

# REMARK ON THE FORMULA BY RAKHMANOV AND STEKLOV'S CONJECTURE

S. DENISOV

ABSTRACT. The conjecture by Steklov was solved negatively by Rakhmanov in 1979. His original proof was based on the formula for orthogonal polynomial obtained by adding point masses to the measure of orthogonality. In this note, we show how this polynomial can be obtained by applying the method developed recently for proving the sharp lower bounds for the problem by Steklov.

## 1. INTRODUCTION: STEKLOV'S CONJECTURE AND RECENT DEVELOPMENT

Consider the weight  $\rho(x)$  on the interval  $[-1, 1]$  and the sequence of polynomials  $\{P_n(x)\}_{n=0}^{\infty}$ , which are orthonormal

$$\int_{-1}^1 P_n(x) P_m(x) \rho(x) dx = \delta_{n,m}, \quad n, m = 0, 1, 2 \dots \quad (1)$$

with respect to  $\rho$ . Assuming that the leading coefficient of  $P_n(x)$  is positive, these polynomials are defined uniquely. The Steklov conjecture dates back to 1921 [13] and it asks *whether a sequence  $\{P_n(x)\}$  is bounded at any point  $x \in (-1, 1)$ , provided that  $\rho(x)$  is positive on  $[-1, 1]$ , i.e.,*

$$\rho(x) \geq \delta, \quad \delta > 0. \quad (2)$$

This conjecture attracted a lot of attention (check, e.g., [3, 4, 5, 7] and a survey [14]). It was solved negatively by Rakhmanov in [10]. All existing proofs use the following connection between the polynomials orthogonal on the segment of the real line and on the unit circle. Let  $\psi, (x \in [-1, 1], \psi(-1) = 0)$  be a non-decreasing bounded function with an infinite number of growth points. Consider the system of polynomials  $\{P_k\}, (k = 0, 1, \dots)$  orthonormal with respect to the measure  $d\psi$  supported on the segment  $[-1, 1]$ . Introduce the function

$$\sigma(\theta) = \begin{cases} -\psi(\cos \theta), & 0 \leq \theta \leq \pi, \\ \psi(\cos \theta), & \pi \leq \theta \leq 2\pi, \end{cases} \quad (3)$$

which is bounded and non-decreasing on  $[0, 2\pi]$ . Consider the polynomials  $\phi_k(z, \sigma) = \lambda_k z^k + \dots$ ,  $\lambda_k > 0$  orthonormal with respect to measure  $d\sigma$ , i.e.,

$$\int_0^{2\pi} \phi_n(e^{i\theta}) \overline{\phi_m(e^{i\theta})} d\sigma = \delta_{n,m}, \quad n, m = 0, 1, 2 \dots \quad (4)$$

These polynomials can be thought of as polynomials orthonormal on the unit circle  $\mathbb{T}$  with respect to a measure  $\sigma$  given on  $\mathbb{T}$  as well.

Later, we will use the following notation: for every polynomial  $Q_n(z) = q_n z^n + \dots + q_0$  of degree at most  $n$ , we introduce the  $(*)$ -operation:

$$Q_n(z) \xrightarrow{(*)} Q_n^*(z) = \bar{q}_0 z^n + \dots + \bar{q}_n$$

This  $(*)$  depends on  $n$ . Then, we have the Lemma.

**Lemma 1.1.** ([6, 15]) *The polynomial  $\phi_n$  is related to  $P_k$  by the formula*

$$P_k(x, \psi) = \frac{\phi_{2k}(z, \sigma) + \phi_{2k}^*(z, \sigma)}{\sqrt{2\pi} [1 + \lambda_{2k}^{-1} \phi_{2k}(0, \sigma)]} z^{-k}, \quad k = 0, 1, \dots, \quad (5)$$

where  $x = (z + z^{-1})/2$ .

This reduction also works in the opposite direction: given a measure  $\sigma$ , defined on  $\mathbb{T}$  and symmetric with respect to  $\mathbb{R}$ , we can map it to the measure on the real line and the corresponding polynomials will be related by (5).

For every  $\delta \in (0, 1)$ , let  $S_\delta$  denote the class of probability measures  $\sigma$  which satisfy

$$\sigma'(\theta) \geq \delta/(2\pi), \quad \text{a.e. } \theta \in [0, 2\pi). \quad (6)$$

We will call  $S_\delta$  the Steklov class. The version of Steklov's conjecture for the unit circle then reads as follows:

*Given  $\sigma \in S_\delta$ , is it true that the sequence  $\{\phi_n(z, \sigma)\}$  is bounded for every  $z \in \mathbb{T}$ ?*

The normalization

$$\int d\sigma = 1$$

is not restrictive because of the scaling:  $\phi_n(z, \sigma) = \alpha^{1/2} \phi_n(z, \alpha\sigma)$ ,  $\alpha > 0$ . The negative answer to this question (see [10]) implied the solution to Steklov's conjecture on the real line due to Lemma 1.1.

Besides the orthonormal polynomials, we can define the monic orthogonal ones  $\{\Phi_n(z, \sigma)\}$  by requiring

$$\text{coeff}(\Phi_n, n) = 1, \quad \int_0^{2\pi} \Phi_n(e^{i\theta}, \sigma) \overline{\Phi_m(e^{i\theta}, \sigma)} d\sigma = 0, \quad m < n,$$

where  $\text{coeff}(Q, j)$  denotes the coefficient in front of  $z^j$  in the polynomial  $Q$ .

The original argument by Rakhmanov was based on the following formula for the orthogonal polynomial that one gets after adding several point masses to a "background" measure at particular locations on the circle.

**Lemma 1.2. (Rakhmanov's formula, [10])** *Let  $\mu$  be a positive measure on  $\mathbb{T}$  with infinitely many growth points and*

$$K_n(\xi, z, \mu) = \sum_{j=0}^n \overline{\phi_j(\xi, \mu)} \phi_j(z, \mu)$$

be the Christoffel-Darboux kernel, i.e.,

$$P(\xi) = \langle P(z), K_n(\xi, z, \mu) \rangle_{L^2(\mathbb{T}, \mu)}, \quad \forall P : \deg P \leq n.$$

Then, if  $\{\xi_j\} \in \mathbb{T}$ ,  $j = 1, \dots, m$ ,  $m < n$  are chosen such that

$$K_{n-1}(\xi_j, \xi_l, \mu) = 0, \quad j \neq l \quad (7)$$

then

$$\Phi_n(z, \eta) = \Phi_n(z, \mu) - \sum_{k=1}^m \frac{m_k \overline{\Phi_n(\xi_k, \mu)}}{1 + m_k K_{n-1}(\xi_k, \xi_k, \mu)} K_{n-1}(\xi_k, z, \mu) \quad (8)$$

where

$$\eta = \mu + \sum_{k=1}^m m_k \delta_{\theta_k}, \quad \xi_k = e^{i\theta_k}, \quad m_k \geq 0.$$

It is known ([10]) that for every  $\widehat{z} \in \mathbb{T}$ , the function  $K_{n-1}(\xi, \widehat{z}, \mu)$  has exactly  $n - 1$  different roots  $\{\xi_j(\widehat{z})\}, j = 1, \dots, n - 1$  and they all lie on  $\mathbb{T}$ . Moreover,  $K_{n-1}(\xi_i, \xi_j, \mu) = 0$  for  $i \neq j$  (see [10]). The limitation that  $\{\xi_j\}$  must be the roots is quite restrictive and the direct application of this formula with background  $d\mu = \delta(2\pi)^{-1}d\theta$  yields only logarithmic lower bound in the following variational problem:

$$M_{n,\delta} = \sup_{\sigma \in S_\delta} \|\phi_n(z, \sigma)\|_{L^\infty(\mathbb{T})} \geq C(\delta) \log n, \quad n > n_0(\delta) \quad (9)$$

The straightforward iteration of this “fixed- $n$ , varying  $\sigma$ ” construction gave the negative solution to the original conjecture of Steklov ([10]).

**Remark.** It is known [12] that for probability measures  $\sigma$  in the Szegő class, i.e., those  $\sigma$  for which

$$\int_0^{2\pi} \log \sigma' d\theta > -\infty,$$

we have

$$\exp\left(\frac{1}{4\pi} \int_{\mathbb{T}} \log(2\pi\sigma'(\theta)) d\theta\right) \leq \left| \frac{\Phi_n(z, \sigma)}{\phi_n(z, \sigma)} \right| \leq 1, \quad \forall z \in \mathbb{C}$$

Thus, for measures in Steklov class, i.e., those satisfying (6), the following estimate holds

$$\sqrt{\delta} \leq \left| \frac{\Phi_n(z, \sigma)}{\phi_n(z, \sigma)} \right| \leq 1, \quad \forall z \in \mathbb{C}$$

so, is  $\Phi_n$  or  $\phi_n$  grow in  $n$ , they grow simultaneously.

The upper bound for  $M_{n,\delta}$  is easy to obtain

$$M_{n,\delta} \leq C(\delta)\sqrt{n} \quad (10)$$

and the corresponding result for fixed  $\sigma \in S_\delta$  and  $n \rightarrow \infty$  is contained in the following Lemma.

**Lemma 1.3.** *If  $\sigma \in S_\delta$ , then*

$$\|\phi_n(z, \sigma)\|_{L^\infty(\mathbb{T})} = o(\sqrt{n}), \quad n \rightarrow \infty. \quad (11)$$

This result follows from, e.g., [8], Theorem 4 (see also [9],[16] for the real-line case).

The gap between  $\log n$  and  $\sqrt{n}$  was nearly closed in the second paper by Rakhmanov [11] where the following bound was obtained:

$$M_{n,\delta} \geq C(\delta) \sqrt{\frac{n}{\log^3 n}}$$

under the assumption that  $\delta$  is small.

In the recent paper [2], the following two Theorems were proved.

**Theorem 1.1.** ([2]) *If  $\delta \in (0, 1)$  is fixed, then*

$$M_{n,\delta} > C(\delta)\sqrt{n}. \quad (12)$$

and

**Theorem 1.2.** ([2]) *Let  $\delta \in (0, 1)$  be fixed. Then, for every positive sequence  $\{\beta_n\} : \lim_{n \rightarrow \infty} \beta_n = 0$ , there is a probability measure  $\sigma^* : d\sigma^* = \sigma'^* d\theta$ ,  $\sigma^* \in S_\delta$  such that*

$$\|\phi_{k_n}(z, \sigma^*)\|_{L^\infty(\mathbb{T})} \geq \beta_{k_n} \sqrt{k_n} \quad (13)$$

for some sequence  $\{k_n\} \subset \mathbb{N}$ .

These two results completely settle the problem by Steklov on the sharpness of estimates (10) and (11). The method used in the proof was very different from those of Rakhmanov. In the current paper, we will show that it can be adjusted to the cover construction by Rakhmanov. This new modification is interesting in its own as it contains certain cancellation different from the one used in [2].

The structure of the paper is as follows. The second section contains the explanation of the main idea used in [2] to prove Theorem 1.1. In the third one, we show how it can be used to cover the Rakhmanov's construction.

We will use the following notation. The Cauchy kernel  $C(z, \xi)$  is defined as

$$C(z, \xi) = \frac{\xi + z}{\xi - z}, \quad \xi \in \mathbb{T}.$$

The function analytic in  $\mathbb{D} = \{z : |z| < 1\}$  is called Caratheodory function if its real part is nonnegative in  $\mathbb{D}$ . Given a set  $\Omega$ ,  $\chi_\Omega$  denotes the characteristic function of  $\Omega$ . If two positive functions  $f_{1(2)}$  are given, we write  $f_1 \lesssim f_2$  if there is an absolute constant  $C$  such that

$$f_1 < C f_2$$

for all values of the argument. We define  $f_1 \gtrsim f_2$  similarly. Writing  $f_1 \sim f_2$  means  $f_1 \lesssim f_2 \lesssim f_1$ .

## 2. METHOD USED TO PROVE THEOREM 1.1

In this section we explain an idea used in the proof of Theorem 1.1. We start with recalling some basic facts about the polynomial orthogonal on the unit circle. With any probability measure  $\mu$ , which is defined on the unit circle and have infinitely many growth points, one can associate the orthonormal polynomials of the first and second kind,  $\{\phi_n\}$  and  $\{\psi_n\}$ , respectively.  $\{\phi_n\}$  satisfy the following recursions ([12], p. 57) with Schur parameters  $\{\gamma_n\}$ :

$$\begin{cases} \phi_{n+1} = \rho_n^{-1}(z\phi_n - \bar{\gamma}_n\phi_n^*), & \phi_0 = 1 \\ \phi_{n+1}^* = \rho_n^{-1}(\phi_n^* - \gamma_n z\phi_n), & \phi_0^* = 1 \end{cases} \quad (14)$$

and  $\{\psi_n\}$  satisfy the same recursion but with Schur parameters  $\{-\gamma_n\}$ , i.e.,

$$\begin{cases} \psi_{n+1} = \rho_n^{-1}(z\psi_n + \bar{\gamma}_n\psi_n^*), & \psi_0 = 1 \\ \psi_{n+1}^* = \rho_n^{-1}(\psi_n^* + \gamma_n z\psi_n), & \psi_0^* = 1 \end{cases} \quad (15)$$

The coefficient  $\rho_n$  is defined as

$$\rho_n = \sqrt{1 - |\gamma_n|^2}$$

The following Bernstein-Szegő approximation is valid:

**Lemma 2.1.** ([6],[12]) *Suppose  $d\mu$  is a probability measure and  $\{\phi_j\}$  and  $\{\psi_j\}$  are the corresponding orthonormal polynomials of the first/second kind, respectively. Then, for any  $N$ , the Caratheodory function*

$$F_N(z) = \frac{\psi_N^*(z)}{\phi_N^*(z)} = \int_{\mathbb{T}} C(z, e^{i\theta}) d\mu_N(\theta), \quad \text{where} \quad d\mu_N(\theta) = \frac{d\theta}{2\pi|\phi_N(e^{i\theta})|^2} = \frac{d\theta}{2\pi|\phi_N^*(e^{i\theta})|^2}$$

*has the first  $N$  Taylor coefficients identical to the Taylor coefficients of the function*

$$F(z) = \int_{\mathbb{T}} C(z, e^{i\theta}) d\mu(\theta).$$

*In particular, the polynomials  $\{\phi_j\}$  and  $\{\psi_j\}$ ,  $j \leq N$  are the orthonormal polynomials of the first/second kind for the measure  $d\mu_N$ .*

We also need the following Lemma which can be verified directly:

**Lemma 2.2.** *The polynomial  $P_n(z)$  of degree  $n$  is the orthonormal polynomial for a probability measure with infinitely many growth points if and only if*

1.  $P_n(z)$  has all  $n$  zeroes inside  $\mathbb{D}$  (counting the multiplicities).
2. The normalization conditions

$$\int_{\mathbb{T}} \frac{d\theta}{2\pi |P_n(e^{i\theta})|^2} = 1, \quad \text{coeff}(P_n, n) > 0$$

are satisfied.

*Proof.* Take  $2\pi |P_n(e^{i\theta})|^{-2} d\theta$  itself as a probability measure. The orthogonality is then immediate.  $\square$

We continue with a Lemma which paves the way for constructing the measure giving, in particular, the optimal bound (12). It is a special case of a solution to the truncated moment's problem.

**Lemma 2.3.** *Suppose we are given a polynomial  $\phi_n$  and Caratheodory function  $\tilde{F}$  which satisfy the following properties*

1.  $\phi_n^*(z)$  has no roots in  $\overline{\mathbb{D}}$ .
2. Normalization on the size and "rotation"

$$\int_{\mathbb{T}} |\phi_n^*(z)|^{-2} d\theta = 2\pi, \quad \phi_n^*(0) > 0. \quad (16)$$

3.  $\tilde{F} \in C^\infty(\mathbb{T})$ ,  $\text{Re } \tilde{F} > 0$  on  $\mathbb{T}$ , and

$$\frac{1}{2\pi} \int_{\mathbb{T}} \text{Re } \tilde{F}(e^{i\theta}) d\theta = 1. \quad (17)$$

Denote the Schur parameters given by the probability measures  $\mu_n$  and  $\tilde{\sigma}$

$$d\mu_n = \frac{d\theta}{2\pi |\phi_n^*(e^{i\theta})|^2}, \quad d\tilde{\sigma} = \tilde{\sigma}' d\theta = \frac{\text{Re } \tilde{F}(e^{i\theta})}{2\pi} d\theta,$$

as  $\{\gamma_j\}$  and  $\{\tilde{\gamma}_j\}$ , respectively. Then, the probability measure  $\sigma$ , corresponding to Schur coefficients

$$\gamma_0, \dots, \gamma_{n-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \dots$$

is purely absolutely continuous with the weight given by

$$\sigma' = \frac{4\tilde{\sigma}'}{|\phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n)|^2} = \frac{2 \text{Re } \tilde{F}}{\pi |\phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n)|^2}. \quad (18)$$

The polynomial  $\phi_n$  is the orthonormal polynomial for  $\sigma$ .

The proof of this Lemma is contained in [2]. We, however, prefer to give its sketch here.

*Proof.* First, notice that  $\{\tilde{\gamma}_j\} \in \ell^1$  by Baxter's Theorem (see, e.g., [12], Vol.1, Chapter 5). Therefore,  $\sigma$  is purely absolutely continuous by the same Baxter's criterion. Define the orthonormal polynomials of the first/second kind corresponding to measure  $\tilde{\sigma}$  by  $\{\tilde{\phi}_j\}, \{\tilde{\psi}_j\}$ . Similarly, let  $\{\phi_j\}, \{\psi_j\}$  be orthonormal polynomials for  $\sigma$ . Since, by construction,  $\mu_n$  and  $\sigma$  have identical first  $n$  Schur parameters,  $\phi_n$  is  $n$ -th orthonormal polynomial for  $\sigma$ .

Let us compute the polynomials  $\phi_j$  and  $\psi_j$ , orthonormal with respect to  $\sigma$ , for the indexes  $j > n$ . By (15), the recursion can be rewritten in the following matrix form

$$\begin{pmatrix} \phi_{n+m} & \psi_{n+m} \\ \phi_{n+m}^* & -\psi_{n+m}^* \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} \phi_n & \psi_n \\ \phi_n^* & -\psi_n^* \end{pmatrix} \quad (19)$$

where  $\mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{D}_m$  satisfy

$$\begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} = \frac{1}{\tilde{\rho}_0 \cdots \tilde{\rho}_{m-1}} \begin{pmatrix} z & -\tilde{\gamma}_{m-1} \\ -z\tilde{\gamma}_{m-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} z & -\tilde{\gamma}_0 \\ -z\tilde{\gamma}_0 & 1 \end{pmatrix}$$

and thus depend only on  $\tilde{\gamma}_0, \dots, \tilde{\gamma}_{m-1}$ . Moreover, we have

$$\begin{pmatrix} \tilde{\phi}_m & \tilde{\psi}_m \\ \tilde{\phi}_m^* & -\tilde{\psi}_m^* \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus,  $\mathcal{A}_m = (\tilde{\phi}_m + \tilde{\psi}_m)/2$ ,  $\mathcal{B}_m = (\tilde{\phi}_m - \tilde{\psi}_m)/2$ ,  $\mathcal{C}_m = (\tilde{\phi}_m^* - \tilde{\psi}_m^*)/2$ ,  $\mathcal{D}_m = (\tilde{\phi}_m^* + \tilde{\psi}_m^*)/2$  and their substitution into (19) yields

$$2\phi_{n+m}^* = \phi_n(\tilde{\phi}_m^* - \tilde{\psi}_m^*) + \phi_n^*(\tilde{\phi}_m + \tilde{\psi}_m) = \tilde{\phi}_m^* \left( \phi_n + \phi_n^* + \tilde{F}_m(\phi_n^* - \phi_n) \right) \quad (20)$$

where

$$\tilde{F}_m(z) = \frac{\tilde{\psi}_m^*(z)}{\tilde{\phi}_m^*(z)}.$$

Since  $\{\tilde{\gamma}_n\} \in \ell^1$  and  $\{\gamma_n\} \in \ell^1$ , we have ([12], p. 225)

$$\tilde{F}_m \rightarrow \tilde{F} \text{ as } m \rightarrow \infty \text{ and } \phi_j^* \rightarrow \Pi, \tilde{\phi}_j^* \rightarrow \tilde{\Pi} \text{ as } j \rightarrow \infty.$$

uniformly on  $\bar{\mathbb{D}}$ . The functions  $\Pi$  and  $\tilde{\Pi}$  are the Szegő functions of  $\sigma$  and  $\tilde{\sigma}$ , respectively, i.e., they are the outer functions in  $\mathbb{D}$  that satisfy

$$|\Pi|^{-2} = 2\pi\sigma', \quad |\tilde{\Pi}|^{-2} = 2\pi\tilde{\sigma}' \quad (21)$$

on  $\mathbb{T}$ . In (20), send  $m \rightarrow \infty$  to get

$$2\Pi = \tilde{\Pi} \left( \phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n) \right) \quad (22)$$

and we have (18) after taking the square of absolute values and using (21).  $\square$

In [2], to prove (12) with small  $\delta$ , the polynomial  $\phi_n$  and  $\tilde{F}$  were chosen to satisfy extra conditions (see Decoupling Lemma in [2]):

$$|\phi_n(1)| > C\sqrt{n} \quad (23)$$

and

$$|\phi_n^*(z)| + |\tilde{F}(z)(\phi_n^*(z) - \phi_n(z))| \leq C\sqrt{\operatorname{Re} \tilde{F}(z)}, \quad z \in \mathbb{T} \quad (24)$$

(23) yields the  $\sqrt{n}$ -growth claimed in Theorem 1.1. The last inequality guarantees that  $\sigma$  belongs to Steklov class due to (18) and (21). However, as will be made clear in the next section, (24) is not necessary for polynomials to have large uniform norm.

### 3. RAKHMANOV'S CONSTRUCTION VIA NEW APPROACH

Our goal in this section is twofold. Firstly, we use the method explained in section 2 to reproduce Rakhmanov's polynomial and polynomials with the similar structure that have large uniform norm and which are orthogonal with respect to a measure in Steklov class. Secondly, we show that the last condition in the Decoupling Lemma ([2], formula (3.6), or, what is the same, the bound (24) above) is not really necessary for the orthogonal polynomial to have large uniform norm. Instead, that can be achieved by a different sort of cancellation which might be of its own interest.

We start with recalling the construction by Rakhmanov [10]. In Lemma 1.2, take the Lebesgue measure  $\mu : d\mu = d\theta/(2\pi)$ . We have the following expression for the kernel

$$K_{n-1}(\xi, z, \mu) = \sum_{j=0}^{n-1} \bar{\xi}^j z^j = \frac{(z\bar{\xi})^n - 1}{z\bar{\xi} - 1}$$

Given two parameters  $\epsilon, (0 < \epsilon < 1)$  and  $m, (m < n - 1)$ , we add the mass  $m_k = \epsilon m^{-1}$  to each of the points  $\xi_k = e^{i2\pi k/n}, k = 0, \dots, m - 1$ . Then Lemma 1.2 gives

$$\Phi_n(z, \mu) = z^n - \frac{\epsilon m^{-1}}{1 + \epsilon n m^{-1}} \sum_{j=0}^{m-1} \left( (\bar{\xi}_j)^{n-1} z^{n-1} + \dots + \bar{\xi}_j z + 1 \right)$$

and therefore

$$\Phi_n^*(z, \mu) = 1 - \frac{\epsilon m^{-1}}{1 + \epsilon n m^{-1}} (d_1 z + d_2 z^2 + \dots + d_n z^n) \quad (25)$$

$$d_l = \sum_{j=0}^{m-1} \xi_j^{n-l} = \sum_{j=0}^{m-1} \xi_j^{-l}, \quad l = 1, \dots, n$$

Thus, if  $n$  is even and  $m = n/2$ , we have

$$d_n = m, \quad d_l = \frac{(-1)^l - 1}{e^{-i2\pi l/n} - 1}, \quad l = 1, \dots, n-1 \quad (26)$$

and  $\bar{d}_{n-l} = d_l, l = 1, \dots, n-1$ . Then,

$$\Phi_n^* - \Phi_n = \left( \frac{1 + 3\epsilon}{1 + 2\epsilon} \right) (1 - z^n), \quad \|\Phi_n^* - \Phi_n\|_{L^\infty(\mathbb{T})} < C \quad (27)$$

Since

$$e^{-i2\pi l/n} - 1 = -i \frac{2\pi l}{n} + O\left(\frac{l^2}{n^2}\right), \quad l < 0.01n, \quad (28)$$

it is clear that  $\|\Phi_n\|_{L^\infty(\mathbb{T})} \sim 1 + \epsilon \log n$  and this growth occurs around the points  $z = 1$  and  $z = -1$ . The choice of  $\{m_j\}$  can be rather arbitrary and does not have to be given by equal mass distribution to provide the logarithmic growth. Since  $\|\eta\| = 1 + \epsilon$  and  $\eta' = (2\pi)^{-1}$ , the normalized measure  $\eta/\|\eta\| \in S_\delta, \delta = (1 + \epsilon)^{-1}$

$$\|\phi_n(z, \eta/\|\eta\|)\|_{L^\infty(\mathbb{T})} \sim 1 + \epsilon \ln n$$

This argument proves (9).

The next theorem is the main result of the paper. It is not new (see, e.g., [1]). However, we give a different proof. In particular, it explains how the polynomial of the structure similar to (25) can be obtained by the method described in the previous section.

**Theorem 3.1.** *For every  $\epsilon \in (0, 1)$ , there is  $\sigma = \sigma' d\theta$ :*

$$\int_0^{2\pi} d\sigma = 1, \quad |\sigma'(\theta) - (2\pi)^{-1}| \lesssim \epsilon$$

and

$$\|\phi_n(z, \sigma)\|_{L^\infty(\mathbb{T})} \sim \epsilon \log n$$

*Proof.* We will consider an analytic polynomial  $M_n$  of degree  $n - 1$  satisfying two conditions

$$\int_0^{2\pi} \operatorname{Re} M_n(e^{i\theta}) d\theta = 0, \quad \|\operatorname{Re} M_n(e^{i\theta})\|_{L^\infty(\mathbb{T})} < C, \quad \|\operatorname{Im} M_n(e^{i\theta})\|_{L^\infty(\mathbb{T})} \sim \log n \quad (29)$$

This  $M_n$  is easy to find. Consider  $l(\theta) = \chi_{0 < \theta < \pi} - \chi_{\pi < \theta < 2\pi}$  and take

$$L(z) = \mathfrak{C}(l) = \frac{1}{2\pi} \int_0^{2\pi} C(z, e^{i\theta}) l(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} l(e^{i\theta}) d\theta$$

$\operatorname{Re} C$  is the Poisson kernel so  $\operatorname{Re} L(e^{i\theta}) = l(\theta)$ ,  $\theta \neq 0, \pi$ . Then, we take  $M_n = \mathcal{F}_n * L$ , where  $\mathcal{F}_n$  is the Fejer kernel. Since  $\mathcal{F}_n$  is real, nonnegative trigonometric polynomial of degree  $n - 1$  and  $\|\mathcal{F}_n\|_{L^1[0, 2\pi]} = 1$ , we have

$$|\operatorname{Re} M_n| = |\mathcal{F}_n * l| \leq 1, \quad z \in \mathbb{T}; \quad \int_0^{2\pi} \operatorname{Re} M_n(e^{i\theta}) d\theta = 0, \quad M_n(0) = 0$$

The logarithmic growth of  $M_n$  around the points  $\theta = 0$  and  $\theta = \pi$  is a standard exercise, e.g.,

$$|\operatorname{Im} M_n(e^{i\theta})| \sim \log n, \quad |\theta| < Cn^{-1} \quad (30)$$

with arbitrary large fixed  $C$ . Now, take a small positive  $\epsilon$  and define

$$\tilde{F} = 1 - 2\epsilon M_n, \quad D_n = M_n + b, \quad \phi_n^* = a(1 + \epsilon(D_n + D_n^*)) \quad (31)$$

where  $a$  and  $b$  are positive parameters to be chosen later so that all conditions of the Lemma 2.3 are satisfied. We have

$$\phi_n = a(z^n + \epsilon(D_n + D_n^*))$$

(Notice that  $\phi_n^* - \phi_n = a(1 - z^n)$  and compare it with (27).) Since  $D_n(0) = b$  and  $\deg D_n = n - 1$ , we have  $D_n^*(0) = 0$  and then  $\phi_n^*(0) = a(1 + \epsilon b) > 0$ . Let us check other normalization conditions for these functions.

$$\operatorname{Re} \tilde{F}(e^{i\theta}) = 1 + O(\epsilon) > 0, \quad \int_0^{2\pi} \operatorname{Re} \tilde{F}(e^{i\theta}) d\theta = 2\pi \quad (32)$$

Choose  $b$  such that  $\operatorname{Re} D_n \in [C_1, C_2]$  with  $C_1 > 0$ . For example, if  $b = 2$ , then  $\operatorname{Re} D_n \in [1, 3]$ . We can write

$$1 + \epsilon(D_n + D_n^*) = D_n \left( \epsilon(1 + e^{i(n\theta - 2\Theta_n)}) + D_n^{-1} \right), \quad z = e^{i\theta} \in \mathbb{T}$$

where  $\Theta_n = \arg D_n$ . Notice that  $D_n$  is zero free in  $\overline{\mathbb{D}}$  since it has positive real part on  $\mathbb{T}$ . Since

$$\operatorname{Re} \left( 1 + e^{i(n\theta - 2\Theta_n)} \right) \geq 0, \quad \operatorname{Re} D_n^{-1} = \frac{\operatorname{Re} D_n}{|D_n|^2} > 0$$

we have that  $\phi_n^*$  is zero free in  $\overline{\mathbb{D}}$ . Then, for  $z \in \mathbb{T}$ ,

$$|1 + \epsilon(D_n + D_n^*)| \geq \frac{|\operatorname{Re} D_n|}{|D_n|} \sim |D_n|^{-1}$$

and

$$\int_0^{2\pi} |1 + \epsilon(D_n + D_n^*)|^{-2} d\theta \lesssim \int_0^{2\pi} |D_n|^{-2} d\theta \lesssim 1 \quad (33)$$

since  $\|D_n\|_{L^2(\mathbb{T})} \lesssim b + \|L\|_{L^2(\mathbb{T})} \lesssim 1$ . On the other hand,

$$\left\| \left( 1 + \epsilon(D_n + D_n^*) \right) - 1 \right\|_{L^2(\mathbb{T})} \leq 2\epsilon \|D_n\|_{L^2(\mathbb{T})} \lesssim \epsilon$$

and so  $\|1 + \epsilon(D_n + D_n^*)\|_{L^2(\mathbb{T})} = \sqrt{2\pi} + O(\epsilon)$ . From Cauchy-Schwarz inequality, we get

$$2\pi \leq \|1 + \epsilon(D_n + D_n^*)\|_{L^2(\mathbb{T})} \left\| \left( 1 + \epsilon(D_n + D_n^*) \right)^{-1} \right\|_{L^2(\mathbb{T})}$$

and so

$$\frac{2\pi}{\sqrt{2\pi} + O(\epsilon)} \leq \left\| \left( 1 + \epsilon(D_n + D_n^*) \right)^{-1} \right\|_{L^2(\mathbb{T})} \lesssim 1$$



Let us choose  $a$  so that

$$\int_0^{2\pi} |\phi_n^*|^{-2} d\theta = 2\pi$$

which implies  $a \sim 1$ . We satisfied all conditions of the Lemma 2.3. Consider the formula (18). We can write

$$\begin{aligned} \phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n) &= 2\phi_n^* - 2\epsilon M_n(\phi_n^* - \phi_n) = \\ 2\phi_n^* - 2a\epsilon(M_n - M_n z^n) &= 2a\left((1 + \epsilon(D_n + D_n^*)) - \epsilon(M_n - M_n z^n)\right) = \\ 2a\left((1 + \epsilon(D_n + D_n^*)) - \epsilon(M_n - (M_n + \overline{M}_n - \overline{M}_n)z^n)\right) &= 2a\left(1 + \epsilon b(1 + z^n) + 2\epsilon z^n \operatorname{Re} M_n\right) \end{aligned} \quad (34)$$

Let us control the deviation of  $\sigma'$  from the constant. We get

$$2\pi\sigma' = 4(2a)^{-2} \cdot \operatorname{Re} \tilde{F} \cdot |1 + O(\epsilon)|^{-2} = a^{-2}(1 + O(\epsilon)) \cdot |1 + O(\epsilon)|^{-2}$$

where we used  $|\operatorname{Re} M_n| \leq 1$  and (32). Since  $a \sim 1$ , we have that the deviation of  $2\pi\sigma'$  from  $a^{-2}$  is at most  $C\epsilon$ . Since  $\sigma$  is a probability measure, this implies  $a = 1 + O(\epsilon)$ . We are left to show that  $\|\phi_n\|_{L^\infty(\mathbb{T})} \sim \log n$ . By construction, it is sufficient to prove

$$\|M_n + z^n \overline{M}_n\|_{L^\infty(\mathbb{T})} \sim \log n$$

Indeed,

$$|M_n(\tilde{z}_n) + \tilde{z}_n^n \overline{M}_n(\tilde{z}_n)| \sim \log n, \quad \tilde{z}_n = e^{i\pi/n}$$

as follows from (30). □

**Remark.** Our analysis covers the polynomial (25) constructed by Rakhmanov too. If  $d_0 = m$ , we can rewrite (25) as

$$\begin{aligned} \Phi_n^* &= 1 + \frac{\epsilon}{1 + \epsilon n m^{-1}} - \frac{\epsilon m^{-1}}{1 + \epsilon n m^{-1}}(d_0 + d_1 z + d_2 z^2 + \dots + d_n z^n) \\ &= 1 + \frac{\epsilon}{1 + 2\epsilon} - \epsilon(b + b z^n + M_n + M_n^*) \end{aligned} \quad (35)$$

with

$$b = \frac{1}{1 + 2\epsilon}, \quad M_n = \frac{m^{-1}}{2(1 + 2\epsilon)}(d_1 z + d_2 z^2 + \dots + d_{n-1} z^{n-1})$$

The straightforward analysis shows that (26) implies (29). The formula (35) differs from (31), in essence, only by the negative sign and the normalization factor. Different sign makes checking conditions (1) and (2) in the Lemma 2.3 harder when compared to the argument in the proof of Theorem 3.1. However, in this particular case, this can be done directly by analyzing the polynomial  $d_0 + d_1 z + \dots + d_n z^n$  around points  $z = 1$  and  $z = -1$ . Indeed, we have

$$\left| \sum_{j=1}^N \frac{\sin(j\theta)}{j} \right| < C$$

uniformly over  $\theta$  and  $N$ . Then (28), (26), and (25) imply  $\operatorname{Re} \Phi_n^* = 1 + O(\epsilon)$ ,  $z \in \mathbb{T}$ . Therefore,

$$\int_0^{2\pi} |\Phi_n^*(e^{i\theta})|^{-2} d\theta < C_1$$

and the opposite estimate

$$\int_0^{2\pi} |\Phi_n^*(e^{i\theta})|^{-2} d\theta > C_2 > 0$$

follows from the analysis of  $\Phi_n^*$  away from  $z = \pm 1$ , i.e., on the arcs  $z = e^{i\theta}$ ,  $\epsilon < |\theta| < \pi - \epsilon$ . Now, we can normalize  $\Phi_n^*$  and define  $\phi_n^* = a\Phi_n^*$  so that

$$\int_0^{2\pi} |\phi_n^*(e^{i\theta})|^{-2} d\theta = 2\pi$$

For the constant  $a$ , we then have  $a \sim 1$ . Next, to check that  $\phi_n^*$  corresponds to a Steklov measure, one only needs to modify the choice of  $\tilde{F}$  by changing the sign in front of  $\epsilon$ :

$$\tilde{F} = 1 + C\epsilon M_n$$

and repeating (34) with properly chosen  $C$ .

**Remark.** As one can see from the proof of Theorem 3.1, the different sort of cancellation has been used to show the Steklov condition of the measure. In particular, the estimate (24) is violated as  $\phi_n^* - \phi_n = a(1 - z^n)$  does not provide the strong cancellation around  $z = 1$ .

## Acknowledgement.

The work on section 3 was supported by RSF-14-21-00025. The research done in the rest of the paper was supported by the grant NSF-DMS-1464479.

## REFERENCES

- [1] M. U. Ambroladze, On the possible rate of growth of polynomials that are orthogonal with a continuous positive weight (Russian), *Mat. Sb.* 182 (1991), no. 3, 332–353; English translation in: *Math. USSR-Sb.* 72 (1992), no. 2, 311–331.
- [2] A. Aptekarev, S. Denisov, D. Tulyakov, On a problem by Steklov, to appear in *Journal of the AMS*.
- [3] Ya. L. Geronimus, Some estimates of orthogonal polynomials and the problem of Steklov, *Dokl. Akad. Nauk SSSR*, 236 (1977), no. 1, 14–17.
- [4] Ya. L. Geronimus, The relation between the order of growth of orthonormal polynomials and their weight function. *Mat. Sb. (N.S.)* 61 (103), 1963, 65–79.
- [5] Ya. L. Geronimus, On a conjecture of V.A. Steklov. *Dokl. Akad. Nauk SSSR*, 142, 1962, 507–509.
- [6] Ya. L. Geronimus, *Polynomials orthogonal on the circle and on the interval*, GIFML, Moscow, 1958 (in Russian); English translation: *International Series of Monographs on Pure and Applied Mathematics*, Vol. 18 Pergamon Press, New York-Oxford-London-Paris, 1960.
- [7] B. L. Golinskii, The problem of V.A. Steklov in the theory of orthogonal polynomials. *Mat. Zametki*, 15 (1974), 21–32.
- [8] A. Máté, P. Nevai, V. Totik, Extensions of Szegő's theory of orthogonal polynomials. II, III. *Constr. Approx.* 3 (1987), 1, 51–72, 73–96.
- [9] P. Nevai, J. Zhang, V. Totik, Orthogonal polynomials: their growth relative to their sums. *J. Approx. Theory* 67 (1991), 2, 215–234.
- [10] E. A. Rahmanov, On Steklov's conjecture in the theory of orthogonal polynomials, *Matem. Sb.*, 1979, 108(150), 581–608; English translation in: *Math. USSR, Sb.*, 1980, 36, 549–575.
- [11] E. A. Rahmanov, Estimates of the growth of orthogonal polynomials whose weight is bounded away from zero, *Matem. Sb.*, 1981, 114(156):2, 269–298; English translation in: *Math. USSR, Sb.*, 1982, 42, 237–263.
- [12] B. Simon, *Orthogonal polynomials on the unit circle*, volumes 1 and 2, AMS 2005.
- [13] V. A. Steklov, Une methode de la solution du probleme de developement des fonctions en series de polynomes de Tchebysheff independante de la theorie de fermeture, *Izv. Rus. Ac. Sci.*, 1921, 281–302, 303–326.
- [14] P. K. Suetin, V.A. Steklov's problem in the theory of orthogonal polynomials, *Itogi Nauki i Tech. Mat. Anal.*, VINITI, 1977, 15, 5–82; English translation in: *Journal of Soviet Mathematics*, 1979, 12(6), 631–682.
- [15] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., 23, Providence RI, 1975 (fourth edition).
- [16] R. Szwarc, Uniform subexponential growth of orthogonal polynomials. *J. Approx. Theory* 81 (1995), 2, 296–302.

UNIVERSITY OF WISCONSIN–MADISON  
 MATHEMATICS DEPARTMENT  
 480 LINCOLN DR., MADISON, WI, 53706, USA  
 DENISSOV@MATH.WISC.EDU

AND

KELDysh INSTITUTE FOR APPLIED MATHEMATICS, RUSSIAN ACADEMY OF SCIENCES  
MIUSSKAYA PL. 4, 125047 MOSCOW, RUSSIA