WAVE PROPAGATION THROUGH SPARSE POTENTIAL BARRIERS

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ABSTRACT. We prove that 3-dimensional Schrödinger operator with slowly decaying sparse potential has an a.c. spectrum that fills $\mathbb{R}^+$. A new kind of WKB asymptotics for Green's function is obtained. The absence of positive eigenvalues is established as well.

Consider the Schrödinger operator

$$H = -\Delta + V, x \in \mathbb{R}^d \tag{1}$$

We are interested in studying the scattering properties of H for the slowly decaying real potential V. This problem is classical [23, 27] but very little is known in comparison with one-dimensional case for which the transparent theory can be built. The following conjecture is due to Barry Simon [26]

Conjecture. If $V(x)$ is such that

$$\int_{\mathbb{R}^d} \frac{V^2(x)}{1+|x|^{d-1}}\, dx < \infty \tag{2}$$

then $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$.

Some progress was recently made for slowly decaying oscillating potentials [16, 17, 25, 6] and potentials asymptotically close to the spherically symmetric one [21]. But even for V satisfying the bound $|V(x)| < C\langle x\rangle^{-\gamma}$, $(\gamma < 1, \langle x\rangle = (1+|x|^2)^{0.5})$, there are no results. We also want to mention that for the problem on the Bethe lattice (Caley tree), we have relatively good understanding [7, 8]: consider a rooted Caley tree and denote the root by O. Assume that each point has three neighbors and O has only two. Consider the discrete Schrödinger operator with potential V and denote by $d\sigma_O$ the spectral measure corresponding to the discrete delta function at O. Let $w(\lambda) = (4\pi)^{-1}(8-\lambda^2)^{1/2}$ on $[-2\sqrt{2}, 2\sqrt{2}]$, and $\rho_O = \sigma'_O(\lambda)[w(\lambda)]^{-1}$, a relative density of the spectral measure at the point O. Consider all paths that go from O to infinity without self-intersections and define the probability space on the set of these paths by assigning to each of them the same weight (i.e. as we go from O to infinity, we toss the coin at any vertex and move to one of the neighbors further from O depending on the result). In [8], we proved the following

Theorem 0.1. *For any bounded V, define*

$$s_O = \int_{-2\sqrt{2}}^{2\sqrt{2}} \ln \rho_O(\lambda) w(\lambda) d\lambda$$

Then the following inequality is true

$$\exp s_O \geq \mathbb{E}\left\{\exp\left[-\frac{1}{4}\sum_{n=1}^{\infty} V^2(x_n)\right]\right\} \tag{3}$$

where the expectation is taken with respect to all paths $\{x_n\}$ going from O to infinity without self-intersections. In particular, if the r.h.s. of (3) is positive, then $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$.

Notice that for the r.h.s. to be positive we just need to make sure that there are "enough" paths over which the potential is square summable. This is much weaker than (2) and the result is rather deep.

In the current paper, we consider the following model on $\mathbb{R}^3$, which is both interesting and very important for understanding the general picture. Let $\{R_n\}$ be a sparse sequence of real numbers, i.e. $R_n \to +\infty, R_{n+1}/R_n \to \infty$ as $n \to \infty$. Consider concentric spherical layers $\Sigma_n = \{x : R_n < |x| < R_n + 1\}$ and assume that the measurable functions $v_n(x) = 0$ outside these layers and $|v_n(x)| < v_n, x \in \Sigma_n$, $v_n \to 0$ as $n \to \infty$. Take

$$V_n(x) = \sum_{j=1}^{n} v_j(x), V(x) = \sum_{j=1}^{\infty} v_j(x)$$

Since $V(x)$ decays at infinity, Weyl's theorem on essential spectrum [22] implies that $\sigma_{ess}(H) = [0, \infty)$. We will study the scattering properties of H. In particular, we want to study the a.c. spectrum and the spatial asymptotics of the Green function. In our opinion, the most interesting result of this paper is totally new WKB-type asymptotics. It is different from the classical results for long-range potentials obtained earlier [11, 27, 21] where the WKB correction was given just by the integration of potential over the segments. Our calculations will show that, for irregular potentials, the WKB asymptotics in the given direction should depend, quite naturally, on the potential's behavior in the paraboloid-like domains corresponding to that direction. The method to obtain this asymptotics is not the direct application of perturbation technique. Instead, we use certain induction formula for the amplitude which we iterate to get certain multiplicative structure. In earlier papers [5, 6], new robust methods were developed that allow one to prove the presence of a.c. spectrum in very general situations. We apply this approach in our case and obtain the optimal result with respect to decay of v_n. Then, we also prove the theorem on the absence of positive embedded eigenvalues. For that, the sparse potential does not have to decay at infinity.

The sparse functions are classical for both Harmonic Analysis and Spectral Theory. The Schrödinger operators with sparse potentials were extensively studied earlier (see, e.g., [18, 19, 15] and references there). The one-dimensional result is as follows (see [20, 14]).

Theorem 0.2. (see [14]) *Assume that $d = 1$, $R_n/R_{n+1} \to 0$ and $v_n \to 0$. Then for*

$$V(x) = \sum_{n=1}^{\infty} v_n \phi(x - R_n)$$

($\phi(x)$–a nonzero, nonnegative bump function), we have the following: if $\{v_n\} \in \ell^p(\mathbb{Z}^+), p \leq 2$, the spectrum of H is purely a.c. on $\mathbb{R}^+$. For $p > 2$, it is singular continuous.

We are interested in studying the same phenomena in the multidimensional case. Clearly, by taking the potential spherically symmetric, we can separate variables and apply the one-dimensional result above (see [23] for the discussion on spherically symmetric potentials) thus showing that the spectrum can be singular continuous for $\{v_n\} \in \ell^p, p > 2$ even in multidimensional case. We will address the following question: assume that the sequence $\{R_n\}$ is as sparse as we like and $\{v_n\}$ is an arbitrary sequence from $\ell^2(\mathbb{Z}^+)$. Is it true that the a.c. spectrum exists? Of course, we do not assume the potential to be spherically-symmetric in this case.

First, we need to introduce certain notations and quantities. Symbol Σ will denote the unit sphere in $\mathbb{R}^3$ centered at origin and $\mathbb{T}_\theta-$ the tangent plane at point $\theta \in \Sigma$. For any sequence $\{\alpha_m\} \in \ell^p$, $\|\alpha_m\|_p$ (or just $\|\alpha\|_p$) stands for its ℓ^p–norm. Let $z = k^2$ and $\Pi = \{k \in \mathbb{C}^+, 0 < \operatorname{Im} k < 1, \operatorname{Re} k > 0\}$. If $f(x)$ is nonzero L^2 function with compact support, say, within the unit ball, then $u_n(x,k) = (-\Delta + V_n - k^2)^{-1} f$ has the following asymptotics at infinity

$$\begin{aligned} u_n(x,k) &= \frac{\exp(ikr)}{r}\left(A_n(f,k,\theta) + \bar{o}(1)\right), \\ \frac{\partial u_n(x,k)}{\partial r} &= ik\frac{\exp(ikr)}{r}\left(A_n(f,k,\theta) + \bar{o}(1)\right), \qquad \text{(Sommerfeld's radiation conditions)} \\ & r = |x|, \theta = \frac{x}{|x|}, |x| \to \infty \end{aligned} \tag{4}$$

as long as $k \in \mathbb{C}^+$ and $k^2 \notin \sigma(H_n)$. Since V_n has compact support, A_n is continuous in k up to the real line. It follows, for example, from the following representations

$$u_n(x,k) = (-\Delta - k^2)^{-1}\left[f - V_n u_n\right] = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{|x-y|}\left[f - V_n u_n\right](y)dy$$

$$A_n(f,k,\theta) = \frac{1}{4\pi}\int\limits_{|y|<R_n+1} e^{-ik\langle\theta,y\rangle}[f - V_n u_n](y)dy$$

and from the Agmon absorption principle for short-range potentials (see, e.g., [24]). This principle says that $u_n(x,k)$, as the function in the weighted space $L^2(\mathbb{R}^3, \langle x\rangle^{1/2+\epsilon})$, has continuous extension in k up the real line with zero excluded.

The physical meaning of $A_n(f,k,\theta)$ is the amplitude of the outgoing spherical wave in the direction $\theta \in \Sigma$ after propagation through n concentric barriers. For the free case (i.e., when $V = 0$) the amplitude of function f is given by the formula

$$A^0(f,k,\theta) = (4\pi)^{-1}\int\limits_{\mathbb{R}^3} \exp(-ik\langle\theta,x\rangle)f(x)dx$$

Notice that for the fixed θ this function is entire in k (nonzero f has compact support) and so has finite number of zeroes inside any compact in $\mathbb{C}$.

Instead of proving the asymptotics of the Green function of H, we will study an asymptotical behavior of the sequence $A_n(f,k,\theta)$, as $k \in \Pi$, and $n \to \infty$. That will contain all relevant information on the scattering mechanism. The following is the main result of the paper

Theorem 0.3. *If the sequence $\{R_n\}$ is sparse enough*[1]*, then for any $\{v_n\} \in \ell^2(\mathbb{Z}^+)$, we have*

$$A_n(f,k,\theta) = WKB_n(k,\theta)\cdot \tilde{A}_n(f,k,\theta) \tag{5}$$

where

$$WKB_n(k,\theta) = \exp\left[-(4\pi)^{-1}\int\limits_{\mathbb{R}^3} \frac{\exp(ik(|t|-\langle\theta,t\rangle))}{|t|} V_n(t)dt\right] \tag{6}$$

The second factor $\tilde{A}_n(f,k,\theta)$ has a finite limit as $n\to\infty$ and

$$|\tilde{A}_n(f,k,\theta)| < C(k) \tag{7}$$

uniformly in n for $k : k = \tau + i\epsilon \in \Pi$. If k is fixed, then

$$\tilde{A}_n(f,k,\theta) = A^0(f,k,\theta) + \delta(\|v\|_2, k) \tag{8}$$

We have $\delta(\|v\|_2,k) \to 0$ as $\|v\|_2 \to 0$ and this convergence is uniform in n and $\theta \in \Sigma$.

Moreover, $\sigma_{ac}(H) = \mathbb{R}^+$.

Notice that $WKB_n(k,\theta)$ can behave badly as $n\to\infty$ for fixed k and θ. For instance, it can be unbounded or go to zero. Thus, since $\tilde{A}_n(f,k,\theta)$ converges and (8) holds, we actually obtain an asymptotics of the sequence $A_n(f,k,\theta)$. Notice also that the correction factor $WKB_n(k,\theta)$ does not depend on f.

Proof. For simplicity, assume $\|v\|_2 < 1$. From now on, we will reserve the symbol C for the constant whose value can change from one formula to another. We need two simple lemmas first.

Lemma 0.1. *Consider a potential $V(x) : V(x) = 0$ for $|x| > R$ and $|V(x)| < 1$. Denote the Green function (i.e. the kernel of $(-\Delta+V-k^2)^{-1}$) by $G(x,y,k)$. Then, for any positive L and for $k : k = \tau + i\epsilon; \quad 0 < \tau, \epsilon < L$, we have*

$$G(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|} + \delta(x,y,k) \tag{9}$$

where

$$|\delta(x,y,k)| \le C\frac{R^3 e^{2\epsilon R - \epsilon|x| - \epsilon|y|}}{\tau\epsilon(|x|-R)(|y|-R)}; \quad |x|,|y| > 2R, R > 1 \tag{10}$$

Moreover, if

$$B(\hat{x},y,k) = \lim_{|x|\to\infty}\Bigl(|x|e^{-ik|x|}G(x,y,k)\Bigr)$$

then

$$|\partial B(\hat{x},y,k)| \le (4\pi)^{-1}|k||y|\exp(\epsilon\langle\hat{x},y\rangle) + CR|k|\left(\frac{R^3e^{2\epsilon R-\epsilon|y|)}}{\tau\epsilon(|y|-R)}\right), \hat{x} = x/|x| \tag{11}$$

and the derivative is taken with respect to $\hat{x}$ on the unit sphere Σ. The constants C might depend on L.

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[1]For example, an estimate $R_{n+1} > e^{\alpha R_n}, \alpha > 1, n \in \mathbb{Z}^+$ will be sufficient, we do not try to minimize the sparseness.

Proof. Iterating the second resolvent identity

$(H_0+V-z)^{-1} = (H_0-z)^{-1}-(H_0-z)^{-1}V(H-z)^{-1} = (H_0-z)^{-1}-(H-z)^{-1}V(H_0-z)^{-1}$

we get the following equation for the kernel

$$G(x,y,k) = (4\pi)^{-1}\frac{e^{ik|x-y|}}{|x-y|} - (4\pi)^{-2}\int \frac{e^{ik|x-u|}}{|x-u|}V(u)\frac{e^{ik|u-y|}}{|u-y|}du \tag{12}$$
$$+(4\pi)^{-2}\int \frac{e^{ik|x-u|}}{|x-u|}V(u)\int G(u,t,k)V(t)\frac{e^{ik|t-y|}}{|t-y|}dtdu$$

To estimate the second term, we use $V(x)=0$ for $|x|>R$. That gives

$$\left|\int \frac{e^{ik|x-u|}}{|x-u|}V(u)\frac{e^{ik|u-y|}}{|u-y|}du\right| < C\frac{R^3 e^{2\epsilon R-\epsilon|x|-\epsilon|y|}}{(|x|-R)(|y|-R)}$$

For the third term, we use the Cauchy-Schwarz inequality for the first integral which gives us

$$\left|\int \frac{e^{ik|x-u|}}{|x-u|}V(u)\int G(u,t,k)V(t)\frac{e^{ik|t-y|}}{|t-y|}dtdu\right|$$
$$\leq \left(\int\limits_{|u|<R} \frac{e^{-2\epsilon|x-u|}}{|x-u|^2}du\right)^{1/2} \left\|\int G(u,t,k)V(t)\frac{e^{ik|t-y|}}{|t-y|}dt\right\|_2$$

For the second factor written above, we apply the simple estimate ([1], p.90)

$$\|(A-z)^{-1}f\| \leq (\operatorname{Im} z)^{-1}\|f\| \tag{13}$$

which is true for any $z\in\mathbb{C}^+$ and any selfadjoint operator acting in Hilbert space. Since $G(x,y,k)$ is the kernel of $(H-k^2)^{-1}$ and $\operatorname{Im}(k^2)=2\tau\epsilon$, we have

$$\left|\int \frac{e^{ik|x-u|}}{|x-u|}V(u)\int G(u,t,k)V(t)\frac{e^{ik|t-y|}}{|t-y|}dtdu\right|$$
$$< (2\tau\epsilon)^{-1}\left(\int\limits_{|u|<R} \frac{e^{-2\epsilon|x-u|}}{|x-u|^2}du\right)^{1/2}\left(\int\limits_{|t|<R} \frac{e^{-2\epsilon|t-y|}}{|t-y|^2}dt\right)^{1/2}$$

Now (10) is straightforward.

To obtain (11), multiply (12) by $|x|\exp(-ik|x|)$ and take $|x|\to\infty$. Then, we get the following bounds for any unit vector $\nu\in\mathbb{T}_{\hat{x}}$,

$$|\partial_\nu B(\hat{x},y,k)| \leq (4\pi)^{-1}|k\langle\nu,y\rangle e^{-ik\langle\hat{x},y\rangle}| + (4\pi)^{-2}\left|\int k\langle\nu,u\rangle e^{-ik\langle\hat{x},u\rangle}V(u)\frac{e^{ik|u-y|}}{|u-y|}du\right|$$
$$+(4\pi)^{-2}\left|\int k\langle\nu,u\rangle e^{-ik\langle\hat{x},u\rangle}V(u)\int G(u,t,k)V(t)\frac{e^{ik|t-y|}}{|t-y|}dtdu\right|$$

Then, we estimate the second and the third terms as before. $\square$

Lemma 0.2. *Under the conditions of lemma 0.1, consider nonzero $f(x)\in L^2(\mathbb{R}^3)$ with compact support (say, inside the unit ball). Define the corresponding amplitude $A(f,k,\hat{x})$. Then,*

$$u(x,k) = (H-k^2)^{-1}f = \frac{\exp(ik|x|)}{|x|}\left(A(f,k,\hat{x})+\rho(x,k)\right) \tag{14}$$

where

$$|\rho(x,k)| < C(\tau\epsilon)^{-1}\frac{R^{3.5}e^{\epsilon R}}{|x|-R}; \quad \text{and} \quad |x| > 3R^2 \tag{15}$$

Proof. We have $-\Delta u + Vu = f + k^2 u$ which can be rewritten $u = (-\Delta - k^2)^{-1}\mu$, where $\mu = f - Vu$ has compact support within the ball or radius R. So,

$$u(x,k) = (4\pi)^{-1}\int \frac{e^{ik|x-t|}}{|x-t|}\mu(t)dt$$

Clearly,

$$A(f,k,\hat{x}) = (4\pi)^{-1}\int e^{-ik\langle \hat{x},t\rangle}\mu(t)dt$$

and Cauchy-Schwarz inequality yields

$$\int \left| \left(\frac{|x|}{|x-t|} e^{ik(|x-t|-|x|)} - e^{-ik\langle\hat{x},t\rangle} \right) \mu(t) \right| dt$$

$$\leq \|\mu\|_2 \left[\int\limits_{|t|<R} \left| \frac{|x|}{|x-t|} e^{ik(|x-t|-|x|)} - e^{-ik\langle\hat{x},t\rangle} \right|^2 dt \right]^{1/2} \tag{16}$$

To estimate $\|\mu\|_2$, we use (13)

$$\|\mu\|_2 \leq \|f\|_2 \left[1 + C(\tau\epsilon)^{-1}\right]$$

In (16), the function under the integral will be estimated by its maximum. That gives

$$\text{l.h.s. of (16)} < C(\tau\epsilon)^{-1}R^{1.5}\max_{|t|<R}\left| \left(\frac{|x|}{|x-t|} - 1 \right) e^{ik(|x-t|-|x|)} + (e^{ik|x-t|-ik|x|} - e^{-ik\langle\hat{x},t\rangle}) \right|$$

Let us estimate the second term under the max, the first one allows an elementary bound based on triangle inequality:

$$\left| \left(\frac{|x|}{|x-t|} - 1 \right) e^{ik(|x-t|-|x|)} \right| \leq \frac{|t|}{|x-t|} e^{\epsilon||x-t|-|x||} \leq \frac{R}{|x|-R} e^{\epsilon R}$$

We have

$$e^{ik|x-t|-ik|x|} - e^{-ik\langle\hat{x},t\rangle} = e^{-ik\langle\hat{x},t\rangle}\left[e^{ik\delta} - 1\right], \quad \delta = |x-t| + \langle\hat{x},t\rangle - |x|$$

It is a simple exercise to see that

$$0 < \delta \leq \frac{R^2}{|x-t|} \leq \frac{R^2}{|x|-R}$$

Since $|x| > 3R^2$, we have $\delta < 0.5$. Therefore,

$$\left| e^{ik|x-t|-ik|x|} - e^{-ik\langle\hat{x},t\rangle} \right| \leq C\frac{R^2 e^{\epsilon R}}{|x|-R}$$

Now, (15) is straightforward. $\square$

Now, let us proceed to the proof of the theorem. The strategy is rather simple. We want to obtain recursion for A_n. To do that, we will consecutively perturb H_n by $v_{n+1}(x)$, H_{n+1} by $v_{n+2}(x)$, etc. That will allow us to obtain almost multiplicative representation for the amplitude, so well-known in the one-dimensional case. On the level of physical intuition, by requiring the sparseness of barriers, we make sure that the wave, propagated through n barriers, hits the $n+1$–th barrier almost like an outgoing spherical wave. That makes an analysis doable. To be able to apply lemma 0.2 later on, we make the first assumption on sparseness

$$R_{n+1} > 3R_n^2 \tag{17}$$

Let us write the second resolvent identity for $H_{n+1} = -\Delta + V_n(x) + v_{n+1}(x)$ just like we did before for (12):

$$G_{n+1}(x,y,k) = G_n(x,y,k) - \int G_n(x,u,k)v_{n+1}(u)G_n(u,y,k)du \tag{18}$$
$$+ \int G_n(x,u,k)v_{n+1}(u) \int G_{n+1}(u,s,k)v_{n+1}(s)G_n(s,y,k)dsdu$$

Therefore, for fixed nonzero $f \in L^2(\mathbb{R}^3)$ with compact support:

$$u_{n+1}(x) = \int G_{n+1}(x,y,k)f(y)dy = u_n(x) - \int G_n(x,y,k)v_{n+1}(y)u_n(y)dy$$
$$+ \int G_n(x,y,k)v_{n+1}(y) \int G_{n+1}(y,s,k)v_{n+1}(s)u_n(s)dsdy \tag{19}$$

The second term is the only term linear in $v_{n+1}(x)$. Moreover, since v_{n+1} is supported very far from the the ball $\{|x| < R_n\}$, $u_n(y)$ is essentially a spherical wave

$$(4\pi)^{-1}\frac{e^{ik|y|}}{|y|}A_n(k,\hat{y})$$

on the $(n+1)$–th layer[2]. Then, due to (9) and (10), $G_n(x,y,k)$ is essentially the free Green function $(4\pi)^{-1}e^{ik|x-y|}/(|x-y|)$ as long as both x and y are far from the support of $V_n(x)$. Therefore, by taking $|x|$ to infinity in (19), we get

$$A_{n+1}(k,\hat{x}) = A_n(k,\hat{x}) - (4\pi)^{-1}\int e^{-ik\langle \hat{x},u\rangle}v_{n+1}(u)\frac{e^{ik|u|}}{|u|}A_n(k,\hat{u})du + r_n(k,\hat{x}) \tag{20}$$

and r_n will be of smaller order.

The following lemma is elementary

Lemma 0.3. *For any $\epsilon > 0, r > 1, \theta \in \Sigma$, we have*

$$\int\limits_{|u|=\rho} e^{-\epsilon(|u|-\langle\theta,u\rangle)}du \le C\epsilon^{-1}\rho$$

Proof. Without loss of generality, assume $\theta = (0,0,1)$. Introducing the spherical coordinates $u_1 = \rho\cos\theta\cos\varphi, u_2 = \rho\cos\theta\sin\varphi, u_3 = \rho\sin\theta$, we get

$$\rho^2 \int\limits_{-\pi}^{\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta \cos\theta \exp\left[-\epsilon\rho(1-\sin\theta)\right] < C\epsilon^{-1}\rho$$

after the change of variables. □

[2]Since f is fixed, we will suppress the dependence of $A_n(f,k,\theta)$ on it and will simply write $A_n(k,\theta)$ for shorthand.

Applying this lemma to the second term in (20), we get

$$\left|(4\pi)^{-1}\int e^{-ik\langle\hat{x},u\rangle}v_{n+1}(u)\frac{e^{ik|u|}}{|u|}A_n(k,\hat{u})du\right| < C\|A_n\|_{L^\infty(\Sigma)}v_{n+1}\epsilon^{-1} \tag{21}$$

since the width of each layer is not greater than one.

We will see that r_n can be regarded as a small correction to the recurrence relation which basically looks like this (compare with recurrence (20)):

$$l_{n+1} = \left[I - \int\limits_{R_{n+1}}^{R_{n+1}+1} O_t q_t dt\right] l_n \tag{22}$$

where $l_n(\theta)$ are the functions on Σ,

$$O_t f(\theta) = (4\pi)^{-1} t \int_\Sigma e^{ikt(1-\langle\theta,s\rangle)} f(s)ds \tag{23}$$

and q_t is just an operator of multiplication by the function $q_t(\theta)$ given on the unit sphere. The difficulty of the problem comes from noncommutativity of O_t and q_t. Without the correction r_n, we would have just the product of operators. But to get the needed asymptotics for this product, we will have to essentially use the sparseness condition again.

The rest of the proof goes as follows: we first obtain rough apriori estimates on $A_n(k,\theta)$ and $\partial A_n(k,\theta)$. Then, we will use them to obtain an accurate asymptotics for $A_n(k,\theta)$ as $n\to\infty$. In the last part, this asymptotics will be used to show the presence of a.c. component of the spectral measure.

For r_n, we have $r_n = I_1 + \ldots + I_7$. Each of I_k was obtained from the formula (19) by:

- firstly, plugging in an expression for $G_n(x,y,k)$ from (9) and (10) and an expression for $u_n(x,k)$ coming from (14), (15)
- secondly, taking $|x|\to\infty$.

We have exactly seven terms to consider because one combination was singled out as the second term in (20), the main contribution in recursion.

Thus, applying lemma 0.1 and lemma 0.2, we get the following estimates for each of I_j. The quantities δ_n and ρ_n in the calculations below are taken from lemmas 0.1 and 0.2, respectively.

$$I_1 = -\lim_{|x|\to\infty}|x|e^{-ik|x|}\int \frac{e^{ik|x-u|}}{4\pi|x-u|}v_{n+1}(u)\frac{e^{ik|u|}}{|u|}\rho_n(u,k)du$$

$$|I_1| < C\left|\int e^{-ik\langle\hat{x},u\rangle}v_{n+1}(u)\frac{e^{ik|u|}}{|u|}\rho_n(u,k)du\right|$$

$$< C\sigma_n(\tau\epsilon)^{-1}v_{n+1}\int\limits_{R_{n+1}<|u|<R_{n+1}+1}\frac{e^{\epsilon(\langle\hat{x},u\rangle-|u|)}}{|u|}du \tag{24}$$

where

$$\sigma_n = R_n^{3.5}e^{R_n}(R_{n+1}-R_n)^{-1} \tag{25}$$

An application of lemma 0.3 to the integral in (24) yields

$$|I_1| < C(\tau\epsilon^2)^{-1}\sigma_n v_{n+1} \tag{26}$$

For I_2:

$$I_2 = -\lim_{|x|\to\infty} |x|e^{-ik|x|} \int \delta_n(x,t,k) v_{n+1}(t) \frac{e^{ik|t|}}{|t|} A_n(\hat{t},k)dt$$

$$|I_2| < C \int \frac{R_n^3 e^{2\epsilon R_n - \epsilon|t|}}{\tau\epsilon(t-R_n)} v_{n+1}(t) \frac{e^{-\epsilon|t|}}{|t|} |A_n(\hat{t},k)|dt$$

$$< C(\tau\epsilon)^{-1} R_n^3 v_{n+1} e^{2\epsilon(R_n - R_{n+1})} \|A_n(\hat{t},k)\|_{L^\infty(\Sigma)} \tag{27}$$

due to (17).

Define I_3 as

$$I_3 = -\lim_{|x|\to\infty} |x|e^{-ik|x|} \int \delta_n(x,t,k) v_{n+1}(t) \frac{e^{ik|t|}}{|t|} \rho_n(t,k)dt$$

Then, using the estimates on $\delta_n(x,t,k)$ and $\rho_n(t,k)$ given by lemma 0.1 and lemma 0.2, we have

$$|I_3| < C(\tau\epsilon)^{-1} R_n^3 \sigma_n \int \frac{e^{2\epsilon R_n - 2\epsilon|t|}}{\tau\epsilon(|t|-R_n)|t|} |v_{n+1}(t)|dt < C\tau^{-2}\epsilon^{-2} v_{n+1}\sigma_n R_n^3 e^{2\epsilon(R_n-R_{n+1})} \tag{28}$$

For the other terms, we will be using the so-called Combes-Thomas inequality [10], which says the following[3]. Assume that potential Q is bounded. Then, for the *operator kernel*, we have (see [10], Theorem 1 with $n=1$, $\gamma = 1/2$)

$$\|\chi(y)(-\Delta + Q - z)^{-1}\chi(x)\| < C(\operatorname{Im} z)^{-1} e^{-\upsilon \operatorname{Im} z|x-y|} \tag{29}$$

where $\chi(x)$ is characteristic function of the unit cube centered at x, υ is fixed positive parameter and the norm is understood as the norm of operator acting in $L^2(\mathbb{R}^3)$. In this estimate, we also assume that $0<|z|<L$ ($L-$ any positive constant), then υ and C depend only on $\|Q\|_\infty$ and on L. Although an estimate on the operator kernel is not the same as pointwise estimate on the Green function, it is almost the same in our case. Let us accurately show that for I_4. For the other terms, we will be skipping details. We have

$$I_4 = \lim_{|x|\to\infty} |x|e^{-ik|x|} \int \frac{e^{ik|x-u|}}{4\pi|x-u|} v_{n+1}(u) \int G_{n+1}(u,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} A_n(\hat{s},k)dsdu$$

$$|I_4| < C\int e^{\epsilon\langle\hat{x},u\rangle} |v_{n+1}(u)| \left| \int G_{n+1}(u,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} A_n(\hat{s},k)ds \right| du$$

$$< \sum_{u_i,s_j} \int_{C(u_i)} e^{\epsilon\langle\hat{x},u\rangle} |v_{n+1}(u)| \left| \int_{C(s_j)} G_{n+1}(u,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} A_n(\hat{s},k)ds \right| du \tag{30}$$

where $C(u_i)$ are all unit cubes $\{(s_1,s_2,s_3) : n_l \le s_l \le n_l+1, l=1,2,3\}$ (defined for each integer triple (n_1,n_2,n_3)) that intersect Σ_{n+1}. Points $u_i = (n_1+0.5, n_2+0.5, n_3+0.5)$ (same as s_j) are centers of these cubes and will be used to index them. Application of (29) with $\operatorname{Im} z = 2\tau\epsilon$ and Cauchy-Schwarz inequality for each pair of cubes yields

[3] The actual estimate obtained in [10] is stronger

$$|I_4| < Cv_{n+1}^2 \|A_n\|_{L^\infty(\Sigma)} \sum_{u_i,s_j} e^{\epsilon\langle \hat{x}, u_i\rangle} \frac{e^{-\upsilon\epsilon\tau|u_i-s_j|}}{\tau\epsilon} \frac{e^{-\epsilon|s_j|}}{|s_j|}$$

$$< Cv_{n+1}^2 \|A_n\|_{L^\infty(\Sigma)} \int\limits_{R_{n+1}-2<|u|<R_{n+1}+3} e^{\epsilon\langle \hat{x}, u\rangle} \int\limits_{R_{n+1}-2<|s|<R_{n+1}+3} \frac{e^{-\upsilon\tau\epsilon|u-s|}}{\tau\epsilon} \frac{e^{-\epsilon|s|}}{|s|} ds du$$

$$< Cv_{n+1}^2 \|A_n\|_{L^\infty(\Sigma)} \frac{e^{-\epsilon R_{n+1}}}{R_{n+1}} \int\limits_{R_{n+1}-2<|u|<R_{n+1}+3} e^{\epsilon\langle \hat{x}, u\rangle} \int\limits_{R_{n+1}-2<|s|<R_{n+1}+3} \frac{e^{-\upsilon\tau\epsilon|u-s|}}{\tau\epsilon} ds du$$

We will bound the second integral by the integral over the whole space

$$\int\limits_{\mathbb{R}^3} \frac{e^{-\upsilon\tau\epsilon|u-s|}}{\tau\epsilon} ds < C(\tau\epsilon)^{-4}$$

Then, an application of lemma 0.3 to the integral over u yields

$$|I_4| < Cv_{n+1}^2 \|A_n\|_{L^\infty(\Sigma)} \tau^{-4}\epsilon^{-5} \tag{31}$$

In the same way, we have

$$I_5 = \lim_{|x|\to\infty} |x| e^{-ik|x|} \int \delta_n(x,y,k) v_{n+1}(y) \int G_{n+1}(y,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} A_n(\hat{s},k) ds dy$$

$$|I_5| < C \int \frac{R_n^3 e^{2\epsilon R_n - \epsilon|y|}}{\tau\epsilon(|y|-R_n)} |v_{n+1}(y)| \left| \int G_{n+1}(y,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} A_n(\hat{s},k) ds \right| dy$$

$$< C(\tau\epsilon)^{-5} R_n^3 v_{n+1}^2 \|A_n\|_{L^\infty(\Sigma)} e^{2\epsilon(R_n - R_{n+1})} \tag{32}$$

For I_6:

$$I_6 = \lim_{|x|\to\infty} |x| e^{-ik|x|} \int \delta_n(x,y,k) v_{n+1}(y) \int G_{n+1}(y,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} \rho_n(s,k) ds dy$$

Apply estimates from lemma 0.1 and 0.2 and then the Combes-Thomas inequality. Combining the factors, we see that $|I_6|$ is bounded by

$$\frac{CR_n^{6.5} e^{3\epsilon R_n} v_{n+1}^2}{(\tau\epsilon)^3} \int\limits_{R_{n+1}-2<|y|<R_{n+1}+3} \frac{e^{-\epsilon|y|}}{|y|-R_n} \int\limits_{R_{n+1}-2<|s|<R_{n+1}+3} \frac{e^{-\upsilon\tau\epsilon|u-s|}}{|u-s|} \frac{e^{-\epsilon|s|}}{|s|} ds dy$$

Estimating the integrals as before and recalling the formula (25) for σ_n, we get

$$|I_6| < C(\tau\epsilon)^{-6} R_n^3 \sigma_n v_{n+1}^2 e^{2\epsilon(R_n - R_{n+1})} \tag{33}$$

Define I_7:

$$I_7 = \lim_{|x|\to\infty} |x| e^{-ik|x|} \int \frac{e^{ik|x-y|}}{4\pi|x-y|} v_{n+1}(y) \int G_{n+1}(y,s,k) v_{n+1}(s) \frac{e^{ik|s|}}{|s|} \rho_n(s,k) ds dy$$

In the same way, we obtain

$$|I_7| < C(\tau\epsilon)^{-6} \sigma_n v_{n+1}^2 \tag{34}$$

From now on we assume that τ changes within the interval $I = [a,b], a > 0$. Then, we can disregard dependence on τ and keep track on ϵ only (the bounds we are getting of course might depend on a and b). The estimates on I_j and (21) amount to

$$\|A_{n+1}\|_{L^\infty(\Sigma)} \leq \|A_n\|_{L^\infty(\Sigma)} \left(1 + Cv_{n+1}\epsilon^{-5} + Cv_{n+1}\epsilon^{-5}R_n^3 e^{2\epsilon(R_n - R_{n+1})}\right)$$

$$+C\epsilon^{-6}\sigma_n v_{n+1} + C\epsilon^{-6}\sigma_n R_n^3 v_{n+1} e^{2\epsilon(R_n - R_{n+1})} \tag{35}$$

The following lemma is trivial

Lemma 0.4. *If* $x_n, a_n, b_n \geq 0$ *and* $x_{n+1} \leq a_n x_n + b_n$, $n = 0, 1, \ldots,$ *then*

$$x_{n+1} \leq (x_0 + \sum_{j=0}^{n} b_j) \max_{j=0,1,\ldots,n} \{1, a_j \cdot a_{j+1} \cdot \ldots \cdot a_{n-1} \cdot a_n\}$$

Proof. The proof follows from the iteration of the given inequality. □

Lemma 0.5. *The following estimates*

$$x^j e^{-\epsilon x} \leq (j/e)^j \epsilon^{-j}$$

hold for any $x > 0, j > 0, \epsilon > 0$.

Proof. The function $f(x) = x^j e^{-\epsilon x}$ has maximum at the point $x^* = j\epsilon^{-1}$. □

Lemma 0.6. *The bound below is true for any* $\alpha, v_k \geq 0$

$$\prod_{j=1}^{n}(1 + \alpha v_j) \leq \exp\left[\alpha \sum_{j=1}^{n} v_j\right] \leq \exp\left[\alpha \cdot \|v\|_2 \cdot \sqrt{n}\right]$$

Proof. The first estimate follows from the bound $1 + x \leq e^x, x \geq 0$. The second is an application of Cauchy-Schwarz inequality. □

From (17), we get $R_{n+1} - R_n > R_n$. Therefore, lemma 0.5 with $j = 3$ yields

$$R_n^3 e^{2\epsilon(R_n - R_{n+1})} < C\epsilon^{-3}$$

Thus, (35) implies

$$\|A_{n+1}\|_{L^\infty(\Sigma)} \leq \|A_n\|_{L^\infty(\Sigma)} \left(1 + Cv_{n+1}\epsilon^{-8}\right) + C\epsilon^{-9}\sigma_n v_{n+1} \tag{36}$$

From lemmas 0.4 and 0.6,

$$\|A_n\|_{L^\infty(\Sigma)} \leq \left[\|A_0\|_{L^\infty(\Sigma)} + C\epsilon^{-9}\sum_{j=0}^{n-1} v_{j+1}\sigma_j\right] \exp(C\epsilon^{-8}\|v\|_2 n^{0.5}) \tag{37}$$

Let us make another rather strong assumption on sparseness of $\{R_n\}$:

$$\sigma_n < e^{-n} \tag{38}$$

Then, by Cauchy-Schwarz, we have the following apriori bound on $\|A_n\|_{L^\infty(\Sigma)}$:

$$\|A_n\|_{L^\infty(\Sigma)} \leq g_n = (\|A_0\|_{L^\infty(\Sigma)} + C\|v\|_2\epsilon^{-9}) \exp(C\epsilon^{-8}\|v\|_2 n^{0.5}) \tag{39}$$

We will need analogous estimate for the derivatives of A_n. Let us use formula (20) and the same bounds as before together with estimate (11). From (11), we see that we only pick an extra R_{n+1} in the inequalities. So,

$$\|\partial A_{n+1}\|_{L^\infty(\Sigma)} \leq \|\partial A_n\|_{L^\infty(\Sigma)} + C\|A_n\|_{L^\infty(\Sigma)} R_{n+1} v_{n+1}\epsilon^{-8} + C\epsilon^{-9} R_{n+1}\sigma_n v_{n+1}$$

Summing these estimates yields

$$\|\partial A_{n+1}\|_{L^\infty(\Sigma)} \le \|\partial A_0\|_{L^\infty(\Sigma)} + C\epsilon^{-8}R_{n+1}\max_{j=0,\ldots n}\|A_j\|_{L^\infty(\Sigma)}\sum_{j=1}^{n+1} v_j + C\epsilon^{-9}R_{n+1}\sum_{j=0}^{n} v_{j+1}\sigma_j$$

For the sums, we will use Cauchy-Schwarz inequality and an assumption (38). To estimate $\|A_j\|_{L^\infty(\Sigma)}$, (39) can be applied. Thus,

$$\|\partial A_{n+1}\|_{L^\infty(\Sigma)} \le g'_{n+1} = \|\partial A_0\|_{L^\infty(\Sigma)} + CR_{n+1}\epsilon^{-9}\|v\|_2 +$$
$$+ n^{0.5}R_{n+1}\|v\|_2\epsilon^{-8}\exp(C\epsilon^{-8}\|v\|_2 n^{0.5})\left(\|A_0\|_{L^\infty(\Sigma)} + C\|v\|_2\epsilon^{-9}\right) \tag{40}$$

Now we are going to use these apriori estimates to obtain asymptotics of the sequence A_n. Consider the second term in (20). It can be written as

$$(4\pi)^{-1}A_n(k,\hat{x})\int \frac{v_{n+1}(u)}{|u|}e^{-ik\langle\hat{x},u\rangle+ik|u|}du \tag{41}$$
$$+(4\pi)^{-1}\int \frac{v_{n+1}(u)}{|u|}(A_n(k,\hat{u}) - A_n(k,\hat{x}))e^{-ik\langle\hat{x},u\rangle+ik|u|}du$$

Denote

$$\kappa_n = (4\pi)^{-1}\int \frac{v_{n+1}(u)}{|u|}e^{-ik\langle\hat{x},u\rangle+ik|u|}du \tag{42}$$

By lemma 0.3,

$$|\kappa_n| < C\epsilon^{-1}v_{n+1} \in \ell^2(\mathbb{Z}^+) \tag{43}$$

The second term in (41) can be bounded using an apriori estimate on the derivative. It is not greater than

$$C\|\partial A_n\|_{L^\infty(\Sigma)}\int e^{-\epsilon(|u|-\langle\hat{x},u\rangle)}|v_{n+1}(u)|\frac{|\hat{u}-\hat{x}|}{|u|}du$$

By introducing the spherical coordinates we see that the last expression is less than

$$Cg'_n v_{n+1}R_{n+1}\left[\int\limits_{\pi/2-\delta}^{\pi/2} e^{-\epsilon R_{n+1}(1-\sin\theta)}\cos^2\theta d\theta + \int\limits_{-\pi/2}^{\pi/2-\delta} e^{-\epsilon R_{n+1}(1-\sin\theta)}\cos\theta d\theta\right]$$
$$< Cg'_n v_{n+1}R_{n+1}\left[\epsilon^{-1.5}R_{n+1}^{-1.5} + \epsilon^{-1}R_{n+1}^{-1}e^{-\delta_1\epsilon R_{n+1}}\right] \tag{44}$$
$$< Cg'_n v_{n+1}R_{n+1}\left[\epsilon^{-1.5}R_{n+1}^{-1.5} + \epsilon^{-2}R_{n+1}^{-2}\right] \tag{45}$$

At the last step, lemma 0.5 was applied to the second term in the square brackets in (44). Now, we will pay special attention to I_4. The other terms will be of little importance. We have

$$I_4 = \beta_n A_n(k,\hat{x})$$
$$+(4\pi)^{-1}\int e^{-ik\langle\hat{x},u\rangle}v_{n+1}(u)\int G_{n+1}(u,s,k)v_{n+1}(s)\frac{e^{ik|s|}}{|s|}(A_n(k,\hat{s}) - A_n(k,\hat{x}))dsdu \tag{46}$$

where

$$\beta_n = (4\pi)^{-1}\int e^{-ik\langle\hat{x},u\rangle}v_{n+1}(u)\int G_{n+1}(u,s,k)v_{n+1}(s)\frac{e^{ik|s|}}{|s|}dsdu$$

and upon using Combes-Thomas bound (just like when we were estimating I_4 in (30)), we get

$$|\beta_n| < C\epsilon^{-5}v_{n+1}^2 \tag{47}$$

Here, it is crucial that $\beta_n \in \ell^1(\mathbb{Z}^+)$. This is the only place where we essentially use $\{v_m\} \in \ell^2$. The other term in (46) can be estimated by

$$C\epsilon^{-1}v_{n+1}^2\|\partial A_n\|_{L^\infty(\Sigma)} \int\limits_{R_{n+1}-2<|u|<R_{n+1}+3} e^{\epsilon\langle \hat{x},u\rangle}$$

$$\times \int\limits_{R_{n+1}-2<|s|<R_{n+1}+3} e^{-v\epsilon|u-s|}\frac{e^{-\epsilon|s|}}{|s|}(|\hat{s}-\hat{u}|+|\hat{u}-\hat{x}|)dsdu \qquad (48)$$

Here, we bounded the factor $|\hat{s}-\hat{x}|$ from the Lagrange Theorem by $|\hat{s}-\hat{u}|+|\hat{u}-\hat{x}|$ using the triangle inequality. That gives us two terms. The first one, corresponding to $|\hat{s}-\hat{u}|$, is estimated by

$$Cv_{n+1}^2\epsilon^{-2}g_n'R_{n+1}^2 \max_{\hat{u}\in\Sigma}\int_\Sigma e^{-v\epsilon R_{n+1}|\hat{u}-\hat{s}|}|\hat{u}-\hat{s}|d\hat{s}$$

where we used lemma 0.3 for the integral in u and the bound (40) for ∂A_n. The last expression is not greater than

$$Cv_{n+1}^2\epsilon^{-2}g_n'R_{n+1}^2\left[\int\limits_{\pi/2-\delta}^{\pi/2} e^{-v\epsilon R_{n+1}\cos\theta}\cos^2\theta d\theta + e^{-v\epsilon\delta_1 R_{n+1}}\right]$$

$$< Cv_{n+1}^2\epsilon^{-5}g_n'R_{n+1}^{-1} \qquad (49)$$

where δ,δ_1- some small positive constants and lemma 0.5 was applied to the second term in square brackets.

The second term in (48) is estimated by

$$C\epsilon^{-1}v_{n+1}^2g_n'\int_{R_{n+1}-2<|u|<R_{n+1}+3} e^{\epsilon<\hat{x},u>}|\hat{u}-\hat{x}|\int_{R_{n+1}-2<|s|<R_{n+1}+3} e^{-v\epsilon|u-s|}\frac{e^{-\epsilon|s|}}{|s|}dsdu \qquad (50)$$

The integral in s allows the following bound

$$\int_{R_{n+1}-2<|s|<R_{n+1}+3} e^{-v\epsilon|u-s|}\frac{e^{-\epsilon|s|}}{|s|}ds < C\epsilon^{-3}\frac{e^{-\epsilon R_{n+1}}}{R_{n+1}}$$

Thus,

$$(50) < C\epsilon^{-4}v_{n+1}^2g_n'R_{n+1}\left[\int\limits_{\pi/2-\delta}^{\pi/2} e^{-\epsilon R_{n+1}(1-\sin\theta)}\cos^2\theta d\theta + \int\limits_{-\pi/2}^{\pi/2-\delta} e^{-\epsilon R_{n+1}(1-\sin\theta)}\cos\theta d\theta\right]$$

$$< C\epsilon^{-6}v_{n+1}^2g_n'R_{n+1}^{-0.5} \qquad (51)$$

For the other I_j, we are using estimates obtained before and get the following recursion

$$A_{n+1}(k,\theta) = A_n(k,\theta)(1-\kappa_n+\beta_n)+\eta_n \qquad (52)$$

For η_n, we apply estimates (26),(27),(28),(32),(33),(34),(45),(48),(49),(51) to get

$$|\eta_n| < C\epsilon^{-d}v_{n+1}\left\{g_n'R_{n+1}^{-0.5}+\sigma_n+R_n^3g_ne^{2\epsilon(R_n-R_{n+1})}\right\} \qquad (53)$$

with some d that will be reserved for the positive constant that might change from formula to formula, we will not care for its particular value. From the estimates on g_n and g_n', we get

$$|\eta_n| < C\epsilon^{-d} v_{n+1}\left[\exp\left[C\epsilon^{-8}n^{0.5}\right]\left\{nR_nR_{n+1}^{-0.5} + R_n^3 e^{2\epsilon(R_n-R_{n+1})}\right\} + \sigma_n\right]$$

Consider the second term in the curly brackets. Notice that (17) implies $R_{n+1} > 2R_n$ and $R_n > 2n$. Then, by lemma 0.5,

$$R_n^3 e^{-\epsilon R_n} \le C\epsilon^{-3}, e^{-\epsilon R_n} \le e^{-2\epsilon n}$$

Assuming

$$nR_nR_{n+1}^{-0.5} < e^{-2n} \tag{54}$$

and using (38), we have

$$|\eta_n| < C\epsilon^{-d}v_{n+1}\exp(C\epsilon^{-8}n^{0.5} - 2\epsilon n) = C\epsilon^{-d}v_{n+1}\exp(C\epsilon^{-8}n^{0.5} - \epsilon n)\exp(-\epsilon n)$$

Now, let us apply an estimate

$$C\epsilon^{-\alpha}x - \epsilon x^2 \le \frac{C^2\epsilon^{-2\alpha-1}}{4} \tag{55}$$

which is true for any $x > 0$ and $\alpha > 0$ and, in particular, for $x = \sqrt{n}$ and $\alpha = 8$. By choosing new constant d, we get

$$|\eta_n| < \exp(C\epsilon^{-d})v_{n+1}\exp(-\epsilon n) \tag{56}$$

Now, we are going to use the following lemma

Lemma 0.7. *Assume that* $x_{n+1} = x_n(1 + q_n) + d_n$ *where* $q_n \in \ell^2(\mathbb{Z}^+)$, $|d_n| < \omega e^{-\alpha n}, \alpha > 0$, *and* $n = 0, 1, \ldots$. *Then, we have*

$$x_n = \exp\left[\sum_{j=0}^{n-1} q_j\right]\cdot \tilde{x}_n, \quad |\tilde{x}_n| \le C\exp(C\|q\|_2^2)\cdot\left|x_0 + \omega\sum_{j=1}^{n}\exp(-\alpha j + \|q\|_2\sqrt{j})\right| \tag{57}$$

and $\tilde{x}_n$ *has a finite limit.*

Proof. Iterating, we get

$$x_{n+1} = (1+q_n)\ldots(1+q_0)x_0 + (1+q_n)\ldots(1+q_1)d_0 + \ldots + (1+q_n)d_{n-1} + d_n \tag{58}$$

Clearly,

$$(1+q_n)\ldots(1+q_k) = \exp\left[\sum_{j=0}^{n} q_j\right] r_{k,n}$$

where

$$r_{k,n} = \exp\left[-\sum_{j=0}^{k-1} q_j\right]\prod_{j=k}^{n} l(q_j)$$

and

$$l(z) = (1+z)e^{-z}$$

It is obvious that $|l(z)| < \exp\left[C|z|^2\right]$ for any $z \in \mathbb{C}$ and $l(z) = 1 + \underline{O}(|z|^2)$ for small $|z|$. Then, by Cauchy-Schwarz,

$$\left|\exp\left[-\sum_{j=0}^{k-1} q_j\right]\right| \le \exp\left[\sqrt{k}\|q\|_2\right] \tag{59}$$

and

$$\lim_{n\to\infty}\prod_{j=k}^{n} l(q_j)=\prod_{j=k}^{\infty} l(q_j),\quad \prod_{j=k}^{n}|l(q_j)|<\exp\left[C\|q\|_2^2\right] \tag{60}$$

Now, introduce

$$\tilde{x}_{n+1}=r_{0,n}x_0+r_{1,n}d_0+\ldots+r_{n,n}d_{n-1}+\exp\left[-\sum_{j=0}^{n}q_j\right]d_n$$

Estimates (59), (60) show that

$$\tilde{x}_n\to r_{(0,\infty)}x_0+\sum_{k=0}^{\infty}r_{(k+1,\infty)}d_k$$

and (57) also easily follows. □

Let us now apply this lemma to (52) bearing in mind (56), (43), (47). Then, we get

$$A_n(k,\theta)=\exp\left[-\sum_{j=0}^{n-1}\kappa_j\right]\cdot\tilde{A}_n(k,\theta),\quad \tilde{A}_n(k,\theta)=\underline{O}\left(\exp\left[\epsilon^{-d}\|v\|_2^2\right]\right)\left[A^0(k,\theta)+\nu_n\right] \tag{61}$$

with

$$|\nu_n|<\|v\|_2\exp(C\epsilon^{-d})\sum_{j=1}^{n}\exp(-\epsilon j+C\epsilon^{-d}\sqrt{j}) \tag{62}$$

$$=\|v\|_2\exp(C\epsilon^{-d})\sum_{j=1}^{n}\exp(-(\epsilon/2)j)\cdot\exp(-(\epsilon/2)j+C\epsilon^{-d}\sqrt{j})<\|v\|_2\exp(C\epsilon^{-d}) \tag{63}$$

where (55) was used for the last inequality. Then, $\tilde{A}_n(k,\theta)$ converges for fixed k,θ and

$$A_n(k,\theta)=\exp\left[-\sum_{j=0}^{n-1}\kappa_j\right]\underline{O}\left(\exp[C\epsilon^{-d}]\right) \tag{64}$$

which proves (5), (7) (recall the expression (42) for κ_j). Now, we need to show (8). Indeed, fix k. Then, consider the first term in (58). It can be written as

$$x_0\cdot\exp\left[\sum_{j=0}^{n}q_j\right]\cdot\prod_{j=0}^{n}l(q_j)$$

and

$$\prod_{j=0}^{n}l(q_j)=1+\underline{O}(\|q\|_2^2)$$

as long as $\|q\|_2$ is small. Applying this representation to (52) and using (63), we easily get (8) as $\|v\|_2\to 0$.

Let us show that the a.c. spectrum of H fills $\mathbb{R}^+$. We will prove that the interval $I^2=\{k^2,k\in I=[a,b]\}$ supports the a.c. component of the spectrum. Since the positive a and b were chosen arbitrarily, that will finish the proof. Following [12, 6], consider an isosceles triangle T in Π with the base equal to I and the adjacent angles both equal to π/γ_1, $\gamma_1 > d$ with d from (64). Then, for simplicity, fix f– nonzero L^2 spherically-symmetric function with compact support within, say, a unit ball centered at origin. Clearly, we can find some point k_0 within this triangle T at which

$$|A^0(f, k_0, \theta)| > C > 0 \tag{65}$$

for $\theta \in \Sigma$. It follows just from the analyticity of $A^0(f, k, \theta)$ in k for fixed θ and spherical symmetry in θ for fixed k. Let us fix this k_0. Then, consider a new potential $\hat{V} = \chi_{|x|>R}V(x)$ with R large enough. The corresponding sequence $\hat{v}_k = 0, k < k(R)$ and $\hat{v}_k = v_k, k \geq k(R)$ and $\|\hat{v}\|_2 \to 0$ as $R \to \infty$. By Rozenblum-Kato theorem [23], the a.c. spectrum of $-\Delta + \hat{V}$ coincides with a.c. spectrum of the original operator $-\Delta + V$. Now, by (8) and (65), we can choose R large enough (that means $\|\hat{v}\|_2$ is small) to guarantee that

$$|\tilde{A}_n(f, k_0, \theta)| > C > 0 \tag{66}$$

uniformly in n and all $\theta \in \Sigma$. Then, we will prove that $\sigma'_f(k^2) > 0$ for a.e. $k \in I$, where $d\sigma_f(E)$ is the spectral measure of f with respect to operator $\hat{H} = -\Delta + \hat{V}$. That will show that I^2 is in the support of the a.c. spectrum of H. Since I is arbitrary, that will mean $\sigma_{ac}(H) = \mathbb{R}^+$.

To implement this strategy, we use the following factorization identity ([27], pages 40-42)

$$\sigma'_{n,f}(E) = k\pi^{-1}\|A_n(f, k, \theta)\|^2_{L^2(\Sigma)}, E = k^2 \tag{67}$$

where $\sigma_{n,f}(E)$ is the spectral measure of f with respect to the operator with potential $\hat{V}_n = \chi_{|x|>R}V_n$ and A_n is an amplitude with respect to the same potential. Let $\omega(k_0, s), s \in \partial T$ denote the value at k_0 of the Poisson kernel associated to T. One can easily show that

$$0 \leq \omega(k_0, s) < C|s - s_{1(2)}|^{\gamma_1 - 1},\ s \in \partial T \tag{68}$$

where $s_{1(2)}$ are endpoints of I. It is also nonnegative function. We have

$$\int_I \omega(k_0, s) \ln \|A_n(f, s, \theta)\|^2_{L^2(\Sigma)} ds$$

$$= \int_I \omega(k_0, s) \ln \int_\Sigma |WKB_n(s, \theta)\tilde{A}_n(f, s, \theta)|^2 d\theta ds \geq 2(J_1 + J_2)$$

by Jensen's inequality, where

$$J_1 = \int_I \omega(k_0, s) \int_\Sigma \ln |WKB_n(s, \theta)| d\theta ds$$

$$J_2 = \int_I \omega(k_0, s) \int_\Sigma \ln |\tilde{A}_n(f, s, \theta)| d\theta ds = \int_\Sigma \int_I \omega(k_0, s) \ln |\tilde{A}_n(f, s, \theta)| ds d\theta$$

We will estimate from below each of these terms. The function $\ln |\tilde{A}_n(f, k, \theta)|$ is subharmonic in $k \in T$ for fixed θ, so we have the mean-value inequality

$$\int_I \omega(k_0, s) \ln |\tilde{A}_n(f, s, \theta)| ds \geq \ln |\tilde{A}_n(f, k_0, \theta)| - \int_{I_1 \bigcup I_2} \omega(k_0, s) \ln |\tilde{A}_n(f, s, \theta)| ds$$

where $I_{1(2)}$ are the sides of triangle T. From (64), (66), and (68), we obtain the bound

$$\int\limits_I \omega(k_0,s)\ln|\tilde{A}_n(f,s,\theta)|ds > C > -\infty$$

uniformly in n and $\theta\in\Sigma$. Here, the possible growth of $\ln|\tilde{A}_n(f,s,\theta)|$ near the real line is compensated by the zero of the kernel $\omega(k_0,s)$. Therefore, $J_2 > C > -\infty$ uniformly in n. Thus, we are left to show that the same is true for J_1.

$$\begin{aligned} J_1 &= -(4\pi)^{-1}\int\limits_I \omega(k_0,s)\int\limits_\Sigma d\theta\int \frac{\cos(s(|u|-\langle\theta,u\rangle))}{|u|}\hat{V}_n(u)du \\ &= -(4\pi)^{-1}\int\limits_I \omega(k_0,s)\int\frac{\sin(2s|u|)}{s|u|^2}\hat{V}_n(u)duds \end{aligned}$$

Since $\omega(k_0,s)$ is smooth on I and equals to 0 at the endpoints, we have

$$|J_1| < C\int|\hat{V}_n(u)|(1+|u|^2)^{-1.5}du < C \tag{69}$$

upon integration by parts in s.

Thus, from (67), we get

$$\int\limits_I \ln\sigma'_{n,f}(k^2)dk > C \tag{70}$$

uniformly in n. We need the following simple result.

Lemma 0.8. *If $n\to\infty$, then $d\sigma_{n,f}\to d\sigma_f$ in the weak–$(*)$ sense.*

Proof. For any $z\in\mathbb{C}^+$, we have

$$\lim_{n\to\infty}((-\Delta+\hat{V}_n-z)^{-1}f,f)\to((-\Delta+\hat{V}-z)^{-1}f,f) \tag{71}$$

Indeed, by the second resolvent identity,

$$((-\Delta+\hat{V}_n-z)^{-1}f,f) = ((-\Delta+\hat{V}-z)^{-1}f,f)-((-\Delta+\hat{V}-z)^{-1}(\hat{V}_n-\hat{V})(-\Delta+\hat{V}_n-z)^{-1}f,f)$$

and the last term can be written as

$$((-\Delta+\hat{V}_n-z)^{-1}f,(\hat{V}_n-\hat{V})g),\quad g=(-\Delta+\hat{V}-\bar{z})^{-1}f$$

Since g is a fixed vector in $L^2(\mathbb{R}^3)$ and $\|\hat{V}_n-\hat{V}\|_{L^\infty(\mathbb{R}^3)}\to 0$ as $n\to\infty$, we have $\|(\hat{V}_n-\hat{V})g\|_{L^2(\mathbb{R}^3)}\to 0$. Moreover, $\|(-\Delta+\hat{V}_n-z)^{-1}f\|_{L^2(\mathbb{R}^3)}\le(\operatorname{Im}z)^{-1}\|f\|_{L^2(\mathbb{R}^3)}$ by (13). Thus, (71) holds.

By the Spectral Theorem [1], (71) can be written as

$$\int_{\mathbb{R}^+}\frac{d\sigma_{n,f}(\lambda)}{\lambda-z}\to\int_{\mathbb{R}^+}\frac{d\sigma_f(\lambda)}{\lambda-z}$$

for any $z\in\mathbb{C}^+$ as long as $n\to\infty$. The simple approximation argument implies the weak–$(*)$ convergence of $d\sigma_{n,f}$ to $d\sigma_f$. □

Now, $(*)-\lim_{n\to\infty}d\sigma_{n,f}=d\sigma_f$ and (70) (this argument is based on the semi-continuity of the entropy, see [13], Section 5) imply

$$\int\limits_I \ln\sigma'_f(k^2)dk > -\infty$$

Therefore, $[a^2,b^2]\in\sigma_{ac}(H)$.

Let us collect the conditions on sparseness that we used:

$$R_{n+1}>3R_n^2,\quad \text{(formula (17) above)}$$

$$\sigma_n=R_n^{3.5}e^{R_n}(R_{n+1}-R_n)^{-1}<e^{-n},\quad nR_nR_{n+1}^{-0.5}<e^{-2n}\quad \text{(formulas (38), (54) above)}$$

Obviously, the condition on σ_n is the strongest one and we can satisfy all of them by requiring

$$R_{n+1}>e^{\alpha R_n},\alpha>1$$

and R_0 is big enough. In particular, $R_n=g^{(n)}(R_0)=g(g(\ldots g(R_0))$ will work for $g(x)=e^{2x}$. □

We believe that the restrictions on sparseness can be relaxed by more detailed, rather straightforward analysis. It is also likely that by controlling the oscillatory integrals for real k one can show that the spectrum is purely a.c. on $\mathbb{R}^+$. We do not want to pursue that technically difficult problem in this paper. It also might be that formula (5.4) from [6] can be used to obtain the asymptotics of Green's function in a simpler way. Notice also that condition (2) is satisfied under the assumptions of the theorem.

Consider the following randomized model: e.g., $v_n(x)=\sum_{k\in\Delta_n}\omega_k^n\nu_k^n(x)$ where $\{\omega_k^n\}$ are, e.g., independent Bernoulli random variables, functions $\nu_k^n(x)$, ($k\in\Delta_n$– set of indexes)– bump functions living within the small nonintersecting balls all located inside the n–th spherical layer, which also satisfy the bound $\max_{k\in\Delta_n}\|\nu_k^n\|_\infty\in\ell^2(\mathbb{Z}^+)$ in n. Then,

$$\mathbb{E}_\omega\left|\int\limits_{\mathbb{R}^3}\frac{\exp(ik(|t|-\langle\theta,t\rangle))}{|t|}V_n(t)dt\right|^2=\int\limits_{\mathbb{R}^3}\frac{V_n^2(t)}{|t|^2}dt<C$$

uniformly in $n\in\mathbb{Z}^+$, $k\in\overline{\mathbb{C}^+}$, and $\theta\in\Sigma$. So, by Kolmogorov's theorem on series of random variables, the corresponding $WKB_n(k,\theta)$ converges almost surely (k and θ are fixed) and one does not have any modification to the asymptotics of the Green function. That is due to oscillations and was observed before [6].

Now, we want to discuss the following issue. In the last theorem, we established the WKB asymptotics away from the real line. Recall that in the one-dimensional situation [3, 14], the corresponding WKB correction to the asymptotics of the generalized eigenfunctions was given by

$$WKB(k)=\exp\left(-\frac{i}{2k}\int\limits_0^\infty V(r)dr\right)$$

and its absolute value is equal to one for real k. Clearly, this is not the case for the multidimensional WKB that we have got. We had an estimate (69) that was sufficient to conclude the presence of a.c. spectrum but rather than that this WKB can exhibit quite a bad behavior. Apparently, the actual WKB asymptotics should be understood differently. The level sets of the function $|t|-\langle t,\theta\rangle$ from the formula (7) are paraboloids and it suggests that some evolution equation of the heat-transfer type might be involved. Consider operators O_t given by the formula (23).

Lemma 0.9. *The following parametrix representation is true for any $k \in \mathbb{C}^+$*

$$-2ikO_t f = \exp\left[-\frac{B}{2ikt}\right] f + \underline{O}(t^{-1})\|f\|_2, \quad t > 1 \tag{72}$$

where B is the Laplace-Beltrami operator on the unit sphere. The constant in $\underline{O}$ depends on k.

Proof. First, let us show that $-2ikO_t \to I$ in the strong sense as $t \to \infty$. Indeed, if $\operatorname{Im} k = \epsilon > 0$, then

$$|\left[O_t f\right](\theta)| \le \|f\|_{L^\infty(\Sigma)} \cdot (4\pi)^{-1} t \int\limits_{\Sigma} e^{-\epsilon t(1-\langle\theta,s\rangle)} ds < C\|f\|_{L^\infty(\Sigma)}$$

where lemma 0.3 was applied to the integral. Thus, the operator norm $\|O_t\|_{\infty,\infty} < C$ and by duality $\|O_t\|_{1,1} < C$. Application of Riesz-Thorin interpolation theorem yields $\|O_t\|_{2,2} < C$. Moreover, for $f \in C^\infty(\Sigma)$, the simple integration by parts shows $-2ik\left[O_t f\right](\theta) \to f(\theta)$ uniformly as $t \to \infty$. By the standard approximation argument, $-2ikO_t f \to f$ in $L^2(\Sigma)$ for any $f \in L^2(\Sigma)$.

Then, consider $g_t = \exp\left[-\dfrac{B}{2ikt}\right] f$. It solves the following problem

$$g' = \frac{B}{2ikt^2}\, g,\ g(\infty) = f$$

Take $\psi(t) = -2ikO_t f - g(t)$. Then,

$$\upsilon(t) = \psi' - \frac{B}{2ikt^2}\psi = C\left[\int\limits_{\Sigma} e^{ikt(1-\langle x,y\rangle)}(1-\langle x,y\rangle) f(y)dy + \right.$$

$$\left. + ikt\int\limits_{\Sigma} e^{ikt(1-\langle x,y\rangle)}[1-\langle x,y\rangle + \frac{1}{2}(\langle x,y\rangle^2 - 1)] f(y)dy\right]$$

As before, we estimate these integral operators in $L^{1,1}(\Sigma)$, $L^{\infty,\infty}(\Sigma)$ norms first. That can be done by introducing spherical variables. Then we interpolate by Riesz-Thorin theorem to get

$$\|\upsilon\|_2 \le Ct^{-2}\|f\|_2, \psi(\infty) = 0$$

By integration,

$$\psi(t) = -\int\limits_t^\infty \exp\left[\frac{B}{2ik}(\tau^{-1} - t^{-1})\right]\upsilon(\tau)d\tau$$

Since B is nonpositive selfadjoint operator,

$$\|\psi(t)\|_2 \le \int\limits_t^\infty \|\upsilon(\tau)\|_2 d\tau$$

and we get (72)

□

An interesting open question is: what is the WKB correction for the *real* k? Consider the following evolution equations

$$\frac{d}{dt}U_0(\tau,t,k) = -(2ik)^{-1}\frac{B}{t^2}U_0(\tau,t,k), U_0(\tau,\tau,k) = I \tag{73}$$

$$\frac{d}{dt}U(\tau,t,k)=-(2ik)^{-1}\left[\frac{B}{t^2}-V(t)\right]U(\tau,t,k),U(\tau,\tau,k)=I \tag{74}$$

Then,

$$\exp\left[-\frac{B}{2ikt}\right]=U_0(t,\infty,k)$$

Since A_{R_n} changes slowly with the respect to the $(n+1)$–th layer, an expression

$$\int\limits_{R_{n+1}}^{R_{n+1}+1} O_t q_t dt$$

from (22) basically coincides with the linear in V term for the Duhamel expansion of $U(R_{n+1},R_{n+2},k)$. Therefore, the likely candidate for WKB might be a solution to the following evolution problem

$$2iku_r=-\frac{B}{r^2}u(r,k)+V(r)u(r,k) \tag{75}$$

Unfortunately, we cannot control the terms corresponding to the multiple collisions within the same layer (e.g., higher order in v_n terms) for the real k. But if one considers only those that are linear in v_n, then the conjecture seems to be reasonable. The same evolution equation can be obtained via the formal asymptotical expansion for the 3-dim Schrödinger operator written as the one-dimensional Schrödinger operator with an operator-valued potential. This new candidate for the correct WKB and modification of wave operators preserves the $L^2(\Sigma)$ norm. In support of our guess, we want to mention that the recent preprint [9] shows that the evolution equations (75) appear in scattering problems for another important class of potentials.

Although theorem 0.3 proves the asymptotics of Green's function for $k\in\mathbb{C}^+$ only, some analysis is possible on the real line too. The following simple result on the absence of embedded eigenvalues is true.

Theorem 0.4. *If R_n is sparse enough, then there are no positive eigenvalues for any bounded v_n.*

Proof. The idea is quite simple and was used before to treat the one-dimensional problem: we show that the solutions can not decay too fast between the layers where the potential is zero. Then, since the solution and its gradient are square summable in $\mathbb{R}^3$, we get the contradictions for R_n large enough. To make the argument work, we also need some apriori bounds from below that we borrow from [2].

Assume that $\psi(x)$ is a real-valued eigenfunction corresponding to an eigenvalue $E>0$. Since $\psi(x)\to 0$ as $|x|\to\infty$ (see Chapter 2, [4]), there is a point x_0 such that $\psi(x_0)=\max_{x\in\mathbb{R}^3}|\psi(x)|=1$, this is our normalization of ψ. We also know that $\|\psi\|_2$ is finite but we have no control over this quantity.

Introduce the spherical change of variables and consider $\omega(r)=r\psi(r\sigma)$, $\sigma\in\Sigma$, $x=r\sigma$. Now, $\omega(r)\in L^2(\mathbb{R}^+,L^2(\Sigma))$. Expanding in the spherical harmonics,

$$\omega(r)=\sum_{m=0}^{\infty}\sum_{l_m=-m}^{m}Y_{m,l_m}(\sigma)f_{m,l_m}(r)$$

Then,

$$\int\limits_{R_1<|x|<R_2} |\psi(x)|^2 dx = \int\limits_{R_1}^{R_2} \sum_{m=0}^{\infty} \sum_{l_m=-m}^{m} |f_{m,l_m}(r)|^2 dr \tag{76}$$

$$\left[-\frac{d^2}{dr^2} - \frac{B}{r^2} + V\right]\omega = E\omega$$

and we have for any l_m

$$-f''_{m,l_m}(r) + m(m+1)r^{-2} f_{m,l_m}(r) = E f_{m,l_m}(r),\ R_n + 1 < r < R_{n+1} \tag{77}$$

since the spherical harmonics are the eigenfunctions of B and $V = 0$ between the layers. From [2], lemma 3.10, we infer the following bound

$$\int\limits_{R_n+2<|x|<R_n+3} \psi^2(x)dx > CR_n^2 e^{-\gamma R_n^{4/3}\ln R_n}, \quad \text{and } n \text{ is large} \tag{78}$$

with $\gamma(E) > 0$ which is an independent constant (at this point we used the normalization of ψ at x_0). The bounds (78) and (76) lead to the existence of $r_n \in [R_n + 2, R_n + 3]$ such that

$$\sum_{m=0}^{\infty} \sum_{l_m=-m}^{m} |f_{m,l_m}(r_n)|^2 > CR_n^2 e^{-\gamma R_n^{4/3}\ln R_n} \tag{79}$$

Since u is bounded, we have $\|\Delta^j\psi\|_\infty < C(j), j = 1, 2, \ldots$ between the spherical layers. This implies

$$|(B\omega(r), \omega(r))_{L^2(\Sigma)}| \le Cr^2, R_n + 2 < r < R_{n+1} - 1$$

with constant C independent of n. In particular, for $r = r_n$, that reads

$$\sum_{m=0}^{\infty} (m+1)^2 \sum_{l_m=-m}^{m} |f_{m,l_m}(r_n)|^2 < CR_n^2 \tag{80}$$

Consider

$$p_m = \sum_{k=m}^{\infty} \sum_{l_k=-k}^{k} |f_{k,l_k}(r_n)|^2$$

Discrete integration by parts (or Abel's lemma) gives

$$\sum_{m=1}^{\infty} mp_m < CR_n^2$$

and

$$p_m < CR_n^2 m^{-1}$$

Take integer k_n so large that $R_n^2 k_n^{-1} < R_n \exp(-\gamma R_n^{4/3}\ln R_n)$. From (79), we have an estimate

$$\sum_{m=0}^{k_n} \sum_{l_m=-m}^{m} |f_{m,l_m}(r_n)|^2 > (C_1 R_n^2 - C_2 R_n)\exp(-\gamma R_n^{4/3}\ln R_n) \tag{81}$$

with $C_1 > 0$.

Next, we will use the following estimate true for general one-dimensional Schrödinger operators, a simple corollary from formula (2.4), [14]. If $u(r,k)$ is real–valued and $-u''+Vu=k^2u$, (V is real), then

$$\frac{R(r,k)}{R(r_0,k)} \geq \exp\left[-\frac{1}{2k}\int\limits_{r_0}^{r}|V(\rho)|d\rho\right], \quad R(r,k)=\sqrt{[u(r,k)]^2+k^{-2}[u'(r,k)]^2},\, r>r_0$$

Since all $f_{m,k}(r)$ are real-valued, we can apply this estimate to (77) if $r\in[r_n,R_{n+1}]$ and get

$$|f_{m,l_m}(r)|^2+E^{-1}|f'_{m,l_m}(r)|^2$$
$$> \left[|f_{m,l_m}(r_n)|^2+E^{-1}|f'_{m,l_m}(r_n)|^2\right]\exp\left[-(\sqrt{E})^{-1}m(m+1)\int\limits_{r_n}^{r}\rho^{-2}d\rho\right]$$
$$> \left[|f_{m,l_m}(r_n)|^2+E^{-1}|f'_{m,l_m}(r_n)|^2\right]\exp\left[-(\sqrt{E})^{-1}m(m+1)(r-r_n)r^{-1}r_n^{-1}\right]$$
$$> |f_{m,l_m}(r_n)|^2\exp\left[-(\sqrt{E})^{-1}(m+1)^2r_n^{-1}\right]$$

Integrating the last estimate in r from r_n to R_{n+1} and summing over the indices, we have

$$\int\limits_{r_n<|x|<R_{n+1}}[\psi^2(x)+E^{-1}(\psi_r(x)+r^{-1}\psi(x))^2]dx \tag{82}$$
$$> (R_{n+1}-r_n)\sum_{m=0}^{\infty}\sum_{l_m=-m}^{m}|f_{m,l_m}(r_n)|^2\exp\left[-(\sqrt{E})^{-1}(m+1)^2r_n^{-1}\right]$$
$$> (R_{n+1}-r_n)\sum_{m=0}^{k_n}\sum_{l_m=-m}^{m}|f_{m,l_m}(r_n)|^2\exp\left[-(\sqrt{E})^{-1}(m+1)^2r_n^{-1}\right]$$
$$> (R_{n+1}-R_n)(C_1R_n^2-C_2R_n)\exp(-\gamma R_n^{4/3}\ln R_n)\exp\left[-(\sqrt{E})^{-1}(k_n+1)^2(R_n+3)^{-1}\right] \tag{83}$$

where (81) was used to get the last inequality.

Now, choose $\{R_n\}$ so sparse that for any $\gamma>0$ and $E>0$

$$(R_{n+1}-R_n)R_n^2\exp\left[-\gamma R_n^{4/3}\ln R_n-(\sqrt{E})^{-1}(k_n+1)^2(R_n+3)^{-1}\right]\to\infty,\ \text{as}\, n\to\infty$$

Then, for n large enough, we get the contradiction in (83) because the left hand side of (82) is bounded in n. Indeed, it follows from the fact that both ψ and its gradient are square summable over $\mathbb{R}^3$. Notice that the sparseness conditions were chosen independent of the eigenfunction, eigenvalue E, and even of $\|V\|_\infty$. It is satisfied, for instance, if $R_{n+1}>\exp(\exp(R_n^\beta)), \beta>4/3$ and R_0 is large enough. □

Remark. Apparently, the estimates from [2] that we used can be improved in our case. Thus, the sparseness conditions can also be relaxed.

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