8). When $c\to 1$, we can similarly get from the algebraic equation for $h_c(z)$ that $\zeta_{c,\alpha_1}(z)$ converges to

$$\left(\frac{1}{2}\int_{\alpha_1}^z\frac{\mathrm{d}x}{\sqrt{(x-\alpha_1)(x-\beta_1)}}\right)^2=\frac{1}{4}\left(\log\left(\frac{\beta_1+\alpha_1}{2}-z-\sqrt{(z-\alpha_1)(z-\beta_1)}\right)-\log\left(\frac{\beta_1-\alpha_1}{2}\right)\right)^2, \tag{7.17}$$

which allows us to conclude that $\liminf_{c\to 1}\delta_{\alpha_1}(c)>0$ as desired.

Finally, let $D_{\alpha_1}(c) := c^{-1}\max_{|z-\alpha_1|\leqslant\delta_{\alpha_1}c}|\zeta'_{c,\alpha_1}(z)|$. These constants are finite for each $c\in(0,1)$ since each $\zeta_{c,\alpha_1}(z)$ is, in fact, analytic in $\{|z-\alpha_1|<2\delta_{\alpha_1}c\}$. Moreover, since $\zeta_{c,\alpha_1}(z)$ continuously depends on c, so do the constants $D_{\alpha_1}(c)$. Thus, we only need to check their limits as $c\to 0$ and $c\to 1$. The finiteness of $D_{\alpha_1}:=\sup_{c\in(0,1)}D_{\alpha_1}(c)$ now easily follows from (4.8), (7.16), and (7.17). □

7.4.2. *Local maps around $\beta_{c,1}$ when $c\in(0,c^*]$.* Given $c\in(0,c^*]$, define

$$\zeta_{\beta_{c,1}}(z):=\left(-\frac{3}{4}\int_{\beta_{c,1}}^{z}\left(h_c^{(0)}-h_c^{(1)}\right)(s)\mathrm{d}s\right)^{2/3},\quad \alpha_1<\Re z<\alpha_2, \tag{7.18}$$

where the choice of the root function can be made such that $\zeta_{\beta_{c,1}}(z)$ is holomorphic with a simple zero at $\beta_{c,1}$ and is positive for $x>\beta_{c,1}$. Indeed, since $h_c(\boldsymbol{z})$ is bounded at $\boldsymbol{\beta}_{c,1}$, which is a ramification point of order 2, we can write

$$h_c^{(0)}(x)=h_c(\boldsymbol{\beta}_{c,1})-v_c\sqrt{x-\beta_{c,1}}-O(x-\beta_{c,1}) \tag{7.19}$$

for some number v_c and $x>\beta_{c,1}$ sufficiently small. It follows from Proposition 4.2 that $h(\boldsymbol{\beta}_{c,1})$ is a non-zero real number. It is also clear from (4.2) that $h_c^{(0)}(x)$ and $h_c^{(2)}(x)$ assume any non-zero real number somewhere on $(-\infty,\alpha_1)\cup(\beta_2,\infty)$ and $(-\infty,\alpha_2)\cup(\beta_2,\infty)$, respectively. Thus, if $v_c=0$, then the function $h_c(z)-h(\boldsymbol{\beta}_{c,1})$ would have at least four zeros (the zero at $\boldsymbol{\beta}_{c,1}$ would be at least a double one), but only three poles, which is impossible. Hence, $v_c\neq 0$, or more precisely, $v_c>0$ since $h_c^{(0)}(x)$ is a decreasing function on $(\beta_{c,1},\alpha_2)$ as can be seen (4.2). Therefore, the integrand in (7.18) vanishes as a square root at $\beta_{c,1}$. Thus, $\zeta_{\beta_{c,1}}(z)$ has a simple zero there. Again, as in (7.8), we select such a branch of the root function so that $\zeta_{\beta_{c,1}}(z)$ is negative on $\Delta_{c,1}^{\circ}$. Since the difference $h_c^{(0)}(x)-h_c^{(1)}(x)$ is real in the gap $(\beta_{c,1},\alpha_2)$, the map $\zeta_{\beta_{c,1}}(z)$ is positive there.

Lemma 7.5. *There exist $\delta_{\beta_1}>0$ and $A_{\beta_1}>0$, independent of $c\in(0,c^*]$, such that each $\zeta_{\beta_{c,1}}(z)$ is conformal in $\{|z-\beta_{c,1}|<\delta_{\beta_1}c\}$ and $4A_{\beta_1}c^{-1/3}\leqslant\zeta'_{\beta_{c,1}}(\beta_{c,1})$ for all $c\in(0,c^*]$.*

Proof. Since $\zeta'_{\beta_{c,1}}(\beta_{c,1})\neq 0$ for $c\in(0,c^*]$, to prove the second claim, we only need to consider what happens as $c\to 0$. Similarly to considerations preceding (7.15), let $\hat{h}_c(s):=h_c(\beta_{c,1}+|\Delta_{c,1}|(s-1)/2)$ and $\hat{h}(s)$ be the limit of $\hat{h}_c(s)$ as $c\to 0$. Then (7.15) gets replaced by

$$\left(\hat{h}^2+\frac{\hat{h}}{|w_2(\alpha_1)|}+\frac{1}{2(s+1)|w_2(\alpha_1)|^2}\right)\left(\hat{h}-\frac{1}{|w_2(\alpha_1)|}\right)=0.$$

Since each $\hat{h}_c^{(0)}(s)$ has branchpoints at ±1 and is negative for $s<-1$, see (4.2), the same must be true for their limit $\hat{h}^{(0)}(s)$. Thus, solving the above quadratic equation gives us

$$\hat{h}^{(0)}(s)=-\frac{1}{2|w_2(\alpha_1)|}\left(1+\sqrt{\frac{s-1}{s+1}}\right)=-\frac{1}{2|w_2(\alpha_1)|}-\frac{1}{2|w_2(\alpha_1)|}\sqrt{\frac{s-1}{2}}+O(s-1).$$

Plugging the above limit and the substitution $x=\beta_{c,1}+|\Delta_{c,1}|(s-1)/2$ into (7.19) yields

$$\zeta'_{\beta_{c,1}}(\beta_{c,1})=v_c^{2/3}=c^{-1/3}\frac{1+o(1)}{(16)^{1/3}|w_2(\alpha_1)|} \tag{7.20}$$

as $c\to 0$, where v_c was introduced in (7.19). This finishes the proof of the second claim of the lemma. To prove the first one, it is enough to observe that

$$(3c)^{-2/3}\zeta_{\beta_{c,1}}\big(\beta_{c,1}+|\Delta_{c,1}|(s-1)/2\big)\to\left(\int_1^s\sqrt{\frac{t-1}{t+1}}\mathrm{d}t\right)^{2/3}=\left(\sqrt{s^2-1}-\log\left(s+\sqrt{s^2-1}\right)\right)^{2/3} \tag{7.21}$$

as $c\to 0$, where the limit is conformal around 1. □

7.4.3. *Local maps around β_1 for c close to c^* from the right.* This construction will be used only for the ray sequences $\mathcal{N}_{c^*}$ with infinitely many indices $\vec{n}$ such that $c_{\vec{n}}>c^*$. By Proposition 4.2, $h_{c^*}(z)$ is bounded at $\boldsymbol{\beta}_1$ while $h_c(z)$ has a simple pole at $\boldsymbol{\beta}_1$ for all $c>c^*$ and a simple zero z_c that approaches $\boldsymbol{\beta}_1$ as $c\to c^{*+}$. Since the functions $h_c(z)$ converge around $\boldsymbol{\beta}_1$ to $h_{c^*}(z)$ as $c\to c^{*+}$ by (4.2) and Proposition 4.1, we can write

$$-\frac{3}{4}\int_{\beta_1}^{z}\left(h_c^{(0)}-h_c^{(1)}\right)(s)\mathrm{d}s=\sqrt{z-\beta_1}\,(z-\beta_1-\epsilon_c)\,f_c(z),\quad \alpha_1<\Re z<\alpha_2, \tag{7.22}$$

for some $\epsilon_c>0$ such that $\epsilon_c\to 0^+$ as $c\to c^{*+}$, where $f_c(z)$ is a holomorphic function that is real on (α_1,α_2) (observe that the Puisuex expansion of $(h_c^{(0)}-h_c^{(1)})(x)$ around $\beta_{c,1}$ does not have the integral powers of $x-\beta_1$).

Similarly, it holds that $\zeta^{3/2}_{\beta_{c^*,1}}(z) = (z-\beta_1)^{3/2} f_{c^*}(z)$ for some holomorphic function $f_{c^*}(z)$ that is real on (α_1,α_2) and is positive at β_1. Since the right-hand side of (7.22) converges to $\zeta^{3/2}_{\beta_{c^*,1}}(z)$ as $c \to c^{*+}$, the functions $f_c(z)$ converge to $f_{c^*}(z)$ (in particular, $f_c(\beta_1) > 0$ for all c sufficiently close to c^*).

Lemma 7.6. *There exist $c' > c^*$ and a fixed neighborhood of β_1 such that for every $c \in (c^*, c']$ there exists a function $\hat\zeta_{c,\beta_1}(z)$, conformal in this neighborhood, such that*

$$-\frac{3}{4}\int_{\beta_1}^{z}\left(h_c^{(0)} - h_c^{(1)}\right)(s)\mathrm{d}s = \hat\zeta^{3/2}_{c,\beta_1}(z) - \hat\zeta_{c,\beta_1}(\beta_1+\epsilon_c)\hat\zeta^{1/2}_{c,\beta_1}(z) \tag{7.23}$$

(we can adjust the constant $\delta_{\beta_1} > 0$ from Lemma 7.5 so that the neighborhood of conformality is given by $\{|z-\beta_1| < \delta_{\beta_1} c'\}$). Moreover, $\hat\zeta_{c,\beta_1}(z)$ is positive for $x > \beta_1$ and converges to $\zeta_{\beta_{c^*,1}}(z)$ as $c \to c^{*+}$.

Proof. Let $F(z;\epsilon)$ be a family of holomorphic and non-vanishing functions in $\{|z| < r_0\}$ that are positive at the origin and continuously depend on the parameter $\epsilon \in [0,\epsilon_0]$. Consider the equation

$$u(z;\epsilon)(u(z;\epsilon)-3p_\epsilon)^2 = 2g(z;\epsilon), \quad g(z;\epsilon) := z(z-\epsilon)^2 F(z;\epsilon), \tag{7.24}$$

where $p_\epsilon > 0$ is a parameter that we shall fix in a moment. The solution of this cubic equation is formally given by

$$\begin{cases} u(z;\epsilon) = 2p_\epsilon + v^{1/3}(z;\epsilon) + p_\epsilon^2 v^{-1/3}(z;\epsilon), \\ v(z;\epsilon) = g(z;\epsilon) - p_\epsilon^3 + \sqrt{g(z;\epsilon)(g(z;\epsilon)-2p_\epsilon^3)}. \end{cases}$$

Observe that $g'(x;\epsilon) = (x-\epsilon)\big[(3x-\epsilon)F(x;\epsilon) + x(x-\epsilon)F'(x;\epsilon)\big]$. The expression in the square brackets is negative at 0 and positive at ϵ. Since $F(0;\epsilon) > \delta > 0$, independently of $\epsilon \in [0,\epsilon_0]$ for some $\delta, \epsilon_0 > 0$ sufficiently small, the derivative of the expression in the square brackets, that is, $3F(x;\epsilon) + (5x-\epsilon)F'(x;\epsilon) + x(x-\epsilon)F''(x;\epsilon)$, is positive on $[0,\epsilon]$ for all $\epsilon \in [0,\epsilon_0]$, where we might need to decrease ϵ_0 if necessary. Hence, there exists a unique point $x_\epsilon \in (0,\epsilon)$ such that $g'(x_\epsilon) = 0$. Then we let

$$2p_\epsilon^3 := g(x_\epsilon;\epsilon) = \max_{x\in[0,\epsilon]} g(x;\epsilon). \tag{7.25}$$

Since $g'(x;\epsilon) = (x-\epsilon)\big[2xF(x;\epsilon) + (x-\epsilon)(F(x;\epsilon) + xF'(x;\epsilon))\big]$ and $F(0;\epsilon) > \delta > 0$, independently of $\epsilon \in [0,\epsilon_0]$, we can decrease r_0 if necessary so that $g'(x;\epsilon) > 0$ for $x \in (\epsilon, r_0)$ and $\epsilon \in [0,\epsilon_0]$. Thus, there exists a unique $y_\epsilon \in (\epsilon, r_0)$ such that $2p_\epsilon^3 = g(y_\epsilon;\epsilon)$ for all $\epsilon \in [0,\epsilon_0]$, where, again, we might need to decrease ϵ_0. Hence, we can choose $v(z;\epsilon)$ to be holomorphic in $\{|z| < r_0\}\setminus[0,y_\epsilon]$ and $v^{1/3}(z;\epsilon)$ such that $v^{1/3}(x;\epsilon) \to -p_\epsilon$ as $x \to 0^-$.

Now, since $g(x;\epsilon) - p_\epsilon^3$ is real on $[0,y_\epsilon]$ and changes sign exactly once on each interval $[0,x_\epsilon]$, $[x_\epsilon,\epsilon]$, and $[\epsilon,y_\epsilon]$ while the square root vanishes at the endpoint of these intervals, the change of the argument of $v_\pm(x;\epsilon)$ is equal to 3π. Thus, we can define $v^{1/3}(z;\epsilon)$ holomorphically in $\{|z| < r_0\}\setminus[0,y_\epsilon]$ as well, where it also holds that $v_+^{1/3}(x;\epsilon)v_-^{1/3}(x;\epsilon) = p_\epsilon^2$ and $v_\pm(\epsilon;\epsilon) = -e^{\mp 2\pi\mathrm{i}/3}p_\epsilon$. In this case $u(z;\epsilon)$ is in fact holomorphic in $\{|z| < r_0\}$, has a simple zero at the origin, is positive for $x > 0$, and satisfies $u(\epsilon;\epsilon) = 2p_\epsilon$. Since $u(z;0) = z(2F(z;0))^{1/3}$ and $u(z;\epsilon)$ continuously depends on ϵ, we can decrease r_0 if necessary so that all the function $u(z;\epsilon)$ are conformal in $\{|z| < r_0\}$.

Let $u(z;\epsilon_c)$ be the discussed solution of (7.24) and (7.25) with $F(z;\epsilon_c) = f_c^2(z+\beta_1)/2$. Then the desired function $\hat\zeta_{c,\beta_1}(z)$ is given by $u(z-\beta_1;\epsilon_c)$. □

7.4.4. *Local maps around β_1 when $c > c^*$.* This construction will be used only for the ray sequences $\mathcal{N}_{c_\star}$ with $c_\star > c^*$. Similarly to (7.8), given $c \in (c^*, 1)$, define

$$\zeta_{c,\beta_1}(z) := \left(\frac{1}{4}\int_{\beta_1}^{z}\left(h_c^{(0)} - h_c^{(1)}\right)(s)\mathrm{d}s\right)^2, \quad \alpha_1 < \Re z < \alpha_2. \tag{7.26}$$

Then $\zeta_{c,\beta_1}(z)$ is holomorphic in the domain of the definition, has a simple zero at β_1, is real positive for $x > \beta_1$, and is real negative for $x < \beta_1$.

Lemma 7.7. *There exists a continuous and non-vanishing function $\delta_{\beta_1}(c)$ on $(c^*,1)$ with non-zero one-sided limit at 1 such that $\zeta_{c,\beta_1}(z)$ is conformal in $\{|z-\beta_1| < \delta_{\beta_1}(c)\}$. Moreover, the constant A_{β_1} in Lemma 7.5 can be adjusted so that $4A_{\beta_1}(z_c - \beta_1) \leqslant |\zeta'_{c,\beta_1}(\beta_1)|$, where z_c is the zero of $h_c(z)$ described in Proposition 4.2.*

Proof. Since $\zeta_{c,\beta_1}(z)$ has a simple zero at β_1, $\delta_{\beta_1}(c)$ is simply the largest radius of conformality, which is clearly positive. Moreover, when $c \to 1$, the limiting behavior of $\zeta_{c,\beta_1}(z)$ is similar to the one described in (7.17) and therefore $\lim_{c\to 1^-} \delta_{\beta_1}(c) > 0$. To prove the second claim of the lemma observe that $\zeta'_{c,\beta_1}(\beta_1) = u_c^2$, where

$$h_c^{(0)}(x) = u_c(x-\beta_1)^{-1/2} + \tilde{h}_c^{(0)}(x), \quad \tilde{h}_c^{(0)}(x) = O(1) \quad \text{as} \quad x \to \beta_1,$$

exactly as in Lemma 7.4. Thus, we only need to investigate what happens when $c \to c^{*+}$ (existence of a limit of $\zeta_{c,\beta_1}(z)$ as $c \to 1$, which is conformal around β_1, shows that $|\zeta'_{c,\beta_1}(\beta_1)|$ is bounded from below as $c \to 1$). It follows from the second part of Proposition 4.1 and (4.2) that the Puiseux expansion of $h_c^{(0)}(x)$ must converge to the Puiseux expansion of $h_{c^*}^{(0)}(x)$ in some punctured neighborhood of β_1. In particular, we have that $u_c \to 0$ and $\tilde{h}_c^{(0)}(x_c) \to h_{c^*}^{(0)}(\beta_1) = h_{c^*}(\boldsymbol{\beta}_1)$ as $c \to c^{*+}$ for any sequence of points $x_c \to \beta_1^+$ as $c \to c^{*+}$. Since $h_c^{(0)}(z_c) = 0$, it holds that $u_c(z_c-\beta_1)^{-1/2} = -\tilde{h}_c^{(0)}(z_c) \to -h_{c^*}(\boldsymbol{\beta}_1)$ as $c \to c^{*+}$, from which the estimate follows. □

7.4.5. *Estimates of $H_c^{(0)}(z) - H_c^{(1)}(z)$ around $\Delta_{c,1}$.* The following lemma will be used in the proof of Lemma 7.10, but is presented here due to its connection to the conformal maps constructed above.

Lemma 7.8. *Let $H_c(\boldsymbol{z})$ be as in (4.1) and δ_{β_1} as in Lemma 7.5. There exists $\tilde{\delta}_{\beta_1} \in (0, \delta_{\beta_1})$ such that given $c \in (0, c^*)$ and $\delta \in (0, \tilde{\delta}_{\beta_1})$, it holds that*

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x+\mathrm{i}y) \leqslant -B_{\beta_1}\delta^{3/2}c, \quad x \in [\beta_{c,1}+\delta c, \alpha_2 - \delta c],\ y \in [-\delta c/2, \delta c/2], \tag{7.27}$$

where $B_{\beta_1} > 0$ is a constant independent of c and δ. Moreover, for any fixed $\delta > 0$ small enough there exist $c_\delta > 0$ and $\epsilon > 0$ such that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x+\mathrm{i}y) \leqslant -\epsilon \tag{7.28}$$

for all $c \in (0, c_\delta)$, $x \in [\alpha_1+\delta, \alpha_2-\delta]$, and $y \in [-\delta/2, \delta/2]$. Finally, for any $c \in (0,1)$, it holds that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x \pm \mathrm{i}\delta c) \geqslant B_{\beta_1}\delta^{5/2}c, \quad x \in [\alpha_1, \beta_{c,1}]. \tag{7.29}$$

Proof. Since $h_c(\boldsymbol{z}) = 2\partial_z H_c(\boldsymbol{z})$ and $\boldsymbol{\beta}_{c,1}$ is a ramification point of $\mathfrak{R}_c$ belonging to both $\mathfrak{R}_c^{(0)}$ and $\mathfrak{R}_c^{(1)}$, it holds that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(z) = \Re\left(\int_{\beta_{c,1}}^z \left(h_c^{(0)} - h_c^{(1)}\right)(s)\mathrm{d}s\right), \quad \alpha_1 < \Re z < \alpha_2. \tag{7.30}$$

It further follows from (4.2) that

$$\partial_x \Re\left(h_c^{(0)} - h_c^{(1)}\right)(x+\mathrm{i}y) = \int \frac{y^2 - (t-x)^2}{((t-x)^2+y^2)^2}\mathrm{d}(2\omega_{c,1}+\omega_{c,2})(t) < 0$$

when $|y| < \delta c \leqslant \operatorname{dist}(x, \Delta_{c,1} \cup \Delta_{c,2})$. Therefore, it holds that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x+\mathrm{i}y) \leqslant \left(H_c^{(0)} - H_c^{(1)}\right)(\beta_{c,1}+\delta c+\mathrm{i}y)$$

for all $x \in [\beta_{c,1}+\delta c, \alpha_2-\delta c]$ and $y \in [-\delta c/2, \delta c/2]$. Now, by combining (7.18) and (7.30) we get that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(z) = -\frac{4}{3}\Re\left(\zeta_{\beta_{c,1}}^{3/2}(z)\right), \quad \alpha_1 < \Re z < \alpha_2, \tag{7.31}$$

for all $c \in (0, c^*]$. Take $\tilde{\delta}_{\beta_1} \leqslant \sin(\pi/6)\delta_{\beta_1}$. Since each map $\zeta_{\beta_{c,1}}(z)$ is conformal in $|z-\beta_{c,1}| < \delta_{\beta_1}c$ and $\delta < \sin(\pi/6)\delta_{\beta_1}$, every point $\beta_{c,1}+\delta c+\mathrm{i}y$ lies within a disk of conformality of $\zeta_{\beta_{c,1}}(z)$ when $|y| < \delta c/2$. Since $\operatorname{Arg}(\delta c+\mathrm{i}y) \in [-\pi/6, \pi/6]$ when $|y| < \delta c/2$ and $\zeta_{\beta_{c,1}}(x)$ is positive for $x > \beta_{c,1}$, is negative for $x < \beta_{c,1}$ and has a positive derivative at $\beta_{c,1}$, there exists $\delta_c > 0$ such that

$$\Re\left(\zeta_{\beta_{c,1}}^{3/2}(\beta_{c,1}+\delta c+\mathrm{i}y)\right) \geqslant \frac{1}{2}\left|\zeta_{\beta_{c,1}}^{3/2}(\beta_{c,1}+\delta c+\mathrm{i}y)\right|$$

for all $|y| < \delta c/2$ and $\delta < \delta_c$. Since the maps $\zeta_{\beta_{c,1}}(z)$ continuously depend on c and have a rescaled conformal limit as $c \to 0$, see (7.21), the constants δ_c can be chosen so that $\delta_c \geqslant \tilde{\delta}_{\beta_1} > 0$ for all $c \in (0, c^*)$ and some $\tilde{\delta}_{\beta_1} > 0$. Thus,

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x+\mathrm{i}y) \leqslant -\frac{2}{3}\left|\zeta_{\beta_{c,1}}^{3/2}(\beta_{c,1}+\delta c+\mathrm{i}y)\right| \leqslant -B_{\beta_1}\delta^{3/2}c$$

for $x \in [\beta_{c,1} + \delta c, \alpha_2 - \delta c]$, $y \in [-\delta c/2, \delta c/2]$, and a constant $B_{\beta_1} > 0$ independent of c by Lemma 7.5 and (5.1), which finishes the proof of (7.27).

Estimate (7.28) follows in straightforward fashion from the observation that the left-hand side of (7.28) converges to $V^{\omega_2}(\alpha_1) - V^{\omega_2}(x + \mathrm{i}y)$ as $c \to 0$ uniformly on the considered set by Proposition 4.1 and (4.1), where ω_2 is the arcsine distribution on Δ_2.

To prove (7.29), observe that for each $x \in \Delta_{c,1}$ fixed, the functions $\left(H_c^{(0)} - H_c^{(1)}\right)(x \pm \mathrm{i}y)$ are increasing for $y \in [0, \infty)$ and vanish at $y = 0$ by (4.1) and (2.3). Moreover, since these functions have the same value at conjugate-symmetric points, it is enough to consider only the upper half-plane. As the right-hand side of (7.29) is positive whenever $c, \delta > 0$, we can assume without loss of generality that $\delta < \min\{\delta_{\alpha_1}, \delta_{\beta_1}, \min_{c \in [c', 1)} \delta_{\beta_1}(c)\}$, where δ_{α_1}, δ_{β_1}, c', and $\delta_{\beta_1}(c)$ were introduced in Lemmas 7.4, 7.5, 7.6, and 7.7, respectively.

Suppose that $|x + \mathrm{i}\delta c - \alpha_1| < \delta_{\alpha_1} c$. Then it follows from Lemma 7.4 together with (5.1) that

$$\left|\zeta_{c,\alpha_1}^{1/2}(x + \mathrm{i}\delta c)\right| \geqslant (A_{\alpha_1}/4)^{1/2} \delta^{1/2} c. \tag{7.32}$$

It clearly holds that $\operatorname{Arg}(x + \mathrm{i}\delta c) \in \left[\arctan(\delta/\delta_{\alpha_1}), \pi/2\right]$. Since $\zeta_{c,\alpha_1}(z)$ is conformal, negative for $z > \alpha_1$, and positive for $z < \alpha_1$, there exists $\delta_c > 0$ such that

$$\operatorname{Arg}\left(\zeta_{c,\alpha_1}^{1/2}(x + \mathrm{i}\delta c)\right) \in \left(0, (\pi - \arctan(\delta/\delta_{\alpha_1}))/2\right] \tag{7.33}$$

for all $\delta \in (0, \delta_c)$. Since the maps $\zeta_{c,\alpha_1}(z)$ continuously depend on c and have a rescaled conformal limit as $c \to 0$, see (7.16), and a conformal limit as $c \to 1$, see (7.17), the constants δ_c can be chosen so that $\delta_c \geqslant \delta_* > 0$ for all $c \in (0, 1)$. However, as mentioned before, without loss of generality we can consider only $\delta \in (0, \delta_*)$. Furthermore, similarly to (7.31), it holds that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(z) = 4\Re\left(\zeta_{c,\alpha_1}^{1/2}(z)\right), \quad \Re z < \beta_{c,1},$$

by (7.8). Thus, combining the above expression with (7.32) and (7.33) gives us

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x + \mathrm{i}\delta c) \geqslant \sin(\arctan(\delta/\delta_{\alpha_1})/2) \left|\zeta_{c,\alpha_1}^{1/2}(x + \mathrm{i}\delta c)\right| \geqslant B' \delta^{3/2} c \tag{7.34}$$

for some $B' > 0$, independent of c and δ.

Now, we shall examine what happens when x lies in the vicinity of $\beta_{c,1}$. Unfortunately, there are three different constructions of the conformal maps in this case. Thus, we first assume that $c \in (0, c^*]$ and $|x + \mathrm{i}\delta - \beta_{c,1}| < \delta_{\beta_1} c$, see Lemma 7.5. Then it follows from Lemma 7.5 and (5.1) that

$$\left|\zeta_{\beta_{c,1}}^{3/2}(x + \mathrm{i}\delta c)\right| \geqslant (A_{\beta_1}/4)^{3/2} \delta^{3/2} c.$$

In the considered case $\operatorname{Arg}(x + \mathrm{i}\delta c) \in \left[\pi/2, \pi - \arctan(\delta/\delta_{\beta_1})\right]$. Since the conformal maps $\zeta_{\beta_{c,1}}^{3/2}(z)$ continuously depend on c, have a rescaled limit when $c \to 0$, see (7.21), are positive for $z > \beta_{c,1}$ and negative for $x < \beta_{c,1}$, (7.33) gets now replaced by

$$\operatorname{Arg}\left(\zeta_{\beta_{c,1}}^{3/2}(x + \mathrm{i}\delta c)\right) \in \left(5\pi/8, (3\pi - \arctan(\delta/\delta_{\alpha_1}))/2\right] \tag{7.35}$$

for all $\delta \in (0, \delta_*)$ and a possibly adjusted constant $\delta_* > 0$. Thus, combining the above observations with (7.31) gives us that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(x + \mathrm{i}\delta c) \geqslant (4/3) \sin(\arctan(\delta/\delta_{\beta_1})/2) \left|\zeta_{\beta_{c,1}}^{3/2}(x + \mathrm{i}\delta c)\right| \geqslant B'' \delta^{5/2} c \tag{7.36}$$

for some $B'' > 0$, independent of δ and c. Let now c' be the same as in Lemma 7.6 and $|x + \mathrm{i}\delta - \beta_1| < \delta_{\beta_1} c$ for any $c \in (c^*, c']$, again, see Lemma 7.6. Then it follows from (7.23) that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(z) = -\frac{4}{3}\Re\left(\hat{\zeta}_{c,\beta_1}^{3/2}(z) - \hat{\zeta}_{c,\beta_1}(\beta_1 + \epsilon_c)\hat{\zeta}_{c,\beta_1}^{1/2}(z)\right).$$

Since $\hat{\zeta}_{c,\beta_1}(x)$ is positive for $x > \beta_1$ and negative for $x < \beta_1$, it holds that

$$-\frac{4}{3}\Re\left(\hat{\zeta}_{c,\beta_1}^{3/2}(z) - \hat{\zeta}_{c,\beta_1}(\beta_1 + \epsilon_c)\hat{\zeta}_{c,\beta_1}^{1/2}(z)\right) > -\frac{4}{3}\Re\left(\hat{\zeta}_{c,\beta_1}^{3/2}(z)\right)$$

for z with $\operatorname{Arg}(z) \in (0, \pi)$. Since the maps $\hat{\zeta}_{c,\beta_1}(z)$ continuously depend on $c \in [c^*, c']$, where we set $\hat{\zeta}_{c^*,\beta_1}(z) := \zeta_{c^*,\beta_1}(z)$, see Lemma 7.6, the constant δ_* can be adjusted so that (7.35) remains valid with $\zeta_{c,\beta_1}(z)$ replaced by $\hat{\zeta}_{c,\beta_1}(z)$ for $|\delta| < \delta_*$ and $c \in [c^*, c']$. Hence, we can proceed exactly as in the case

$c \in (0, c^*]$, perhaps, at the expense of possibly adjusting the constant B'' in (7.36). Further, when $c \in [c', 1)$, it follows from (7.26) that

$$\left(H_c^{(0)} - H_c^{(1)}\right)(z) = 4\Re\left(\zeta_{c,\beta_1}^{1/2}(z)\right), \quad \alpha_1 < \Re z < \alpha_2.$$

It also follows from Proposition 4.2 and Lemma 7.7 that $|\zeta'_{c,\beta_1}(\beta_1)|$ is bounded away from 0 independently of $c \in [c', 1)$ (the bound does depend on c'). Notice also that in this case (7.33) remains valid with δ_{α_1} replaced by $\min_{c\in[c',1)} \delta_{\beta_1}(c)$. Therefore, (7.34) remains valid as well, where we need to replace $\zeta_{c,\alpha_1}(z)$ by $\zeta_{c,\beta_1}(z)$ and, perhaps, adjust B'.

It only remains to examine what happens when $\alpha_1 + \delta' c \leqslant x \leqslant \beta_{c,1} - \delta' c$ for some $\delta' > 0$. To this end, let us denote by $\tilde{h}_c(x)$ the following function:

$$\begin{aligned}\tilde{h}_c(x) &:= 2\mathrm{i}\Im\left(h_{c+}^{(0)}(x)\right) = h_{c+}^{(0)}(x) - h_{c-}^{(0)}(x) = h_{c+}^{(0)}(x) - h_{c+}^{(1)}(x)\\ &= 2\mathrm{i}\Im\left(h_{c-}^{(1)}(x)\right) = -2\mathrm{i}\Im\left(h_{c+}^{(1)}(x)\right) = -2\mathrm{i}\Im\left(h_{c-}^{(0)}(x)\right), \quad x \in \Delta_{c,1}^{\circ}.\end{aligned}$$

Let us show that $\tilde{h}_c(x) \neq 0$ for $x \in \Delta_{c,1}^{\circ}$. Indeed, if $\tilde{h}_c(x') = 0$ for some $x' \in \Delta_{c,1}^{\circ}$, then $h_{c+}^{(0)}(x') = h_{c-}^{(0)}(x') = h_{c+}^{(1)}(x') = h_{c-}^{(1)}(x')$ and this value is real. That is, there exist $\boldsymbol{x}', \boldsymbol{x}'' \in \boldsymbol{\Delta}_{c,1}$ ($\pi(\boldsymbol{x}') = \pi(\boldsymbol{x}'') = x'$) at which $h_c(\boldsymbol{z})$ assumes the same non-zero real value. On the other hand, when $c \in (c^*, c^{**})$, $h_c(\boldsymbol{z})$ has simple poles at $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2$. Therefore, it can be clearly seen from (4.2) that $h_c^{(0)}(x)$ assumes every non-zero real value twice, once on $(-\infty, \alpha_1) \cup (\beta_2, \infty)$ and once on (β_1, α_2). Furthermore, (4.2) also shows that $h_c^{(1)}(x)$ and $h_c^{(2)}(x)$ assume every non-zero real value once on $(-\infty, \alpha_1) \cup (\beta_1, \infty)$ and $(-\infty, \alpha_2) \cup (\beta_2, \infty)$, respectively. As $h_c(\boldsymbol{z})$ has four zeros/poles, it assumes every value exactly four times. Thus, if $\tilde{h}_c(x')$ were zero, $h_c(\boldsymbol{z})$ would assume a given real value six times, which is impossible. Since the proof for the case $c \in (0, c^*] \cup [c^{**}, 1)$ is quite similar, the claim follows.

For the next step, we would like to argue that

$$\tilde{h}_{\min} := \inf_{c\in(0,1)} \min_{\alpha_1+\delta' c \leqslant x \leqslant \beta_{c,1}-\delta' c} |\tilde{h}_c(x)| > 0.$$

For that, it will be convenient to consider the rescaled function $\hat{\tilde{h}}_c(s) := \tilde{h}_c(\beta_{c,1} + |\Delta_{c,1}|(s-1)/2)$. These functions are purely-imaginary and non-vanishing on $(-1, 1)$. It follows from (4.8) that there exists $\delta'' > 0$ such that

$$\tilde{h}_{\min} \geqslant \inf_{c\in(0,1)} \min_{-1+\delta'' \leqslant s \leqslant 1-\delta''} |\hat{\tilde{h}}_c(s)|.$$

For each c fixed, the minimum over s is clearly non-zero and continuously depends on c. On the other hand, exactly as in Lemma 7.5, it holds that

$$\hat{\tilde{h}}_c(s) \to -\frac{\mathrm{i}}{|w_2(\alpha_1)|}\sqrt{\frac{1-s}{1+s}} \tag{7.37}$$

as $c \to 0$ uniformly on $[-1+\delta'', 1-\delta'']$, which again, has a non-zero minimum of the absolute value. Moreover, a computation similar to the one leading to (7.17) gives us that

$$\hat{\tilde{h}}_c(s) \to -\frac{4\mathrm{i}}{\sqrt{\beta_1 - \alpha_1}}\frac{1}{\sqrt{1-s^2}} \tag{7.38}$$

as $c \to 1$ uniformly on $[-1+\delta'', 1-\delta'']$, which also has a non-zero minimum of the absolute value. Hence, it indeed holds that $\tilde{h}_{\min} > 0$.

Now, observe that $\tilde{h}_c(x)$ is a trace of a function analytic across $\Delta_{c,1}^{\circ}$, namely, of

$$\tilde{h}_c(z) := \begin{cases} h_c^{(0)}(z) - h_c^{(1)}(z), & \Im z > 0,\\ h_c^{(1)}(z) - h_c^{(0)}(z), & \Im z < 0.\end{cases}$$

Therefore, for each $x' \in [\alpha_1 + \delta' c, \beta_{c,1} - \delta' c]$ fixed, there exists $\delta(c; x') > 0$ such that

$$\left|\widetilde{H}_c(z; x')\right| \geqslant (\tilde{h}_{\min}/4)|z - x'|, \quad \widetilde{H}_c(z; x') := \int_{x'}^{z} \tilde{h}_c(s)\mathrm{d}s, \tag{7.39}$$

for all $|z - x'| < \delta(c; x')c$ by (5.1). Notice that $\delta(x')c$ can be taken to be the radius of the largest disk of conformality of $\widetilde{H}_c(z; x')$. Observe also that $\delta(c; x')$ continuously depends on x' and therefore there exists $\delta(c) > 0$ such that $\delta(c; x') \geqslant \delta(c)$ for all $x' \in [\alpha_1 + \delta' c, \beta_{c,1} - \delta' c]$. Since $\delta(c)$ can be made to continuously depend on c and the limits (7.37) and (7.38) hold not only on $(-1, 1)$, but in some neighborhood of $(-1, 1)$ as well, the constant δ_* can be adjusted so that $\delta(c) > \delta_*$ for all $c \in (0, 1)$.

Since the functions $\widetilde{H}_c(z;x')$ are conformal in $|z-x'|<\delta_* c$ for each $x'\in[\alpha_1+\delta' c,\beta_{c,1}-\delta' c]$ and are purely imaginary on the real axis, the same continuity and compactness arguments we have been employing throughout the lemma imply that

$$\Re\left(\widetilde{H}_c(x'+\mathrm{i}y;x')\right)\geqslant C\left|\widetilde{H}_c(x'+\mathrm{i}y;x')\right| \tag{7.40}$$

for all $y\in(0,\delta_* c)$ and $x'\in[\alpha_1+\delta' c,\beta_{c,1}-\delta' c]$, where $C>0$ is constant independent of c. Since $h_c(z)=2\partial_z H_c(z)$, it follows from (7.39) and (7.40) that

$$\left(H_c^{(0)}-H_c^{(1)}\right)(x+\mathrm{i}\delta c)=\Re\left(\widetilde{H}_c(x+\mathrm{i}\delta c;x)\right)\geqslant(C\tilde{h}_{\min}/4)\delta c. \tag{7.41}$$

The estimate in (7.29) now follows from (7.34), (7.36), and (7.41). □

7.5. **Local Parametrices.** Below, we construct solutions of RHP-$\boldsymbol{P}_e$ for $e\in\{\alpha_1,\beta_{\vec{n},1}\}$, $\vec{n}\in\mathcal{N}_{c_\star}$. Recall that the squares U_e have diagonals of length $2\delta c$, where $\delta\leqslant\delta(c_\star)$ see Section 7.2. Additionally, we assume that $\delta\leqslant\min\{\delta_{\alpha_1},\delta_{\beta_1}\}$ or $\delta\leqslant\min\{\delta_{\alpha_1},\delta_{\beta_1}(c_\star)\}$, depending on $c_\star$, see Lemmas 7.4–7.7. Then the maps constructed in Section 7.4 are conformal in the corresponding squares U_e.

7.5.1. *Matrix $\boldsymbol{P}_{\alpha_1}(z)$.* Let $\boldsymbol{\Psi}(\zeta)$ be a matrix-valued function such that

(a) $\boldsymbol{\Psi}(\zeta)$ is holomorphic in $\mathbb{C}\setminus\big(I_+\cup I_-\cup(-\infty,0]\big)$, see (7.2);

(b) $\boldsymbol{\Psi}(\zeta)$ has continuous traces on $I_+\cup I_-\cup(-\infty,0)$ that satisfy

$$\boldsymbol{\Psi}_+(\zeta)=\boldsymbol{\Psi}_-(\zeta)\begin{cases}\begin{pmatrix}0&1\\-1&0\end{pmatrix}, & \zeta\in(-\infty,0),\\ \begin{pmatrix}1&0\\1&1\end{pmatrix}, & \zeta\in I_\pm,\end{cases}$$

where $I_\pm$ are oriented towards the origin;

(c) $\boldsymbol{\Psi}(\zeta)=\boldsymbol{O}(\log|\zeta|)$ as $\zeta\to 0$;

(d) $\boldsymbol{\Psi}(\zeta)$ has the following behavior near ∞:

$$\boldsymbol{\Psi}(\zeta)=\frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1&\mathrm{i}\\\mathrm{i}&1\end{pmatrix}\left(\boldsymbol{I}+\boldsymbol{O}\left(\zeta^{-1/2}\right)\right)\exp\left\{2\zeta^{1/2}\sigma_3\right\}$$

uniformly in $\mathbb{C}\setminus\big(I_+\cup I_-\cup(-\infty,0]\big)$.

Solution of RHP-$\boldsymbol{\Psi}$ was constructed explicitly in [35] with the help of modified Bessel and Hankel functions. Observe that the jump matrices in RHP-$\boldsymbol{\Psi}$(b) have determinant one. Therefore, it follows from RHP-$\boldsymbol{\Psi}$(d) that $\det(\boldsymbol{\Psi}(\zeta))\equiv\sqrt{2}$.

Let $\zeta_{\vec{n},\alpha_1}(z):=\zeta_{c_{\vec{n}},\alpha_1}(z)$, see (7.8), which is conformal in U_{α_1}. It holds due to Lemma 7.4 and (5.1) that

$$\left\{|z|<A_{\alpha_1}\delta n_1^2\right\}\subset|\vec{n}|^2\zeta_{\vec{n},\alpha_1}(U_{\alpha_1}), \tag{7.42}$$

where A_{α_1} is independent of δ and $c_{\vec{n}}=n_1/|\vec{n}|$. It also follows from (4.3) and (7.8) that

$$\zeta_{\vec{n},\alpha_1}(z)=\left(\frac{1}{4|\vec{n}|}\log\left(\Phi_{\vec{n}}^{(0)}(z)/\Phi_{\vec{n}}^{(1)}(z)\right)\right)^2,\quad z\in U_{\alpha_1}. \tag{7.43}$$

Let $\boldsymbol{D}(z)$ be given by (7.6). Note also that the matrix $\sigma_3\boldsymbol{\Psi}(\zeta)\sigma_3$ also satisfies RHP-$\boldsymbol{\Psi}$, but with the orientation of all the rays in RHP-$\boldsymbol{\Psi}$(b) reversed and i replaced by $-\mathrm{i}$ in the asymptotic formula of RHP-$\boldsymbol{\Psi}$(d). Relation (7.43) and RHP-$\boldsymbol{\Psi}$(a,b,c) imply that the matrix

$$\boldsymbol{P}_{\alpha_1}(z):=\boldsymbol{E}_{\alpha_1}(z)\mathsf{T}_1\left(\left(\sigma_3\boldsymbol{\Psi}\sigma_3\right)\left(|\vec{n}|^2\zeta_{\vec{n},\alpha_1}(z)\right)\rho_1^{-\sigma_3/2}(z)\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)^{-\sigma_3/2}(z)\right)\boldsymbol{D}(z), \tag{7.44}$$

satisfies RHP-$\boldsymbol{P}_{\alpha_1}$(a,b,c) for any holomorphic prefactor $\boldsymbol{E}_{\alpha_1}(z)$. As $\zeta_+^{1/4}=\mathrm{i}\zeta_-^{1/4}$ on $(-\infty,0)$, where we take the principal branch, it can be easily checked that

$$\frac{\zeta_+^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}=\frac{\zeta_-^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

there. Then RHP-$\boldsymbol{N}$(b) implies that

$$\boldsymbol{E}_{\alpha_1}(z):=\boldsymbol{M}(z)\mathsf{T}_1\left(\frac{\left(|\vec{n}|^2\zeta_{\vec{n},\alpha_1}(z)\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1&-\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}\rho_1^{-\sigma_3/2}(z)\right)^{-1} \tag{7.45}$$

is holomorphic in $U_{\alpha_1}\backslash\{\alpha_1\}$. Since the first and second columns of $\boldsymbol{M}(z)$ have at most quarter root singularities at α_1 and the third one is bounded, see Lemma 7.3, $\boldsymbol{E}_{\alpha_1}(z)$ is in fact holomorphic in U_{α_1} as desired. Finally, RHP-$\boldsymbol{P}_{\alpha_1}$(d) follows from RHP-$\boldsymbol{\Psi}$(d) and (7.42).

Recall that $\det(\boldsymbol{M}(z)) \equiv \det(\boldsymbol{D}(z)) \equiv 1$ as explained between (7.5) and (7.6). Hence, it holds that $\det(\boldsymbol{E}_{\alpha_1}(z)) \equiv 1/\sqrt{2}$ and respectively $\det(\boldsymbol{P}_{\alpha_1}(z)) \equiv 1$.

7.5.2. *Matrix $\boldsymbol{P}_{\beta_{\vec{n},1}}(z)$ when $c_\star \leqslant c^*$ and $c_{\vec{n}} \leqslant c^*$.* Below, given $\mathcal{N}_{c_\star}$, with $c_\star \leqslant c^*$, we solve RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$ along the subsequence $\mathcal{N}_{c_\star}^{\leqslant} := \{\vec{n} \in \mathcal{N}_{c_\star} : c_{\vec{n}} \leqslant c^*\}$, when such a subsequence is infinite. Clearly, $\mathcal{N}_{c_\star}^{\leqslant}$ only omits finitely many terms from $\mathcal{N}_{c_\star}$ when $c_\star < c^*$.

Given $\sigma \in \mathbb{C}\backslash(-\infty, 0)$ and $s \in (-\infty, \infty)$, let $\boldsymbol{\Phi}_\sigma(\zeta; s)$ be a matrix-valued function such that

(a) $\boldsymbol{\Phi}_\sigma(\zeta; s)$ is holomorphic in $\mathbb{C}\backslash\big(I_+ \cup I_- \cup (-\infty, \infty)\big)$;

(b) $\boldsymbol{\Phi}_\sigma(\zeta; s)$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0) \cup (0, \infty)$ that satisfy

$$\boldsymbol{\Phi}_{\sigma+}(\zeta; s) = \boldsymbol{\Phi}_{\sigma-}(\zeta; s) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \zeta \in I_\pm, \\ \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}, & \zeta \in (0, \infty); \end{cases}$$

(c) $\boldsymbol{\Phi}_1(\zeta; s) = \boldsymbol{O}(1)$ and $\boldsymbol{\Phi}_\sigma(\zeta; s) = \boldsymbol{O}(\log|\zeta|)$ when $\sigma \neq 1$ as $\zeta \to 0$;

(d) $\boldsymbol{\Phi}(\zeta; s)$ has the following behavior near ∞:

$$\boldsymbol{\Phi}_\sigma(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left(\boldsymbol{I} + \boldsymbol{O}\left(\zeta^{-1/2}\right)\right) \exp\left\{-\frac{2}{3}(\zeta + s)^{3/2}\sigma_3\right\}$$

uniformly in $\mathbb{C}\backslash\big(I_+ \cup I_- \cup (-\infty, \infty)\big)$.

As in the previous subsection, notice that $\det(\boldsymbol{\Phi}_\sigma(\zeta; s)) \equiv \sqrt{2}$.

Besides RHP-$\boldsymbol{\Phi}_\sigma$, we shall also need RHP-$\widetilde{\boldsymbol{\Phi}}$ obtained from RHP-$\boldsymbol{\Phi}_0$ by replacing RHP-$\boldsymbol{\Phi}_0$(d) with

(d) $\widetilde{\boldsymbol{\Phi}}(\zeta; s)$ has the following behavior near ∞:

$$\widetilde{\boldsymbol{\Phi}}(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left(\boldsymbol{I} + \boldsymbol{O}\left(\zeta^{-1/2}\right)\right) \exp\left\{-\frac{2}{3}\left(\zeta^{3/2} + s\zeta^{1/2}\right)\sigma_3\right\}.$$

When $\sigma = 1$ and $s = 0$, the Riemann-Hilbert problem RHP-$\boldsymbol{\Phi}_1$ is well known [16] and is solved using Airy functions. In fact, in this case RHP-$\boldsymbol{\Phi}_1$(d) can be improved to

$$\boldsymbol{\Phi}_1(\zeta; 0) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left(\boldsymbol{I} + \boldsymbol{O}\left(\zeta^{-3/2}\right)\right) \exp\left\{-\frac{2}{3}\zeta^{3/2}\sigma_3\right\} \tag{7.46}$$

uniformly in $\mathbb{C}\backslash\big(I_+ \cup I_- \cup (-\infty, \infty)\big)$. More generally, when $\sigma = 1$, the solvability of these two problems for all $s \in (-\infty, \infty)$ was shown in [27] with further properties investigated in [28]. The solvability of the general case $\sigma \in \mathbb{C}\backslash(-\infty, 0)$ was obtained in [44]. In [45, Theorem 4.1] it was shown that RHP-$\boldsymbol{\Phi}_\sigma$(d) can be replaced by

$$\boldsymbol{\Phi}_\sigma(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left(\boldsymbol{I} + \boldsymbol{O}\left(\sqrt{\frac{|s| + 1}{|\zeta| + 1}}\right)\right) \exp\left\{-\frac{2}{3}(\zeta + s)^{3/2}\sigma_3\right\} \tag{7.47}$$

which holds uniformly for $\zeta \in \mathbb{C}\backslash\big(I_+ \cup I_- \cup (-\infty, \infty)\big)$ and $s \in (-\infty, \infty)$ when $\sigma \neq 0$, and uniformly for $s \in [0, \infty)$ when $\sigma = 0$; and that RHP-$\widetilde{\boldsymbol{\Phi}}$(d) can be replaced by

$$\widetilde{\boldsymbol{\Phi}}(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left(\boldsymbol{I} + \boldsymbol{O}\left(\sqrt{\frac{|s| + 1}{|\zeta| + 1}}\right)\right) \exp\left\{-\frac{2}{3}\left(\zeta^{3/2} + s\zeta^{1/2}\right)\sigma_3\right\} \tag{7.48}$$

uniformly for $\zeta \in \mathbb{C}\backslash\big(I_+ \cup I_- \cup (-\infty, 0]\big)$ and $s \in (-\infty, 0]$.

Let $\zeta_{\beta_{\vec{n},1}}(z) := \zeta_{\beta_{c_{\vec{n}},1}}(z)$ be the functions defined in (7.18) that are conformal in $U_{\beta_{\vec{n},1}}$, see Lemma 7.5. It follows from (4.3) and (7.18) that

$$\zeta_{\beta_{\vec{n},1}}(z) = \left(-\frac{3}{4|\vec{n}|} \log\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)\right)^{2/3}, \quad z \in U_{\beta_{\vec{n},1}}. \tag{7.49}$$

According to Lemma 7.5 and (5.1), it holds that

$$\left\{|z| < A_{\beta_1}\delta n_1^{2/3}\right\} \subset |\vec{n}|^{2/3}\zeta_{\beta_{\vec{n},1}}(U_{\beta_{\vec{n},1}}), \tag{7.50}$$

where A_{β_1} is independent of $\vec{n}$ with $\vec{n} \in \mathcal{N}_{c_\star}^{\leqslant}$.

Assume now that $c_\star < c^*$. Recall that is this case $\beta_1 \in U_{\beta_{\vec{n},1}}$ for all $|\vec{n}|$ large enough. Relation (7.49) and RHP-$\mathbf{\Phi}_1$(a,b,c) imply that the matrix

$$\boldsymbol{P}_{\beta_{\vec{n},1}}(z) := \boldsymbol{E}_{\beta_{\vec{n},1}}(z)\mathsf{T}_1\left(\mathbf{\Phi}_1\left(|\vec{n}|^{2/3}\zeta_{\beta_{\vec{n},1}}(z);0\right)\rho_1^{-\sigma_3/2}(z)\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)^{-\sigma_3/2}(z)\right)\boldsymbol{D}(z), \tag{7.51}$$

satisfies RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$(a,b,c) for any holomorphic prefactor $\boldsymbol{E}_{\beta_{\vec{n},1}}(z)$. As in the previous subsection, RHP-$\boldsymbol{N}$(b) implies that

$$\boldsymbol{E}_{\beta_{\vec{n},1}}(z) := \boldsymbol{M}(z)\mathsf{T}_1\left(\frac{\left(|\vec{n}|^{2/3}\zeta_{\beta_{\vec{n},1}}(z)\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\rho_1^{-\sigma_3/2}(z)\right)^{-1} \tag{7.52}$$

is holomorphic in $U_{\beta_{\vec{n},1}}$. Requirement RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$(d) now follows from (7.46) and (7.50).

Assume now that $c_\star = c^*$ and recall (7.3). Observe also that $\beta_{\vec{n},1} \leqslant \beta_1$ for $\vec{n} \in \mathcal{N}_{c^*}^{\leqslant}$ and therefore $s_{\vec{n}} := |\vec{n}|^{2/3}\zeta_{\beta_{\vec{n},1}}(\beta_1) \geqslant 0$. Then, similarly to (7.51), we get from (7.49) and RHP-$\mathbf{\Phi}_0$(a,b,c) that

$$\boldsymbol{P}_{\beta_{\vec{n},1}}(z) := \boldsymbol{E}_{\beta_{\vec{n},1}}(z)\mathsf{T}_1\left(\mathbf{\Phi}_0\left(|\vec{n}|^{2/3}\widetilde{\zeta}_{\beta_{\vec{n},1}}(z);s_{\vec{n}}\right)\rho_1^{-\sigma_3/2}(z)\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)^{-\sigma_3/2}(z)\right)\boldsymbol{D}(z), \tag{7.53}$$

satisfies RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$(a,b,c), where holomorphic prefactor $\boldsymbol{E}_{\beta_{\vec{n},1}}(z)$ is again given by (7.52). Then it follows from (7.47) and (7.49) that

$$\left(\boldsymbol{M}^{-1}\boldsymbol{P}_{\beta_{\vec{n},1}}\boldsymbol{D}^{-1}\right)(s) = \mathsf{T}_1\left(\rho_1^{\sigma_3/2}(s)\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\left(1+\frac{\zeta_{\beta_{\vec{n},1}}(\beta_1)}{\widetilde{\zeta}_{\beta_{\vec{n},1}}(s)}\right)^{\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\times\right.$$
$$\left.\times\left(\boldsymbol{I}+\boldsymbol{O}\left(\sqrt{|\vec{n}|^{-2/3}+\zeta_{\beta_{\vec{n},1}}(\beta_1)}\right)\right)\rho_1^{-\sigma_3/2}(s)\right)$$

for $s \in \partial U_{\beta_{\vec{n},1}}$. Since $\zeta_{\beta_{\vec{n},1}}(\beta_1) \to 0$ as $|\vec{n}| \to \infty$, $\vec{n} \in \mathcal{N}_{c^*}^{\leqslant}$, and $\widetilde{\zeta}_{\beta_{\vec{n},1}}(z)$ is bounded below in modulus on $\partial U_{\beta_{\vec{n},1}}$, RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$(d) follows. As in the previous subsection, we point out that $\det(\boldsymbol{P}_{\beta_{\vec{n},1}}(z)) \equiv 1$.

7.5.3. *Matrix $\boldsymbol{P}_{\beta_{\vec{n},1}}(z)$ when $c_\star = c^*$ and $c_{\vec{n}} > c^*$.* Below, we solve RHP-$\boldsymbol{P}_{\beta_{\vec{n},1}}$ along the subsequence $\mathcal{N}_{c^*}^{>} := \left\{\vec{n} \in \mathcal{N}_{c^*} : c_{\vec{n}} > c^*\right\}$, when such a subsequence is infinite. Let $\hat{\zeta}_{\vec{n},\beta_1}(z) := \hat{\zeta}_{c_{\vec{n}},\beta_1}(z)$ be the conformal map in U_{β_1} constructed in Lemma 7.6. As before, it follows from (4.3) that

$$\hat{\zeta}_{\vec{n},\beta_1}^{3/2}(z) - \hat{\zeta}_{\vec{n},\beta_1}(\beta_1+\epsilon_{\vec{n}})\hat{\zeta}_{\vec{n},\beta_1}^{1/2}(z) = -\frac{3}{4|\vec{n}|}\log\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right), \quad z \in U_{\beta_1}. \tag{7.54}$$

Let $s_{\vec{n}} := -|\vec{n}|^{2/3}\hat{\zeta}_{\vec{n},\beta_1}(\beta_1+\epsilon_{\vec{n}})$. As above, it follows from (7.54) and RHP-$\widetilde{\mathbf{\Phi}}$ that

$$\boldsymbol{P}_{\beta_1}(z) := \boldsymbol{E}_{\beta_1}(z)\mathsf{T}_1\left(\widetilde{\mathbf{\Phi}}\left(|\vec{n}|^{2/3}\hat{\zeta}_{\vec{n},\beta_1}(z);s_{\vec{n}}\right)\rho_1^{-\sigma_3/2}(z)\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)^{-\sigma_3/2}\right)\boldsymbol{D}(z),$$

satisfies RHP-$\boldsymbol{P}_{\beta_1}$, where $\boldsymbol{E}_{\beta_1}(z)$ is given by (7.52) with $\zeta_{\beta_{\vec{n},1}}(z)$ replaced by $\hat{\zeta}_{\vec{n},\beta_1}(z)$, and it follows from (7.48) that RHP-$\boldsymbol{P}_{\beta_1}$(d) is satisfied with

$$o(1) = \boldsymbol{O}\left(\max\left\{\hat{\zeta}_{\vec{n},\beta_1}^{1/2}(\beta_1+\epsilon_{\vec{n}}), |\vec{n}|^{-1/3}\right\}\right).$$

Again, we stress that $\det(\boldsymbol{P}_{\beta_1}(z)) \equiv 1$.

7.5.4. *Matrix $\boldsymbol{P}_{\beta_1}(z)$ when $c_\star > c^*$.* The construction of $\boldsymbol{P}_{\beta_1}(z)$ in the considered case is absolutely identical to the one of $\boldsymbol{P}_{\alpha_1}(z)$ in Section 7.5.1.

Clearly, we can assume that $\vec{n} \in \mathcal{N}_{c_\star}$ is such that $c_{\vec{n}} > c^*$. Let $\zeta_{\vec{n},\beta_1}(z) := \zeta_{c_{\vec{n}},\beta_1}(z)$ be the conformal map defined in (7.26), whose properties were described in Lemma 7.7. It follows from (4.3) and (7.26) that

$$\zeta_{\vec{n},\beta_1}(z) = \left(\frac{1}{4|\vec{n}|}\log\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)\right)^2, \quad z \in U_{\beta_1}.$$

According to Lemma 7.7 and (5.1) theorem and since $n_1^2 \leqslant |\vec{n}|^2$, it holds that

$$\left\{|z| < A_{\beta_1}\delta(z_{c_\star}-\beta_1)n_1^2\right\} \subset |\vec{n}|^2\zeta_{\vec{n},\beta_1}(U_{\beta_1}),$$

where $\delta_{\beta_1}(c)$ is continuous and non-vanishing on $(c^*,1]$. Similarly to (7.44), a solution of RHP-$\boldsymbol{P}_{\beta_1}$ is given by

$$\boldsymbol{P}_{\beta_1}(z) := \boldsymbol{E}_{\beta_1}(z)\mathsf{T}_1\left(\boldsymbol{\Psi}\left(|\vec{n}|^2\zeta_{\vec{n},\beta_1}(z)\right)\rho_1^{-\sigma_3/2}(z)\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(1)}\right)^{-\sigma_3/2}(z)\right)\boldsymbol{D}(z),$$

where

$$\boldsymbol{E}_{\beta_1}(z) := \boldsymbol{M}(z)\mathsf{T}_1\left(\frac{\left(|\vec{n}|^2\zeta_{\vec{n},\beta_1}(z)\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\rho_1^{-\sigma_3/2}(z)\right)^{-1}.$$

It again holds that $\det(\boldsymbol{P}_{\beta_1}(z))\equiv 1$.

7.6. **Solution of RHP-$\boldsymbol{X}$.** Set $U_{\vec{n}} := U_{\alpha_1}\cup U_{\beta_{\vec{n},1}}\cup U_{\alpha_{\vec{n},2}}\cup U_{\beta_2}$ and $\Gamma_{\vec{n}} := \Gamma^+_{\vec{n},1}\cup\Gamma^-_{\vec{n},1}\cup\Gamma^+_{\vec{n},2}\cup\Gamma^-_{\vec{n},2}$. Put

$$\Sigma_{\vec{n},\delta} := \partial U_{\vec{n}}\cup\left(\left(\Gamma_{\vec{n}}\cup[\beta_{\vec{n},1},\beta_1]\cup[\alpha_2,\alpha_{\vec{n},2}]\right)\setminus\overline{U}_{\vec{n}}\right),$$

see Figure 4. For definiteness, we agree that all the segments in $\Sigma_{\vec{n},\delta}$ are oriented from left to right and all

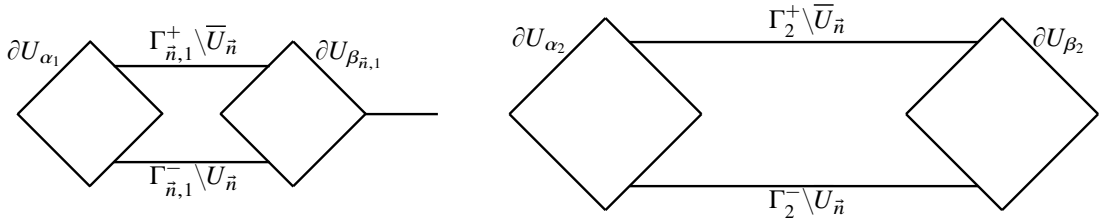

FIGURE 4. Lens $\Sigma_{\vec{n},\delta}$ consisting of two connected components $\Sigma_{\vec{n},\delta,1}$ (the left one) and $\Sigma_{\vec{n},\delta,2}$ (the right one).

the polygons are oriented counter-clockwise. We shall further denote by $\Sigma_{\vec{n},\delta,1}$ and $\Sigma_{\vec{n},\delta,2}$ the left and right, respectively, connected components of $\Sigma_{\vec{n},\delta}$.

For what is to come, we shall need uniform boundedness of the Cauchy operators on $\Sigma_{\vec{n},\delta}$. For convenience, we formulate this claim as a lemma.

Lemma 7.9. *Given $r>1$, there exists a constant $C_r>0$ such that for all $\delta>0$ it holds that*

$$\|C_\pm f\|_{L^r(\Sigma_{\vec{n},\delta})}\leqslant C_r\|f\|_{L^r(\Sigma_{\vec{n},\delta})},$$

where $Cf(z)=\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_{\vec{n},\delta}}\frac{f(t)\mathrm{d}t}{t-z}$ and $C_\pm f(s)$ are the traces of $Cf(z)$ on the left $(-)$ and right $(+)$ hand-sides of $\Sigma_{\vec{n},\delta}$.

Proof. Recall the following known fact, see [15, Equation (7.11)], if R_1,R_2 are two semi-infinite rays with a common endpoint, then

$$\|C_{R_1}f\|_{L^r(R_2)}\leqslant C_r\|f\|_{L^r(R_1)}, \tag{7.55}$$

for some constant $C_r>0$ (we can take $C_2=1$), where C_{R_1} is the Cauchy operator defined on R_1. Moreover, the same estimate holds when $R_2=R_1$ and C_{R_1} is replaced by the trace operators $C_{R_1\pm}$, see [15, Equations (7.5)–(7.7)]. Trivially, the same estimate holds when R_2 is replaced by an interval disjoint from R_1 (may be for an adjusted constant C_r). Since we can embed any two segments with a common endpoint into semi-infinite rays with a common endpoint and embed a function from L^r space of a segment into L^r space of the corresponding ray by extending it by zero, the desired estimate then follows from (7.55) (again, with an adjusted constant C_r). □

Given the global parametrix $\boldsymbol{N}(z)=\boldsymbol{C}(\boldsymbol{MD})(z)$ solving RHP-$\boldsymbol{N}$, see (7.5) and (7.6), and local parametrices $\boldsymbol{P}_e(z)$ solving RHP-$\boldsymbol{P}_e$ and constructed in the previous section, consider the following Riemann-Hilbert Problem (RHP-$\boldsymbol{Z}$):

(a) $\boldsymbol{Z}(z)$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_{\vec{n},\delta}$ and $\boldsymbol{Z}(\infty)=\boldsymbol{I}$;
(b) $\boldsymbol{Z}(z)$ has continuous traces on $\Sigma^\circ_{\vec{n},\delta}$ that satisfy

$$\boldsymbol{Z}_+(s)=\boldsymbol{Z}_-(s)\begin{cases}(\boldsymbol{MD})(s)\mathsf{T}_i\begin{pmatrix}1 & 0\\ 1/\rho_i(s) & 1\end{pmatrix}(\boldsymbol{MD})^{-1}(s), & s\in\Gamma_{\vec{n}}\setminus\overline{U}_{\vec{n}},\\ (\boldsymbol{MD})(s)\mathsf{T}_i\begin{pmatrix}1 & \rho_i(s)\\ 0 & 1\end{pmatrix}(\boldsymbol{MD})^{-1}(s), & s\in\Delta_i\setminus(\Delta_{\vec{n},i}\cup\overline{U}_{\vec{n}}),\\ \boldsymbol{P}_e(s)(\boldsymbol{MD})^{-1}(s), & s\in\partial U_e,\ \ e\in\{\alpha_1,\beta_{\vec{n},1},\alpha_{\vec{n},2},\beta_2\};\end{cases}$$

(c) around the points of $\Sigma_{\vec{n},\delta}\setminus\big(\Sigma^{\circ}_{\vec{n},\delta}\cup\{\beta_1,\alpha_2\}\big)$ the function $\boldsymbol{Z}(z)$ is bounded and around β_1 (resp. α_2) its entries are bounded except for those in the second (resp. third) column that behave like $O(\log|z-\beta_1|)$ (resp. $O(\log|z-\alpha_2|)$).

To show existence and prove size estimates of the matrix function $\boldsymbol{Z}(z)$, let us first estimate the size of its jump:

$$\boldsymbol{V}(s) := \boldsymbol{Z}_-^{-1}(s)\boldsymbol{Z}_+(s) - \boldsymbol{I}, \quad s\in\Sigma_{\vec{n},\delta}. \tag{7.56}$$

More precisely, the following lemma holds.

Lemma 7.10. *Let $\boldsymbol{V}(s)$ be given by (7.56) and RHP-$\boldsymbol{Z}$(b). Then it holds that*

$$\|\boldsymbol{V}\|_{L^\infty(\Sigma_{\vec{n},\delta})} \lesssim \frac{\varepsilon_{\vec{n}}}{\delta^4}\begin{cases} 1, & c_\star\in[0,c^*)\cup(c^{**},1],\\ \min\{z_{c_\star}-\beta_1,\alpha_2-z_{c_\star}\}^{-1/2}, & c_\star\in(c^*,c^{**}),\end{cases} \tag{7.57}$$

with the constant in $\lesssim$ being independent of δ and $\vec{n}$. Moreover, it also holds that $\|\boldsymbol{V}\|_{L^\infty(\Sigma_{\vec{n},\delta})}=o(1)$ when $c_\star\in\{c^*,c^{**}\}$.

Proof. We shall prove (7.57) separately for different parts of $\Sigma_{\vec{n},\delta}$. In fact, we shall do it only on $\Sigma_{\vec{n},\delta,1}$ understanding that the estimates on $\Sigma_{\vec{n},\delta,2}$ can be carried out in the same fashion. For $s\in\partial U_e$, $e\in\{\alpha_1,\beta_{\vec{n},1}\}$, it holds that $\boldsymbol{V}(s)=\boldsymbol{P}_e(s)(\boldsymbol{MD})^{-1}(s)-\boldsymbol{I}$. Therefore, the desired estimate (7.57) follows from Lemma 7.3 and RHP-$\boldsymbol{P}_e$(d). Let now $s=x\in\Delta_1\setminus(\Delta_{\vec{n},1}\cup\overline{U}_{\vec{n}})$, which is non-empty when $c_\star<c^*$. In this case, it holds that

$$\boldsymbol{V}(x)=(\boldsymbol{MD})(x)\mathsf{T}_1\begin{pmatrix}1 & \rho_1(x)\\ 0 & 1\end{pmatrix}(\boldsymbol{MD})^{-1}(x)-\boldsymbol{I}=\rho_1(x)\frac{\Phi^{(0)}_{\vec{n}}(x)}{\Phi^{(1)}_{\vec{n}}(x)}\boldsymbol{M}(x)\boldsymbol{E}_{1,2}\boldsymbol{M}^{-1}(x).$$

Estimate (7.57) now follows from Lemma 7.3 and the estimate

$$\left|\Phi^{(0)}_{\vec{n}}(x)/\Phi^{(1)}_{\vec{n}}(x)\right| = \exp\left\{|\vec{n}|\left(H^{(0)}_{\vec{n}}(x)-H^{(1)}_{\vec{n}}(x)\right)\right\}\leqslant\exp\left\{-B_{\beta_1}\delta^{3/2}n_1\right\}\leqslant\frac{\varepsilon_{\vec{n}}}{B_{\beta_1}\delta^{3/2}}, \tag{7.58}$$

see (4.3) and (7.27). Lastly, let $s\in\Gamma^{\pm}_{\vec{n},1}\setminus U_{\vec{n}}$. Then it holds that

$$\boldsymbol{V}(s)=(\boldsymbol{MD})(s)\mathsf{T}_1\begin{pmatrix}1 & 0\\ 1/\rho_1(s) & 1\end{pmatrix}(\boldsymbol{MD})^{-1}(s)-\boldsymbol{I}=\frac{1}{\rho_1(s)}\frac{\Phi^{(1)}_{\vec{n}}(s)}{\Phi^{(0)}_{\vec{n}}(s)}\boldsymbol{M}(s)\boldsymbol{E}_{2,1}\boldsymbol{M}^{-1}(s).$$

The desired estimate (7.57) can be deduced exactly as in the second step of the proof with (7.29) used instead of (7.27). □

It is essentially a standard argument in the theory of orthogonal polynomials to deduce existence of $\boldsymbol{Z}(z)$ from Lemma 7.10, see [15, Chapter 7].

Lemma 7.11. *Given $\mathcal{N}_{c_\star}$, $c_\star\in[0,1]$, there exists a constant $M(\mathcal{N}_{c_\star})$ such that a solution of RHP-$\boldsymbol{Z}$ exists for all $|\vec{n}|\geqslant M(\mathcal{N}_{c_\star})$ and it satisfies*

$$\max_{i,j}\left|\big[\boldsymbol{Z}(z)-\boldsymbol{I}\big]_{i,j}\right|\lesssim\delta^{-1}\|\boldsymbol{V}\|_{L^\infty(\Sigma_{\vec{n},\delta})} \tag{7.59}$$

for all $z\in\overline{\mathbb{C}}$ when $c_\star\in[c^*,c^{**}]$, $|z-\beta_1|\geqslant\delta/5$ when $c_\star\in(0,c^*)$, $\operatorname{dist}(z,\{\alpha_1,\beta_1\})\geqslant\delta/5$ when $c_\star=0$, $|z-\alpha_2|\geqslant\delta/5$ when $c_\star\in(c^{**},1)$, and $\operatorname{dist}(z,\{\alpha_2,\beta_2\})\geqslant\delta/5$ when $c_\star=1$, where the constant in $\lesssim$ is independent of δ and $\vec{n}$.

Proof. Let C and C_- be the operators defined in Lemma 7.9 and $C_{\boldsymbol{V}}: L^r(\Sigma_{\vec{n},\delta})\to L^r(\Sigma_{\vec{n},\delta})$, $r>1$, be an operator defined by $C_{\boldsymbol{V}}\boldsymbol{F}:=C_-(\boldsymbol{FV})$ for any 2×2 matrix function $\boldsymbol{F}(s)$ in $L^r(\Sigma_{\vec{n},\delta})$. Then it follows from Lemmas 7.9 and 7.10 that

$$\|C_{\boldsymbol{V}}\|_r\leqslant C_r\|\boldsymbol{V}\|_{L^\infty(\Sigma_{\vec{n},\delta})}=o(1). \tag{7.60}$$

Let $M(\mathcal{N}_{c_\star})$ be such that the above norm is less than $1/2$ for all $\vec{n}\in\mathcal{N}_{c_\star}$, $|\vec{n}|\geqslant M(\mathcal{N}_{c_\star})$. Then the operator $\mathcal{I}-C_{\boldsymbol{V}}$ is invertible in $L^r(\Sigma_{\vec{n},\delta})$ for all such $\vec{n}$. Hence, one can readily verify that

$$\boldsymbol{Z}(z)=\boldsymbol{I}+C(\boldsymbol{UV})(z),\quad \boldsymbol{U}(s):=(\mathcal{I}-C_{\boldsymbol{V}})^{-1}(\boldsymbol{I})(s).$$

The above formula and Hölder inequality immediately yield that

$$\max_{i,j}\left|\big[\boldsymbol{Z}(z)-\boldsymbol{I}\big]_{i,j}\right|\lesssim\frac{\|\boldsymbol{UV}\|_{L^r(\Sigma_{\vec{n},\delta})}}{\operatorname{dist}(z,\Sigma_{\vec{n},\delta})}\lesssim\delta^{-1}\|\boldsymbol{V}\|_{L^\infty(\Sigma_{\vec{n},\delta})} \tag{7.61}$$

for $\operatorname{dist}(z,\Sigma_{\vec{n},\delta}) \geqslant \delta/5$, where the constant in $\lesssim$ is independent of $\vec{n}$ and δ (it involves the arclengths of $\Sigma_{\vec{n},\delta}$, but the latter are uniformly bounded above and below).

It can be readily seen from RHP-$\boldsymbol{Z}$(b) that $\boldsymbol{V}(s)$ can be analytically continued off each connected component of $\Sigma^{\circ}_{\vec{n},\delta}$. Hence, solutions of RHP-$\boldsymbol{Z}$ for the same value of $\vec{n}$ and different values of δ are, in fact, analytic continuations of each other. Thus, using (7.61) together with (7.61) where δ is replaced by $\delta/2$, we get that (7.61) in fact holds for $\operatorname{dist}(z,([\beta_{c_\star,1},\beta_1]\cup[\alpha_2,\alpha_{c_\star,2}])\backslash U_{\vec{n}}) \geqslant \delta/5$. The set $([\beta_{c_\star,1},\beta_1]\cup[\alpha_2,\alpha_{c_\star,2}])\backslash U_{\vec{n}}$ is not empty only when $c_\star \in [0,c^*)\cup(c^{**},1]$. In particular, we have finished the proof of the lemma for $c_\star \in [c^*,c^{**}]$. When $c_\star \in (0,c^*)$, set $I_{\vec{n},\delta} := [\beta_1 + \mathrm{i}\delta c_\star/3,\beta_1]\cup(\beta_{c_\star,1}+\mathrm{i}\delta c_\star/3,\beta_1+\mathrm{i}\delta c_\star/3)\backslash\overline{U}_{\vec{n}}$ and let $O_{\vec{n},\delta}$ be the bounded domain delimited by $\partial U_{\vec{n}}$, $I_{\vec{n},\delta}$, and $[\beta_{c_\star,1},\beta_1)\backslash\overline{U}_{\vec{n}}$. Observe that $\boldsymbol{V}(s)$ extends as an analytic matrix function into $O_{\vec{n},\delta}$ and still satisfies (7.58) there by (7.27). Thus, we can analytically continue $\boldsymbol{Z}(s)$ into $O_{\vec{n},\delta}$ by multiplying it by $\boldsymbol{I}+\boldsymbol{V}(z)$ there. This continuation will still have a jump matrix satisfying (7.57) and therefore itself will satisfy (7.61) away from its jump contour. This finishes the proof of the lemma when $c_\star \in (0,c^*)\cup(c^{**},1)$ (the proof for the case $c_\star \in (c^{**},1)$ is identical). The proof in the case $c_\star = 0$ (and therefore in the case $c_\star = 1$) is similar and uses (7.28) instead of (7.27).

The fact that the above constructed matrix $\boldsymbol{Z}(z)$ has behavior as described in RHP-$\boldsymbol{Z}$(c) follows from the fact that it admits an explicit local parametrix around β_1 (resp. α_2) when $c_\star < c^*$ (resp. $c_\star > c^{**}$), see [45, Sections 8.3 and 9.1]. □

The following lemma immediately follows from Lemma 7.11.

Lemma 7.12. *A solution of* RHP-$\boldsymbol{X}$ *is given by*

$$\boldsymbol{X}(z) := \boldsymbol{C}\boldsymbol{Z}(z)\begin{cases}(\boldsymbol{M}\boldsymbol{D})(z), & z\in\overline{\mathbb{C}}\backslash\overline{U}_{\vec{n}},\\ \boldsymbol{P}_e(z), & z\in U_e,\quad e\in\{\alpha_1,\beta_{\vec{n},1},\alpha_{\vec{n},2},\beta_2\},\end{cases} \tag{7.62}$$

where $\boldsymbol{Z}(z)$ solves RHP-$\boldsymbol{Z}$, *$\boldsymbol{N}(z) := \boldsymbol{C}(\boldsymbol{M}\boldsymbol{D})(z)$ solves* RHP-$\boldsymbol{N}$, *see* (7.5)–(7.6), *and $\boldsymbol{P}_e(z)$ solve* RHP-$\boldsymbol{P}_e$, *see Section* 7.5.

7.7. **Proof of Theorems 3.2–3.4.** We are now ready to prove the main results of Section 3. We stop using the notation $c_\star$ and resume writing c as in the statements of Theorems 3.2–3.4.

7.7.1. *Proof of Theorem 3.2.* Let K be a closed subset of $\overline{\mathbb{C}}\backslash(\Delta_{c,1}\cup\Delta_{c,2})$. It follows from Proposition 4.1 that the constant δ in the definition of the contour $\Sigma_{\vec{n},\delta}$ can be adjusted so that K lies outside of each $\overline{\Omega}^{\pm}_{\vec{n},i}$ as well as $\overline{U}_{\vec{n}}$ for all $|\vec{n}|$ large enough. Then it holds that

$$\boldsymbol{Y}(z) = \boldsymbol{C}(\boldsymbol{Z}\boldsymbol{M}\boldsymbol{D})(z),\quad z\in K, \tag{7.63}$$

by (7.4) and Lemma 7.12, where we need to write $\boldsymbol{Y}_\pm(z)$ and $\boldsymbol{Z}_\pm(z)$ for $z\in\Delta_i\backslash\Delta_{c,i}$, $i\in\{1,2\}$. Set

$$B_k(z) := [\boldsymbol{Z}(z)]_{1,k+1} - \delta_{0k} = o(1),\quad k\in\{0,1,2\}, \tag{7.64}$$

where δ_{ij} is the usual Kronecker symbol. Observe that $B_k(\infty)=0$ and

$$|B_k(z)| = \begin{cases} O_{\delta,c}(\varepsilon_{\vec{n}}), & c\notin\{c^*,c^{**}\},\\ o_\delta(1), & c\in\{c^*,c^{**}\},\end{cases} \tag{7.65}$$

uniformly in $\overline{\mathbb{C}}\backslash\{\alpha_1,\beta_1\}$ when $c=0$, in $\overline{\mathbb{C}}\backslash\{\beta_1\}$ when $c\in(0,c^*)$, in $\overline{\mathbb{C}}$ when $c\in[c^*,c^{**}]$, in $\overline{\mathbb{C}}\backslash\{\alpha_2\}$ when $c\in(c^{**},1)$, and in $\overline{\mathbb{C}}\backslash\{\alpha_2,\beta_2\}$ when $c=1$ by (7.57) and (7.59), where the dependence on c of $O_{\delta,c}(\varepsilon_{\vec{n}})$ is uniform on compact subsets of $[0,c^*)\cup(c^{**},1]$. Then it follows from (7.1), (7.63), the definition of $\boldsymbol{M}(z)$ in (7.5), and of $\boldsymbol{C}$, $\boldsymbol{D}(z)$ in (7.6) that

$$\begin{aligned} P_{\vec{n}}(z) &= [\boldsymbol{Y}(z)]_{1,1} = [\boldsymbol{C}]_{1,1}[(\boldsymbol{Z}\boldsymbol{M})(z)]_{1,1}[\boldsymbol{D}(z)]_{1,1}\\ &= \gamma_{\vec{n}}S^{(0)}_{\vec{n}}(z)\left(1+B_0(z)+s_{\vec{n},1}B_1(z)\Upsilon^{(0)}_{\vec{n},1}(z)+s_{\vec{n},2}B_2(z)\Upsilon^{(0)}_{\vec{n},2}(z)\right)\Phi^{(0)}_{\vec{n}}(z),\end{aligned}$$

where $s_{\vec{n},i} := S^{(0)}_{\vec{n}}(\infty)/S^{(i)}_{\vec{n}}(\infty)$, $i\in\{1,2\}$. The first asymptotic formula of the theorem now follows from (7.65), (5.5)–(5.8), and (3.10).

Let now K be a closed subset of $\Delta^{\circ}_{c,1}\cup\Delta^{\circ}_{c,2}$. Again, we can adjust δ so that K does not intersect $\overline{U}_{\vec{n}}$ for all $|\vec{n}|$ large enough. Hence,

$$\boldsymbol{Y}_\pm(x) = \boldsymbol{C}(\boldsymbol{Z}\boldsymbol{M}_\pm\boldsymbol{D}_\pm)(x)(\boldsymbol{I}\pm\rho_i^{-1}(x)\boldsymbol{E}_{i+1,1}),\quad x\in K\cap\Delta_{c,i}, \tag{7.66}$$

for $i \in \{1,2\}$, again by (7.4) and Lemma 7.12. Thus, we get for $x \in K \cap \Delta_{c,i}$ that

$$P_{\vec{n}}(x) = \gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}_{\pm}(x)\left(1 + B_0(x) + B_1(x)\Upsilon^{(0)}_{\vec{n},1\pm}(x) + B_2(x)\Upsilon^{(0)}_{\vec{n},2\pm}(x)\right)$$
$$\pm\, \gamma_{\vec{n}}(\rho_i w_{\vec{n},i\pm})^{-1}(x)\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}_{\pm}(x)\left(1 + B_0(x) + B_1(x)\Upsilon^{(i)}_{\vec{n},1\pm}(x) + B_2(x)\Upsilon^{(i)}_{\vec{n},2\pm}(x)\right).$$

Since $F^{(0)}_{\pm}(x) = F^{(i)}_{\mp}(x)$ on $\Delta_{\vec{n},i}$ for any rational function $F(z)$ on $\mathfrak{R}_{\vec{n}}$, the second asymptotic formula of the theorem now follows from (3.7), (7.65), and (5.5)–(5.8).

7.7.2. *Proof of Theorem 3.3.* Similarly to the matrix $\boldsymbol{Y}(z)$ defined in (7.1), set

$$\widehat{\boldsymbol{Y}}(z) := \begin{pmatrix} L_{\vec{n}}(z) & -A^{(1)}_{\vec{n}}(z) & -A^{(2)}_{\vec{n}}(z) \\ -d_{\vec{n},1}L_{\vec{n}+\vec{e}_1}(z) & d_{\vec{n},1}A^{(1)}_{\vec{n}+\vec{e}_1}(z) & d_{\vec{n},1}A^{(2)}_{\vec{n}+\vec{e}_1}(z) \\ -d_{\vec{n},2}L_{\vec{n}+\vec{e}_2}(z) & d_{\vec{n},2}A^{(1)}_{\vec{n}+\vec{e}_2}(z) & d_{\vec{n},2}A^{(2)}_{\vec{n}+\vec{e}_2}(z) \end{pmatrix}, \tag{7.67}$$

where the constants $d_{\vec{n},i}$ are chosen so that the polynomials $d_{\vec{n},i}A^{(i)}_{\vec{n}+\vec{e}_i}(z)$ are monic. It was shown in [24, Theorem 4.1] that

$$\widehat{\boldsymbol{Y}}(z) = \big(\boldsymbol{Y}^{\mathsf{T}}(z)\big)^{-1}. \tag{7.68}$$

Hence, it follows from (7.63) that on closed subsets of $\overline{\mathbb{C}}\setminus\big(\Delta_{c,1}\cup\Delta_{c,2}\big)$ it holds that

$$\widehat{\boldsymbol{Y}}(z) = \boldsymbol{C}^{-1}\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(z)\big(\boldsymbol{M}^{-1}\big)^{\mathsf{T}}(z)\boldsymbol{D}^{-1}(z)$$

(as before, the contour $\Sigma_{\vec{n},\delta}$ can be adjusted to accommodate any such closed set, moreover, one needs to write $\widehat{\boldsymbol{Y}}_{\pm}(z)$ for $z \in \Delta_i\setminus\Delta_{c,i}$). The above equation and (7.67) yield that

$$A^{(i)}_{\vec{n}}(z) = -\big[\boldsymbol{C}^{-1}\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(z)\big(\boldsymbol{M}^{-1}\big)^{\mathsf{T}}(z)\boldsymbol{D}^{-1}(z)\big]_{1,i+1}, \quad z \in K. \tag{7.69}$$

Let us rewrite (7.7) as

$$\boldsymbol{M}^{-1}(z) =: \operatorname{diag}\left(\frac{1}{S^{(0)}_{\vec{n}}(z)}, \frac{w_{\vec{n},1}(z)}{S^{(1)}_{\vec{n}}(z)}, \frac{w_{\vec{n},2}(z)}{S^{(2)}_{\vec{n}}(z)}\right)\boldsymbol{\Pi}(z)\boldsymbol{S}(\infty),$$

which serves as a definition of the matrix $\boldsymbol{\Pi}(z)$. Notice that $\tau_{\vec{n}}$, defined in the statement of the theorem, is equal to $[\boldsymbol{C}]_{1,1}$. Thus, it follows from (7.69) that

$$A^{(i)}_{\vec{n}}(z) = -\big[\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(z)\boldsymbol{S}(\infty)\boldsymbol{\Pi}^{\mathsf{T}}(z)\big]_{1,i+1}\frac{w_{\vec{n},i}(z)}{\tau_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}(z)}, \quad z \in K. \tag{7.70}$$

Similarly to (7.64), set

$$\widehat{B}_k(z) := \Big[\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(z)\Big]_{1,k+1} - \delta_{0k} = o(1), \quad k \in \{0,1,2\}.$$

Observe that all the jump matrices in RHP-$\boldsymbol{Z}$(b) have determinant one. Since $\boldsymbol{Z}(\infty) = \boldsymbol{I}$, we therefore get that $\det(\boldsymbol{Z}(z)) \equiv 1$. Hence, the functions $\widehat{B}_k(z)$ do obey the estimate of (7.65) as well. Again, it holds that $\widehat{B}_k(\infty) = 0$. Thus,

$$\big[\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(z)\boldsymbol{S}(\infty)\boldsymbol{\Pi}^{\mathsf{T}}(z)\big]_{1,i+1} = S^{(0)}_{\vec{n}}(\infty)\left(\Pi^{(i)}_{\vec{n}}(z) + \widehat{B}_0(z)\Pi^{(i)}_{\vec{n}}(z)\right.$$
$$\left. + s^{-1}_{\vec{n},1}\widehat{B}_1(z)\Pi^{(i)}_{\vec{n},1}(z) + s^{-1}_{\vec{n},2}\widehat{B}_2(z)\Pi^{(i)}_{\vec{n},2}(z)\right), \quad z \in K,$$

where, as before, $s_{\vec{n},l} = S^{(0)}_{\vec{n}}(\infty)/S^{(l)}_{\vec{n}}(\infty)$. Now, observe that

$$\Pi_{\vec{n},l}(z)/\Pi_{\vec{n}}(z) = -A^{-1}_{\vec{n},l}\Upsilon_{\vec{n},l}(z), \quad l \in \{1,2\},$$

which follows from comparing zero/pole divisors and the normalizations at $\infty^{(0)}$ of the left- and right-hand sides of the above equality (recall that $\Pi^{(0)}_{\vec{n}}(\infty) = 1$ and $\Pi^{(0)}_{\vec{n},l}(z) = -z^{-1} + O(z^{-2})$, which can be seen from (5.19)). Therefore, it follows from (7.70) that

$$A^{(i)}_{\vec{n}}(z) = -\left(1 + \widehat{B}_0(z) - \frac{\Upsilon^{(i)}_{\vec{n},1}(z)}{s_{\vec{n},1}A_{\vec{n},1}}\widehat{B}_1(z) - \frac{\Upsilon^{(i)}_{\vec{n},2}(z)}{s_{\vec{n},2}A_{\vec{n},2}}\widehat{B}_2(z)\right)\frac{\big(\Pi^{(i)}_{\vec{n}}w_{\vec{n},i}\big)(z)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}(z)}. \tag{7.71}$$

Hence, the first asymptotic formula of the theorem follows from (7.65), (5.5)–(5.8) (here, one needs to recall that $\widehat{B}_l(\infty) = 0$ and therefore the estimate for $(\Upsilon^{(l)}_{\vec{n},l}\widehat{B}_l)(z)$ around infinity follows from the maximum principle), (3.10), and the fact that $A_{\vec{n},1} \sim c^2_{\vec{n}}$ shown in the proof of Lemma 5.1. When $c = 0$ and $i = 1$, we also deduce from (7.71) and the maximum modulus principle that

$$A^{(1)}_{\vec{n}}(z) = \frac{o(1)}{c^2_{\vec{n}}}\frac{S^{(1)}_{\vec{n}}(\infty)}{S^{(1)}_{\vec{n}}(z)}\frac{\big(\Pi^{(1)}_{\vec{n}}w_{\vec{n},1}\big)(z)}{\tau_{\vec{n}}\Phi^{(1)}_{\vec{n}}(z)} = \frac{o(1)}{c^2_{\vec{n}}}\frac{\big(\Pi^{(1)}_{\vec{n}}w_{\vec{n},1}\big)(z)}{\tau_{\vec{n}}\Phi^{(1)}_{\vec{n}}(z)},$$

where we also used (3.9) and $o(1)$ behaves like the right-hand side of (7.65). Recall that $\Pi^{(1)}_{\vec{n}}(z)$ has a double zero at infinity. Therefore,

$$\big|\big(\Pi^{(1)}_{\vec{n}}w^2_{\vec{n},1}\big)(z)\big| = \left|\left(\Upsilon^{(0)}_{\vec{n},2}\Upsilon^{(2)}_{\vec{n},1} - \Upsilon^{(2)}_{\vec{n},2}\Upsilon^{(0)}_{\vec{n},1}\right)(z)\frac{w_{\vec{n},1}(z)}{w_{\vec{n},2}(z)}\right| = O\big(c^2_{\vec{n}}\big)$$

uniformly on closed subsets of $\mathbb{C}\backslash\Delta_{0,1}$ by (5.20), (5.6)–(5.8), and the maximum modulus principle. Clearly, the last two estimates prove the second asymptotic formula of the theorem (the case $c = 1$ and $i = 2$ can be treated similarly).

Finally, (7.66) and (7.68) give us

$$\widehat{\boldsymbol{Y}}_{\pm}(x) = \boldsymbol{C}^{-1}\big(\boldsymbol{Z}^{-1}\big)^{\mathsf{T}}(x)\big(\boldsymbol{M}^{-1}_{\pm}\big)^{\mathsf{T}}(x)\boldsymbol{D}^{-1}_{\pm}(x)\big(\boldsymbol{I} \mp \rho^{-1}_i(x)\boldsymbol{E}_{1,i+1}\big)$$

on any compact subset of $\Delta^{\circ}_{c,i}$, $i \in \{1,2\}$. Analogously to (7.71), the above formula yields that

$$\begin{aligned} A^{(i)}_{\vec{n}}(x) = -&\left(1 + \widehat{B}_0(x) - \frac{\Upsilon^{(i)}_{\vec{n},1\pm}(x)}{s_{\vec{n},1}A_{\vec{n},1}}\widehat{B}_1(x) - \frac{\Upsilon^{(i)}_{\vec{n},2\pm}(x)}{s_{\vec{n},2}A_{\vec{n},2}}\widehat{B}_2(x)\right)\frac{\big(\Pi^{(i)}_{\vec{n}}w_{\vec{n},i}\big)_{\pm}(x)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}_{\pm}(x)} \\ &\pm \rho^{-1}_i(x)\left(1 + \widehat{B}_0(x) - \frac{\Upsilon^{(0)}_{\vec{n},1\pm}(x)}{s_{\vec{n},1}A_{\vec{n},1}}\widehat{B}_1(x) - \frac{\Upsilon^{(0)}_{\vec{n},2\pm}(x)}{s_{\vec{n},2}A_{\vec{n},2}}\widehat{B}_2(x)\right)\frac{\Pi^{(0)}_{\vec{n}\pm}(x)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}_{\pm}(x)}. \end{aligned}$$

Once again, (7.65) and (5.5)–(5.8) imply that

$$A^{(i)}_{\vec{n}}(x) = -(1+o(1))\frac{\big(\Pi^{(i)}_{\vec{n}}w_{\vec{n},i}\big)_{\pm}(x)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}_{\pm}(x)} \pm (1+o(1))\rho^{-1}_i(x)\frac{\Pi^{(0)}_{\vec{n}\pm}(x)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}_{\pm}(x)}$$

uniformly on compact subsets of $\Delta^{\circ}_{c,i}$. Since

$$\mp\rho^{-1}_i(x)\Pi^{(0)}_{\vec{n}\pm}(x)/\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}_{\pm}(x) = \big(\Pi^{(i)}_{\vec{n}\mp}w_{\vec{n},i\mp}\big)(x)/\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(i)}_{\mp}(x), \quad x \in \Delta_{\vec{n},i},$$

by (3.7), the last asymptotic formula of the theorem follows.

7.7.3. *Proof of Theorem 3.4.* As in the previous two subsections, given a closed set K in $\overline{\mathbb{C}}\backslash(\Delta_1 \cup \Delta_2)$, we can adjust the contour $\Sigma_{\vec{n},\delta}$ so that K lies in the unbounded component of its complement. Hence, using the notation of the previous two subsections, we get from (7.1), (7.5), (7.6), (7.63), and (7.65) that

$$R^{(i)}_{\vec{n}}(z) = \gamma_{\vec{n}}S^{(i)}_{\vec{n}}(z)w^{-1}_{\vec{n},i}(z)\left(1 + B_0(z) + s_{\vec{n},1}B_1(z)\Upsilon^{(i)}_{\vec{n},1}(z) + s_{\vec{n},2}B_2(z)\Upsilon^{(i)}_{\vec{n},2}(z)\right)\Phi^{(i)}_{\vec{n}}(z)$$

for $z \in K$, $i \in \{1,2\}$. The first asymptotic formula of the theorem now follows from (7.65), (5.5)–(5.8), (3.10), and the maximum modulus principle applied to $(\Upsilon^{(i)}_{\vec{n},i}B_i)(z)$ to extend the desired estimates to the neighborhood of infinity. As in the proof of Theorem 3.3, it holds when $c = 0$ and $i = 1$ that

$$R^{(1)}_{\vec{n}}(z) = o(1)\tau_{\vec{n}}\Phi^{(1)}_{\vec{n}}(z)w^{-1}_{\vec{n},1}(z)$$

uniformly on closed subsets of $\overline{\mathbb{C}}\backslash\Delta_{0,1}$ by (5.6)–(5.8) and (3.9)–(3.10). Since an analogous formula holds for $c = 1$ and $i = 2$, the second asymptotic formula of the theorem follows.

Finally, it follows from (7.67) and (7.68) that

$$L_{\vec{n}}(z) = \left(1 + \widehat{B}_0(z) - \frac{\Upsilon^{(0)}_{\vec{n},1}(z)}{s_{\vec{n},1}A_{\vec{n},1}}\widehat{B}_1(z) - \frac{\Upsilon^{(0)}_{\vec{n},2}(z)}{s_{\vec{n},2}A_{\vec{n},2}}\widehat{B}_2(z)\right)\frac{\Pi^{(0)}_{\vec{n}}(z)}{\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}(z)}$$

on closed subsets of $\overline{\mathbb{C}}\backslash(\Delta_{c,1} \cup \Delta_{c,2})$, from which the last asymptotic formula of the theorem follows, as usual, by (7.65) (holding for $\widehat{B}_k(z)$ as well), (5.5)–(5.8), (3.10), and since $A_{\vec{n},1} \sim c^2_{\vec{n}}$ as shown in Lemma 5.1.

8. Proof of Theorem 1.2

While proving Theorem 1.2 we first consider the case of fully marginal sequences and then consider separately the asymptotic behavior of $a_{\vec{n},1}, a_{\vec{n},2}$ and $b_{\vec{n},1}, b_{\vec{n},2}$.

8.1. **Fully Marginal Ray Sequences.** In this section we only consider sequences $\mathcal{N}_0$ and $\mathcal{N}_1$ satisfying (3.1). Again, we present the proof only in the case of $c = 0$. Recurrence formula (1.7) for $P_{\vec{n}}(x)$ can be rewritten as

$$z - b_{\vec{n},i} = \frac{P_{\vec{n}+\vec{e}_i}(z)}{P_{\vec{n}}(z)} + a_{\vec{n},1}\frac{P_{\vec{n}-\vec{e}_1}(z)}{P_{\vec{n}}(z)} + a_{\vec{n},2}\frac{P_{\vec{n}-\vec{e}_2}(z)}{P_{\vec{n}}(x)}, \quad i \in \{1,2\}. \tag{8.1}$$

One can easily see from (8.1) that

$$b_{\vec{n},i} = -\lim_{z\to\infty}\left(\frac{P_{\vec{n}+\vec{e}_i}(z)}{P_{\vec{n}}(z)} - z\right). \tag{8.2}$$

Thus, the limiting behavior of $b_{\vec{n},1}, b_{\vec{n},2}$ follows from Theorem 3.1 and (6.2) in Lemma 6.1. Moreover, since the rays $\{\vec{n} \pm \vec{e}_i : \vec{n} \in \mathcal{N}_0\}$ are also fully marginal, we can use Theorem 3.1 to rewrite (8.1) for $i = 2$ as

$$z - b_{\vec{n},2} = (1 + o(1))\varphi_2(z) + (1 + o(1))\frac{a_{\vec{n},1}}{S(z;\alpha_1)(z-\alpha_1)} + (1 + o(1))\frac{a_{\vec{n},2}}{\varphi_2(z)}. \tag{8.3}$$

Recall that $S(z;\alpha_1) = 1 - (B_{0,1} - \alpha_1)/z + O(z^{-2})$ by (6.7) and (2.7). Hence, if we use (2.6) to obtain the first four terms of the power series expansion of $\varphi_2(z)$ at infinity, we then can rewrite (8.3) as

$$\begin{aligned} z - b_{\vec{n},2} = (1 + o(1))\left(z - B_{0,2} - \frac{A_{0,2}}{z} - \frac{A_{0,2}B_{0,2}}{z^2} + O\left(\frac{1}{z^3}\right)\right) + \\ + \frac{a_{\vec{n},1}}{z}\left(1 + \frac{B_{0,1}}{z} + O\left(\frac{1}{z^2}\right)\right) + \frac{a_{\vec{n},2}}{z}\left(1 + \frac{B_{0,2}}{z} + O\left(\frac{1}{z^2}\right)\right). \end{aligned} \tag{8.4}$$

It follows immediately from (8.4) that

$$a_{\vec{n},1} + a_{\vec{n},2} = (1 + o(1))A_{0,2} \quad \text{and} \quad B_{0,1}a_{\vec{n},1} + B_{0,2}a_{\vec{n},2} = (1 + o(1))B_{0,2}A_{0,2},$$

from which the limits of $a_{\vec{n},1}, a_{\vec{n},2}$ easily follow (recall that $\varphi_2(z)$ is non-vanishing).

8.2. **Asymptotics of $a_{\vec{n},1}, a_{\vec{n},2}$ along non-fully Marginal Sequences.** From now on we are assuming that ray sequences $\mathcal{N}_c$ satisfy (3.11). It can be deduced from orthogonality relations (1.5) and definition (3.14) that

$$R^{(i)}_{\vec{n}}(z) = -\frac{h_{\vec{n},i}}{2\pi\mathrm{i}}\frac{1}{z^{n_i+1}} + O(z^{-n_i-2}), \quad h_{\vec{n},i} := \int P_{\vec{n}}(x)x^{n_i}\mathrm{d}\mu_i(x),$$

$i \in \{1,2\}$. In particular, we have that $m_{\vec{n},i} = -2\pi\mathrm{i}/h_{\vec{n}-\vec{e}_i,i}$ in (7.1). Then it follows from the first and second asymptotic formulae of Theorem 3.4, the definition of constants $\gamma_{\vec{n}}$ and $\tau_{\vec{n}}$ in Theorems 3.2 and 3.3, respectively, and the definition of the matrix $\boldsymbol{C}$ in (7.6) that

$$-\frac{h_{\vec{n},i}}{2\pi\mathrm{i}} = \frac{1 + o(1)}{s_{\vec{n},i}}\frac{[\boldsymbol{C}]_{1,1}}{[\boldsymbol{C}]_{i+1,i+1}} \quad \text{or} \quad -\frac{h_{\vec{n},i}}{2\pi\mathrm{i}} = o(1)\frac{[\boldsymbol{C}]_{1,1}}{[\boldsymbol{C}]_{i+1,i+1}} \tag{8.5}$$

where, as before, $s_{\vec{n},i} = S^{(0)}_{\vec{n}}(\infty)/S^{(i)}_{\vec{n}}(\infty)$, $i \in \{1,2\}$, the first formula holds for $i \in \{1,2\}$ when $c \in (0,1)$, $i = 2$ when $c = 0$, and $i = 1$ when $c = 1$, and the second formula holds for the remaining cases. Furthermore, we get from (7.1) that

$$-\frac{2\pi\mathrm{i}}{h_{\vec{n}-\vec{e}_i,i}} = m_{\vec{n},i} = \lim_{z\to\infty} z^{1-|\vec{n}|}[\boldsymbol{Y}(z)]_{i+1,1}. \tag{8.6}$$

Analogously to the computation after (7.63)–(7.65) we get that $[\boldsymbol{Y}(z)]_{i+1,1}$ is equal to

$$[\boldsymbol{C}]_{i+1,i+1}\frac{S^{(0)}_{\vec{n}}(z)}{S^{(0)}_{\vec{n}}(\infty)}\left(s_{\vec{n},i}\Upsilon^{(0)}_{n,i}(z) + B_{0,i}(z) + s_{\vec{n},1}B_{1,i}(z)\Upsilon^{(0)}_{\vec{n},1}(z) + s_{\vec{n},2}B_{2,i}(z)\Upsilon^{(0)}_{\vec{n},2}(z)\right)\Phi^{(0)}_{\vec{n}}(z) \tag{8.7}$$

in a neighborhood of infinity, where $B_{k,i}(z) := [\boldsymbol{Z}(z)]_{i+1,k+1} - \delta_{ik}$, $k \in \{0,1,2\}$, satisfy (7.65). Since $B_{k,i}(\infty) = 0$ and $\Upsilon^{(0)}_{n,i}(z) = A_{\vec{n},i}z^{-1} + O(z^{-2})$ as $z \to \infty$, see (5.3), we get that

$$-\frac{2\pi\mathrm{i}}{h_{\vec{n}-\vec{e}_i,i}} = \left(s_{\vec{n},i}A_{\vec{n},i} + o(1)\right)\frac{[\boldsymbol{C}]_{i+1,i+1}}{[\boldsymbol{C}]_{1,1}}. \tag{8.8}$$

Now, it is well known, see for example [8, Lemma A.1], that $a_{\vec{n},i} = h_{\vec{n},i}/h_{\vec{n}-\vec{e}_i,i}$. Therefore, it follows from (8.5) and (8.8) that

$$a_{\vec{n},i} = (1+o(1))\big(A_{\vec{n},i} + s_{\vec{n},i}^{-1}o(1)\big) \quad \text{or} \quad a_{\vec{n},i} = o(1)\big(s_{\vec{n},i}A_{\vec{n},i} + o(1)\big)$$

$i \in \{1,2\}$, where the first formula holds for $i \in \{1,2\}$ when $c \in (0,1)$, $i = 2$ when $c = 0$, and $i = 1$ when $c = 1$, and the second formula holds for the remaining cases. The desired limits of $a_{\vec{n},i}$ therefore follow from continuity of the constants $A_{c,i}$ with respect to the parameter c, see Proposition 2.1, asymptotic formulae (3.10), and the estimates $A_{c,1} \sim c^2$ as $c \to 0$ ($A_{c,2} \sim (1-c)^2$ as $c \to 1$), see (5.11) and after.

8.3. **Asymptotics of $b_{\vec{n},1}, b_{\vec{n},2}$ along non-fully Marginal Sequences.** Excluding the cases $i = 1$ when $c = 0$ and $i = 2$ when $c = 1$, we get from (8.6)–(8.8) and (5.6)–(5.8) that

$$P_{\vec{n}-\vec{e}_i}(z) = (1+o(1))A_{\vec{n},i}^{-1}\Upsilon_{\vec{n},i}^{(0)}(z)\gamma_{\vec{n}}\big(S_{\vec{n}}\Phi_{\vec{n}}\big)^{(0)}(z) \tag{8.9}$$

in some neighborhood of the point at infinity. Replacing the sequence $\mathcal{N}_c$ with $\{\vec{n}+\vec{e}_i : \vec{n} \in \mathcal{N}_c\}$, we get from (8.2), Theorem 3.2, and (8.9) that

$$b_{\vec{n},i} = -(1+o(1))\lim_{z\to\infty}\left(\frac{A_{\vec{n}+\vec{e}_i}}{\Upsilon_{\vec{n}+\vec{e}_i,i}^{(0)}(z)} - z\right) = (1+o(1))B_{\vec{n}+\vec{e}_i},$$

where we also used (2.5) and (5.3). The desired claim now follows from Proposition 2.1.

Out of the two exceptional cases, we shall only consider the case $i = 1$ when $c = 0$ understanding that the other one can be treated similarly. Assume for the moment that the measure μ_2 is, in fact, the arcsine distribution on Δ_2, that is,

$$\mathrm{d}\mu_2(x) = \frac{\mathrm{d}x}{2\pi\sqrt{(x-\alpha_2)(\beta_2-x)}} = -\frac{\mathrm{d}x}{2\pi\mathrm{i}w_{2+}(x)}. \tag{8.10}$$

Recall the notation of Section 6 where we wrote $P_{\vec{n}}(z) = P_{\vec{n},1}(z)P_{\vec{n},2}(z)$ with polynomial $P_{\vec{n},i}(z)$ having all its zeros on Δ_i. We would like to show that when μ_2 is of the form (8.10), formula (6.1) still holds along any marginal ray sequence $\mathcal{N}_0$. To this end, we shall use 2×2 Riemann-Hilbert analysis of orthogonal polynomials. Since this method has been described in detail in Section 7, we shall only outline the main steps.

It follows from (1.5) and (8.10) that the Riemann-Hilbert problem

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C}\backslash\Delta_2$ and $\lim_{z\to\infty}\boldsymbol{Y}(z)z^{-n_2} = \boldsymbol{I}$;

(b) $\boldsymbol{Y}(z)$ has continuous traces on each Δ_2° that satisfy $\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x)\begin{pmatrix}1 & (P_{\vec{n},1}/w_{2+})(x)\\ 0 & 1\end{pmatrix}$;

(c) the entries of the first column of $\boldsymbol{Y}(z)$ are bounded and the entries of the second column behave like $O(|z-\xi|^{-1/2})$ as $z \to \xi \in \{\alpha_2,\beta_2\}$;

is solved by

$$\boldsymbol{Y}(z) := \begin{pmatrix} P_{\vec{n},2}(z) & R_{\vec{n}}^{(2)}(z)\\ m_{\vec{n},2}^\star P_{\vec{n},2}^\star(z) & m_{\vec{n},2}^\star R_{\vec{n},2}^\star(z)\end{pmatrix},$$

where $P_{\vec{n},2}^\star(z)$ is the monic polynomial of degree n_2-1 orthogonal to lower degree polynomials with respect to the weight $P_{\vec{n},1}(x)\mathrm{d}\mu_2(x)$ and

$$R_{\vec{n},2}^\star(z) = \frac{1}{2\pi\mathrm{i}}\int\frac{P_{\vec{n},2}^\star(x)P_{\vec{n},1}(x)\mathrm{d}\mu_2(x)}{x-z} = \frac{1}{m_{\vec{n},2}^\star z^{n_2}} + O\big(z^{-n_2-1}\big).$$

Let Γ_2 be a Jordan curve encircling Δ_2 counter-clockwise and containing Δ_1 in its exterior. Set

$$\boldsymbol{X}(z) := \boldsymbol{Y}(z)\begin{cases}\begin{pmatrix}1 & 0\\ -(w_2/P_{\vec{n},1})(z) & 1\end{pmatrix} & z \in \Omega_2,\\ \boldsymbol{I} & \text{otherwise,}\end{cases}$$

where Ω_2 is the interior domain of Γ_2. Then $\boldsymbol{X}(z)$ solves the following Riemann-Hilbert problem:

(a) $\boldsymbol{X}(z)$ is analytic in $\mathbb{C}\backslash(\Delta_2\cup\Gamma_2)$ and $\lim_{z\to\infty}\boldsymbol{X}(z)z^{-n_2} = \boldsymbol{I}$;

(b) $\boldsymbol{X}(z)$ has continuous traces on $\Delta_2^\circ \cup \Gamma_2$ that satisfy

$$\boldsymbol{X}_+(s) = \boldsymbol{X}_-(s) \begin{cases} \begin{pmatrix} 0 & (P_{\vec{n},1}/w_{2+})(s) \\ -(w_{2+}/P_{\vec{n},1})(s) & 0 \end{pmatrix}, & s \in \Delta_2, \\ \begin{pmatrix} 1 & 0 \\ (w_2/P_{\vec{n},1})(s) & 1 \end{pmatrix}, & s \in \Gamma_2; \end{cases}$$

(c) the entries of the first column of $\boldsymbol{X}(z)$ are bounded and the entries of the second column behave like $O(|z-\xi|^{-1/2})$ as $z \to \xi \in \{\alpha_2, \beta_2\}$.

The solution of the above Riemann-Hilbert problem is given by $\boldsymbol{X}(z) = \boldsymbol{C}(\boldsymbol{Z}\boldsymbol{L})(z)$, where

$$\boldsymbol{L}(z) := \begin{pmatrix} 1 & 1/w_2(z) \\ 1/\widetilde{\varphi}_2(z) & \widetilde{\varphi}_2(z)/w_2(z) \end{pmatrix} \left(S_{\vec{n}} \widetilde{\varphi}_2^{n_2}\right)^{\sigma_3}(z)$$

with (compare to (3.5) and observe that $\widetilde{\varphi}_{2+}(x)\widetilde{\varphi}_{2-}(x) \equiv 1$ on Δ_2)

$$\widetilde{\varphi}_2(z) := A_{0,2}^{-1/2}\varphi_2(z) \quad \text{and} \quad S_{\vec{n}}(z) := \prod_{i=1}^{n_1} \left(\frac{\widetilde{\varphi}_2(z) - \widetilde{\varphi}_2(x_{\vec{n},i})}{\widetilde{\varphi}_2(z)\widetilde{\varphi}_2(x_{\vec{n},i}) - 1} \frac{\widetilde{\varphi}_2(z)}{z - x_{\vec{n},i}} \right)^{1/2},$$

$\boldsymbol{C}$ is a diagonal matrix of constants such that $\lim_{z\to\infty} \boldsymbol{C}\boldsymbol{L}(z) z^{-n_2\sigma_3} = \boldsymbol{I}$, and $\boldsymbol{Z}(z)$ solves the following Riemann-Hilbert problem:

(a) $\boldsymbol{Z}(z)$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Gamma_2$ and $\boldsymbol{Z}(\infty) = \boldsymbol{I}$;

(b) $\boldsymbol{Z}(z)$ has continuous traces on Γ_2 that satisfy $\boldsymbol{Z}_+(s) = \boldsymbol{Z}_-(s)\boldsymbol{L}(s) \begin{pmatrix} 1 & 0 \\ (w_2/P_{\vec{n},1})(s) & 1 \end{pmatrix} \boldsymbol{L}^{-1}(s)$.

Indeed, as in Section 7, we only need to verify that the jump of $\boldsymbol{Z}(z)$ on Γ_2 can be estimated as $\boldsymbol{I} + o(1)$ as $n_2 \to \infty$, $\vec{n} \in \mathcal{N}_0$. The latter is equal to

$$\boldsymbol{I} + \frac{1}{(w_2 P_{\vec{n},1} S_{\vec{n}}^2 \widetilde{\varphi}_2^{2n_2})(s)} \begin{pmatrix} \widetilde{\varphi}_2(s) & -1 \\ \widetilde{\varphi}_2^2(s) & -\widetilde{\varphi}_2(s) \end{pmatrix}.$$

Observe that

$$(P_{\vec{n},1} S_{\vec{n}}^2)(s) = \varphi_2^{n_1}(s) \prod_{i=1}^{n_1} b(s; x_{\vec{n},i}), \quad b(z;x_0) := \frac{\widetilde{\varphi}_2(z) - \widetilde{\varphi}_2(x_0)}{\widetilde{\varphi}_2(z)\widetilde{\varphi}_2(x_0) - 1}.$$

Notice that $\inf_{s\in\Gamma_2} |\widetilde{\varphi}_2(s)| > 1$ and $\inf_{s\in\Gamma_2, x_0\in\Delta_1} |b(s;x_0)| > 0$ by the compactness of Δ_1 and Γ_2. Therefore, there exist positive constants $C_1 > 1$ and $C_2 < 1$ such that

$$\sup_{s\in\Gamma} |(w_2 P_{\vec{n},1} S_{\vec{n}}^2 \widetilde{\varphi}_2^{2n_2})(s)|^{-1} \leqslant C_1^{n_1} C_2^{2n_2+n_1} = (C_1^{n_1/(2n_2+n_1)} C_2)^{2n_2+n_1} = o(1)$$

as $n_1/n_2 \to 0$. This finishes the proof of the identity $\boldsymbol{X}(z) = \boldsymbol{C}(\boldsymbol{Z}\boldsymbol{L})(z)$ from which (6.1) easily follows. Observe that μ_2 as in (8.10) is a Szegő weight. Hence, Lemma 6.1 is applicable. Therefore,

$$\lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0} \lim_{z\to\infty} \left(\frac{P_{\vec{n}+\vec{e}_1}(z)}{P_{\vec{n}}(z)} - z \right) = -B_{0,1} \tag{8.11}$$

by (6.2). On the other hand, it should be clear from the above argument that the proof in Section 7 will work if μ_2 is as in (8.10). Therefore, Theorem 3.2 for such a choice of μ_2 gives us that

$$\frac{P_{\vec{n}+\vec{e}_1}(z)}{P_{\vec{n}}(z)} = (1+o(1)) \frac{\gamma_{\vec{n}+\vec{e}_1} (S_{\vec{n}+\vec{e}_1} \Phi_{\vec{n}+\vec{e}_1})^{(0)}(z)}{\gamma_{\vec{n}} (S_{\vec{n}} \Phi_{\vec{n}})^{(0)}(z)} \tag{8.12}$$

in a neighborhood of the point at infinity. It follows from (8.11), (8.12), and (3.9) that

$$\lim_{|\vec{n}|\to\infty,\ \vec{n}\in\mathcal{N}_0} \lim_{z\to\infty} \left(\frac{\tau_{\vec{n}+\vec{e}_1} \Phi_{\vec{n}+\vec{e}_1}^{(0)}(z)}{\tau_{\vec{n}} \Phi_{\vec{n}}^{(0)}(z)} - z \right) = -B_{0,1}, \tag{8.13}$$

where $\tau_{\vec{n}}$ was defined in Theorem 3.3. Observe that (8.13) is a statement about Riemann surfaces $\mathfrak{R}_{\vec{n}}$ for $\vec{n} \in \mathcal{N}_0$ and is independent of the original measures μ_1, μ_2. By Theorem 3.2, (8.12) holds for measures μ_1, μ_2 as in Theorem 1.2, which we are currently proving. Hence, polynomials $P_{\vec{n}}(z)$, $\vec{n} \in \mathcal{N}_0$, satisfy (8.11) by (8.13) and (3.9). The final claim of the theorem now follows from (8.2).

Appendix A.

In this appendix, we will study the operators $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ defined in (2.10). As we have already mentioned in Section 2.2, these operators appear in [8, Formula (4.20)] used with $\vec{\kappa} = \vec{e}_1$ and $\vec{\kappa} = \vec{e}_2$, respectively. The analysis in this section is fairly standard for the Spectral Theory of Jacobi matrices on trees (see, e.g., [33] where the Laplacian and its perturbations were studied for some trees with the finite cone type). However, to make the paper self-contained, we provide complete proofs. That will also emphasize the connection between the quantities used in Spectral Theory, such as m–functions to be defined a few lines below, and the quantities standard in the asymptotical analysis of multiple orthogonal polynomials, e.g., function $\chi_c^{(0)}$.

We denote by $\delta^{(Y)}$ the delta function (Kronecker symbol) of the vertex Y. Consider two functions

$$m_I(z) := \langle (\mathcal{L}_c^{(1)} - z)^{-1}\delta^{(O)}, \delta^{(O)}\rangle, \quad m_{II}(z) := \langle (\mathcal{L}_c^{(2)} - z)^{-1}\delta^{(O)}, \delta^{(O)}\rangle. \tag{A.1}$$

Given the function $\chi_c(z)$ from Proposition 2.1 and $c \in (0,1)$, [8, Equation (4.22)] yields that

$$m_I(z) = \frac{-1}{\chi_c^{(0)}(z) - B_{c,1}}, \quad m_{II}(z) = \frac{-1}{\chi_c^{(0)}(z) - B_{c,2}},$$

where, as usual, $\chi_c^{(0)}(z)$ are the values taken from the zero-th sheet $\mathfrak{R}_c^{(0)}$. By the Spectral Theorem [2], they can also be written in the form

$$m_I(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma_O^{(1)}(x)}{x - z}, \quad m_{II}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma_O^{(2)}(x)}{x - z},$$

where $\sigma_O^{(l)}$ is the spectral measure of $\delta^{(O)}$ with respect to $\mathcal{L}_c^{(l)}$, $l \in \{1,2\}$. The properties of the conformal map $\chi_c(z)$ imply that the functions $m_I(z)$ and $m_{II}(z)$ satisfy:

(A) $m_I(z)$ and $m_{II}(z)$ have no poles since $\chi_c^{(0)}(z) \neq B_{c,j}$ for $z \in \mathfrak{R}_c^{(0)}$ by conformality;

(B) both $m_I(z)$ and $m_{II}(z)$ are Herglotz-Nevanlinna functions in $\mathbb{C}^+$, i.e., they are analytic, have positive imaginary part, and are continuous up to the boundary. Moreover, $\Im m_I(x) = \Im m_{II}(x) = 0$ for $x \in \mathbb{R}\backslash(\Delta_{c,1} \cup \Delta_{c,2})$ and $\Im m_I^+(x) > 0$, $\Im m_{II}^+(x) > 0$ for $x \in \Delta_{c,1}^\circ \cup \Delta_{c,2}^\circ$.

We will use the following notation. If $Y, Z \in \mathcal{V}$ and $Y \sim Z$, then deleting the edge (Y,Z) that connects them leaves us with two subtrees. The one containing Y will be called $\mathcal{T}_{[Z,Y]}$, the other one will be called $\mathcal{T}_{[Y,Z]}$. The restriction of any Jacobi matrix $\mathcal{J}$ to a subtree $\mathcal{T}'$ will be denoted by $\mathcal{J}_{\mathcal{T}'}$.

We learned from **(A)** and **(B)** above that $\sigma_O^{(1)}$ and $\sigma_O^{(2)}$ are absolutely continuous measures with supports equal to $\Delta_{c,1} \cup \Delta_{c,2}$. We need this for the following lemma.

Lemma A.1. *If $c \in (0,1)$, then $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ have no eigenvalues.*

Proof. Suppose that $\mathcal{L}_c^{(l)}$, $l \in \{1,2\}$, has an eigenvector Ψ. Since $\sigma_O^{(l)}$ is purely absolutely continuous as just explained, the restriction of $\mathcal{L}_c^{(l)}$ to the cyclic subspace generated by $\delta^{(O)}$ has no eigenvalues by the spectral theorem. Therefore, we must have $\Psi_O = 0$. Now, consider the restrictions of Ψ to $\mathcal{T}_{[O,O_{(ch),1}]}$ and to $\mathcal{T}_{[O,O_{(ch),2}]}$. One of these functions is not identically equal to zero and the one that is not must be an eigenvector of the corresponding operator: either $\mathcal{J}_{\mathcal{T}_{[O,O_{(ch),1}]}}$ or $\mathcal{J}_{\mathcal{T}_{[O,O_{(ch),2}]}}$. By construction, these operators are identical to either $\mathcal{L}_c^{(1)}$ or $\mathcal{L}_c^{(2)}$ and, as we established earlier, this implies that $\Psi_{O_{(ch),1}} = \Psi_{O_{(ch),2}} = 0$. Repeating the argument, we can now show that $\Psi = 0$ identically on the whole tree which gives a contradiction. □

The following observation holds for a general Jacobi matrix (2.8) and (2.9). Let σ_Y denote the spectral measure of $\delta^{(Y)}$ with respect to $\mathcal{J}$, i.e.,

$$m_Y(z) := \left\langle (\mathcal{J} - z)^{-1}\delta^{(Y)}, \delta^{(Y)} \right\rangle = \int_{\mathbb{R}} \frac{d\sigma_Y(x)}{x - z}, \quad z \in \mathbb{C}^+. \tag{A.2}$$

If we delete all edges connecting Y to its neighbors, say l of them, we will be left with the vertex Y and l subtrees $\{\mathcal{T}_{[Y,Y_j]}\}_{j=1}^l$. The restrictions of $\mathcal{J}$ to these subtrees are also Jacobi matrices and we previously denoted them by $\mathcal{J}_{\mathcal{T}_{[Y,Y_j]}}$. Let

$$m_{[Y,Y_j]}(z) := \left\langle (\mathcal{J}_{\mathcal{T}_{[Y,Y_j]}} - z)^{-1}\delta^{(Y_j)}, \delta^{(Y_j)} \right\rangle = \int_{\mathbb{R}} \frac{d\sigma_{[Y,Y_j]}(x)}{x - z}, \quad z \in \mathbb{C}^+. \tag{A.3}$$

Then the following lemma holds.

Lemma A.2. *For every $z \in \mathbb{C}^+$, we have*

$$m_Y(z) = \frac{1}{V_Y - \sum_{j=1}^{l} W_{Y_j,Y} m_{[Y,Y_j]}(z) - z}. \tag{A.4}$$

Proof. Let $f := (\mathcal{J} - z)^{-1}\delta^{(Y)}$. Clearly, $\mathcal{J}f = zf + \delta^{(Y)}$, that is,

$$(\mathcal{J}f)_X = \begin{cases} V_X f_X + \sum_{Z\sim X} W_{Z,X}^{1/2} f_X = z f_X, & X \neq Y, \\ V_Y f_Y + \sum_{j=1}^{l} W_{Y_j,Y}^{1/2} f_{Y_j} = z f_Y + 1, & X = Y. \end{cases} \tag{A.5}$$

Set $f^{(j)} := -\big(W_{Y,Y_j}^{1/2} f_Y\big)^{-1} f_{|\mathcal{V}_{[Y,Y_j]}}$, which is a renormalized restriction of f to the set of vertices $\mathcal{V}_{[Y,Y_j]}$ of $\mathcal{T}_{[Y,Y_j]}$. Observe that

$$\Big(\mathcal{J}_{\mathcal{T}_{[Y,Y_j]}} f^{(j)}\Big)_X = \begin{cases} \big(\mathcal{J}f^{(j)}\big)_X = z f_X^{(j)}, & X \neq Y_j, \\ V_{Y_j} f_{Y_j}^{(j)} + \sum_{Z\sim Y_j, Z\neq Y} W_{Z,Y_j}^{1/2} f_Z^{(j)} = z f_{Y_j}^{(j)} + 1, & X = Y_j, \end{cases} \tag{A.6}$$

where both relations follow from the first line of (A.5) (for the second relation we need to separate the summand corresponding to $Z = Y$, bring it to the other side of the equation, and then divide by it). It follows immediately from (A.6) that

$$\mathcal{J}_{\mathcal{T}_{[Y,Y_j]}} f^{(j)} = z f^{(j)} + \delta^{Y_j} \quad \Rightarrow \quad f^{(j)} = (\mathcal{J}_{\mathcal{T}_{[Y,Y_j]}} - z)^{-1}\delta^{(Y_j)}.$$

The claim of the lemma follows from the second equality in (A.5) since $f_Y = \langle (\mathcal{J} - z)^{-1}\delta^{(Y)}, \delta^{(Y)}\rangle = m_Y(z)$ and similarly $f_{Y_j} = -\big(W_{Y,Y_j}^{1/2} f_Y\big) f_{Y_j}^{(j)} = -W_{Y_j,Y}^{1/2} m_Y(z) m_{[Y,Y_j]}(z)$. □

Remark. The recursion relations for m-functions, such as the one in formula (A.4), are well-known and have been used previously, e.g., [5, 18, 34].

Let us now return to the operators $\mathcal{J} = \mathcal{L}_c^{(l)}$, $l \in \{1, 2\}$. Take any vertex $Y \neq O$. Deleting the edge $(Y, Y_{(p)})$ leaves us with two subtrees. As before, we denote by $\mathcal{T}_{[Y,Y_{(p)}]}$ the one containing $Y_{(p)}$, and let $m_Y^{(l)}(z)$ and $m_{[Y,Z]}^{(l)}(z)$ to be given by (A.2) and (A.3), respectively (with $\mathcal{J} = \mathcal{L}_c^{(l)}$).

Lemma A.3. *For every $Y \neq O$, the function $m_{[Y,Y_{(p)}]}^{(l)}(z)$ is meromorphic in $\overline{\mathbb{C}}\backslash(\Delta_{c,1} \cup \Delta_{c,2})$ and the function $m_Y^{(l)}(z)$ is analytic there.*

Proof. Recall that the functions $m_I(z)$ and $m_{II}(z)$ are in fact analytic in $\overline{\mathbb{C}}\backslash(\Delta_{c,1} \cup \Delta_{c,2})$. We shall prove the desired claims inductively on n, the distance from Y to the root O. Assume first that $n = 1$. Let ι be the type of Y. Formula (A.4) applied at the vertex O to the operator $\mathcal{L}_c^{(l)}$ restricted to the subtree $\mathcal{T}_{[Y,O]}$ gives

$$m_{[Y,O]}^{(l)}(z) = \frac{1}{B_{c,l} - A_{c,3-\iota} m_{[O,Z]}^{(l)}(z) - z},$$

where Z is the other "child" of O and we used an obvious fact that the restriction of $\mathcal{L}_c^{(l)}$ from $\mathcal{T}_{[Y,O]}$ to the subtree $\mathcal{T}_{[O,Z]}$ is the same as the restriction of $\mathcal{L}_c^{(l)}$ from $\mathcal{T}$ to $\mathcal{T}_{[O,Z]}$. Since the restriction of $\mathcal{L}_c^{(l)}$ to $T_{[O,Z]}$ is $\mathcal{L}_c^{(3-\iota)}$, $m_{[O,Z]}^{(l)}(z)$ is equal to either $m_I(z)$ when $\iota = 2$ or $m_{II}(z)$ when $\iota = 1$. In any case, $m_{[Y,O]}^{(l)}(z)$ is meromorphic outside $\Delta_{c,1} \cup \Delta_{c,2}$.

Suppose now that the claims are true for all vertices up to the distance n. Consider any Y such that its distance from the root is $n + 1$. Let ι be the type of Y. As in the first part of the proof, apply (A.4) at the vertex $Y_{(p)}$ of the subtree $\mathcal{T}_{[Y,Y_{(p)}]}$ to get

$$m_{[Y,Y_{(p)}]}^{(l)}(z) = \frac{1}{B_{c,\iota_p} - A_{c,\iota_{(p)}} m_{[Y_{(p)},(Y_{(p)})_{(p)}]}^{(l)}(z) - A_{c,3-\iota} m_{[Y_{(p)},Z]}^{(l)}(z) - z}.$$

where $\iota_{(p)}$ is the type of $Y_{(p)}$ and Z is the "sibling" of Y. The first function in the denominator is meromorphic outside $\Delta_{c,1} \cup \Delta_{c,2}$ by the inductive assumption and the other one is either $m_I(z)$ or $m_{II}(z)$. Thus, $m_{[Y,Y_{(p)}]}^{(l)}$ is also meromorphic outside $\Delta_{c,1} \cup \Delta_{c,2}$. This way we get the claim for $n + 1$ and so we proved the first statement of the lemma.

Now, apply (A.4) to $m_Y^{(l)}(z)$. The functions involved are $m_{[Y,Y_{(ch),j}]}(z)$, $j \in \{1, 2\}$, and $m_{[Y,Y_{(p)}]}(z)$. The first two are $m_I(z), m_{II}(z)$ and they are analytic in the considered domain. The third one is meromorphic there

by the first statement of the lemma. Notice that $m_Y^{(l)}(z)$ can not have poles by Lemma A.1 thus it is analytic outside $\Delta_{c,1} \cup \Delta_{c,2}$. □

Lemma A.4. *Let $Y \in \mathcal{V}$ and $c \in (0,1)$. If $\sigma_Y^{(l)}$ is the spectral measure of Y with respect to $\mathcal{L}_c^{(l)}$, $l \in \{1,2\}$, then it is absolutely continuous and its support is equal to $\Delta_{c,1} \cup \Delta_{c,2}$.*

Proof. The measure $\sigma_O^{(l)}$ is purely absolutely continuous and is supported on $\Delta_{c,1} \cup \Delta_{c,2}$ as explained before Lemma A.1. Fix $Y \neq O$ and let ι_Y be the type of Y. Further, let $m_Y^{(l)}(z)$ and $m_{[Y,Z]}^{(l)}(z)$ be given by (A.2) and (A.3), respectively, with $\mathcal{J} = \mathcal{L}_c^{(l)}$. Then is follows from (2.10) and (A.4) that

$$\begin{aligned}\Im m_Y^{(l)}(E+\mathrm{i}\epsilon) &= \frac{A_{c,\iota_Y}\Im m^{(l)}_{[Y,Y_{(p)}]}(E+\mathrm{i}\epsilon) + \sum_{i=1}^2 A_{c,i}\Im m^{(l)}_{[Y,Y_{(ch),i}]}(E+\mathrm{i}\epsilon) + \epsilon}{|B_{c,\iota_Y} - A_{c,\iota_Y} m^{(l)}_{[Y,Y_{(p)}]}(E+\mathrm{i}\epsilon) - \sum_{i=1}^2 A_{c,i} m^{(l)}_{[Y,Y_{(ch),i}]}(E+\mathrm{i}\epsilon) - (E+\mathrm{i}\epsilon)|^2} \\ &\leqslant \frac{1}{A_{c,\iota_Y}\Im m^{(l)}_{[Y,Y_{(p)}]}(E+\mathrm{i}\epsilon) + \sum_{i=1}^2 A_{c,i}\Im m^{(l)}_{[Y,Y_{(ch),i}]}(E+\mathrm{i}\epsilon) + \epsilon} \leqslant \frac{1}{A_{c,l}\Im m^{(l)}_{[Y,Y_{(ch),l}]}(E+\mathrm{i}\epsilon)},\end{aligned}$$

because the imaginary parts of all m-functions are positive in $\mathbb{C}^+$. Notice now that the restriction of $\mathcal{L}_c^{(l)}$ to any subtree of the type $\mathcal{T}_{[Z_{(p)},Z]}$ is in fact equal to either $\mathcal{L}_c^{(1)}$ or $\mathcal{L}_c^{(2)}$. Therefore, $m^{(l)}_{[Y,Y_{(ch),l}]}$ is either m_I or m_{II}. The properties **(A)** and **(B)** of m_I and m_{II} listed above can be now applied to get

$$\sup_{E\in I, 0<\epsilon<1} |\Im m_Y^{(l)}(E+\mathrm{i}\epsilon)| < \infty$$

for every interval $I \subset \Delta_{c,1} \cup \Delta_{c,2}$. This implies that $\sigma_Y^{(l)}$ is purely absolutely continuous on I. By Lemma A.3, the measure $\sigma_Y^{(l)}$ is supported inside $\Delta_{c,1} \cup \Delta_{c,2}$ and Lemma A.1 implies that it has no mass points. Therefore, we conclude that $\sigma_Y^{(l)}$ is purely absolutely continuous, as claimed. □

Theorem A.1. *We have that $\sigma(\mathcal{L}_c^{(l)}) = \sigma_{\mathrm{ess}}(\mathcal{L}_c^{(l)}) = \Delta_{c,1} \cup \Delta_{c,2}$, $l \in \{1,2\}$, where, as before, we understand that $\Delta_{0,1} := \{\alpha_1\}$ and $\Delta_{1,2} := \{\beta_2\}$.*

Proof. If $c \in (0,1)$, Lemma A.4 shows that $\delta^{(Y)}$ belongs to the absolutely continuous subspace of $\mathcal{L}_c^{(l)}$ for all Y. Since all linear combinations of $\delta^{(Y)}$ must belong to this subspace and are dense in $\ell^2(\mathcal{V})$, this subspace is in fact the whole space $\ell^2(\mathcal{V})$. Thus, $\sigma(\mathcal{L}_c^{(l)}) = \sigma_{\mathrm{ess}}(\mathcal{L}_c^{(l)})$ and it is equal to $\Delta_{c,1} \cup \Delta_{c,2}$ by Lemma A.4 and the Spectral Theorem.

Let $c \in \{0,1\}$. We shall consider $\mathcal{L}_0^{(2)}$ only, other cases can be handled similarly. By (2.7), we have $A_{0,1} = 0$ and $A_{0,2} > 0$. Thus, the operator $\mathcal{L}_0^{(2)}$ decouples into the following direct sum

$$\mathcal{L}_0^{(2)} = \mathcal{A}_1 \bigoplus \left(\bigoplus_{n=1}^{\infty} \mathcal{A}_2 \right) \tag{A.7}$$

where $\mathcal{A}_1$ is one-sided Jacobi matrix

$$\mathcal{A}_1 := \begin{pmatrix} B_{0,2} & \sqrt{A_{0,2}} & 0 & 0 \\ \sqrt{A_{0,2}} & B_{0,2} & \sqrt{A_{0,2}} & 0 \\ 0 & \sqrt{A_{0,2}} & B_{0,2} & \sqrt{A_{0,2}} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and $\mathcal{A}_2$ is one-sided Jacobi matrix given by

$$\mathcal{A}_2 := \begin{pmatrix} B_{0,1} & \sqrt{A_{0,2}} & 0 & 0 \\ \sqrt{A_{0,2}} & B_{0,2} & \sqrt{A_{0,2}} & 0 \\ 0 & \sqrt{A_{0,2}} & B_{0,2} & \sqrt{A_{0,2}} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

This direct sum decomposition implies that $\sigma(\mathcal{L}_0^{(2)}) = \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2)$. It is well known that $\sigma(\mathcal{A}_1) = [B_{0,2} - 2\sqrt{A_{0,2}}, B_{0,2} + 2\sqrt{A_{0,2}}] = [\alpha_2, \beta_2]$, see (2.7) for the second equality, and that

$$\widehat{m}_1(z) := \langle (\mathcal{A}_1 - z)^{-1}\delta^{(0)}, \delta^{(0)} \rangle = \frac{B_{0,2} - z + \sqrt{(z-B_{0,2})^2 - 4A_{0,2}}}{2A_{0,2}} = \frac{B_{0,2} - z + w_2(z)}{2A_{0,2}}.$$

Furthermore, since the restriction of $\mathcal{A}_2$ from $\ell^2(\mathbb{Z}_{\geqslant 0})$ to $\ell^2(\mathbb{N})$ is equal to $\mathcal{A}_1$ and therefore $m_{[0,1]}(z) = \widehat{m}_1(z)$ in the notation of (A.3), we get from (A.4) that

$$\widehat{m}_2(z) := \langle (\mathcal{A}_2 - z)^{-1}\delta^{(0)}, \delta^{(0)} \rangle = \frac{-1}{A_{0,2}\widehat{m}_1(z) + z - B_{0,1}},$$

where $w_2(z)$ was introduced in the Proposition 2.1. One can readily check that $\Im \widehat{m}_2(x) > 0$ for $x \in (\alpha_2, \beta_2)$, $\Im \widehat{m}_2(x) = 0$ for $x \notin [\alpha_2, \beta_2]$, and that $\widehat{m}_2(z)$ has the unique pole at a point $\check{x} \in \mathbb{R}$ given by

$$A_{0,2}\widehat{m}_1(\check{x}) + \check{x} - B_{0,1} = 0 \tag{A.8}$$

which implies that $\check{x} = \alpha_1$ thanks to (2.7). In other words, $\sigma(\mathcal{A}_2) = \alpha_1 \cup [\alpha_2, \beta_2]$. Now, the statement about the spectrum and essential spectrum follows from direct sum decomposition (A.7). □

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