ON THE SIZE OF THE POLYNOMIALS ORTHONORMAL ON THE UNIT CIRCLE WITH RESPECT TO A MEASURE WHICH IS A SUM OF THE LEBESGUE MEASURE AND P POINT MASSES.

S. DENISOV

ABSTRACT. For the measures on the unit circle that are equal to the sum of Lebesgue measure and p point masses, we give an estimate on the size of the corresponding orthonormal polynomials. As a simple corollary of the method, we obtain a bound for some exponential polynomials.

1. INTRODUCTION

Let $\delta \in (0,1)$. Define S_δ to be the class of the probability measures σ on the unit circle that satisfy the following condition:

$$\sigma'(\theta) \geq \delta/(2\pi)$$

for a.e. $\theta \in [-\pi, \pi)$. We denote the n-th orthonormal polynomial by $\phi_n(z,\sigma)$ and n-th monic orthogonal polynomial by $\Phi_n(z,\sigma)$. The problem of Steklov consists in estimating the size of ϕ_n for $\sigma \in S_\delta$. The sharp bounds for $\|\phi_n\|_{L^\infty(\mathbb{T})}$ were obtained in [1]. The following variational problem played an important role in the proof. Consider

$$M_{n,\delta} = \sup_{\sigma \in S_\delta} \|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})} \,. \tag{1}$$

In [1], the following estimate was established

$$M_{n,\delta} \sim C(\delta)\sqrt{n}$$

for fixed δ. Moreover, it was proved that a maximizer exists and that every $\sigma^* \in \arg\max_{\sigma\in S_\delta} \|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})}$ can be written as

$$\sigma^* = \delta\mu/(2\pi) + \sum_{j=1}^{N} m_j \delta(\theta - \theta_j)\,, \tag{2}$$

where $N \leq n$, $d\mu = d\theta$ is the Lebesgue measure, $m_j > 0$, and $0 < \theta_1 < \ldots < \theta_N \leq 2\pi$. No further information on σ^* was obtained and that motivated us to study the following variational problem.

Given large n and integer $p : \ p < n/2$, we consider the following set of probability measures

$$P_{\delta,p} = \{\sigma = \delta\mu/(2\pi) + \sum_{j=1}^{p} m_j \delta(\theta - \theta_j)\}$$

and define

$$\widehat{M}_{n,p} = \max_{\sigma \in P_{\delta,p}} \|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})} \,.$$

Theorem 1.1. *For every $\kappa > 1.5$, we have*

$$\widehat{M}_{n,p} \leq C(\delta,\kappa)\min\{\alpha_{n,p}\sqrt{p}, p\}, \quad \alpha_{n,p} = \log^\kappa(n/p)\,.$$

Remark. Notice that $p < \alpha_{n,p}\sqrt{p}$ for $p < \widehat{p}(n,\kappa)$ and

$$\widehat{p}(n,\kappa) \sim \Big(\log n\Big)^{2\kappa}.$$

Corollary 1.1. *If N is the number of point masses in* (2)*, then $N \sim n$.*

The proof of the Theorem is given in the next section. It contains several auxiliary variational problems. In the third section, we write the dual formulation for one of them and obtain a slight improvement of the estimate for exponential polynomials discovered recently by Kós [9] and Erdélyi [6]. The last section contains a discussion about the sharpness of the obtained estimated.

2. Proof of the main Theorem

Proof. (Theorem 1.1) Notice that $\sigma \in P_{\delta,p}$ implies (see [15], formula (2.3.1))

$$\prod_{j=0}^{\infty}(1-|\gamma_j|^2) = \delta$$

by the Szegő sum rule. In this formula, $\{\gamma_j\}$ denote the parameters of recursion. Since

$$\Phi_n = \phi_n \left(\prod_{j=0}^{n-1}(1-|\gamma_j|^2)\right)^{1/2}$$

we have

$$\widehat{M}_{n,p} \sim C(\delta) \max_{\sigma \in P_{\delta,p}} \|\Phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})},$$

and we can consider instead the variational problem for Φ_n.

Recall the following characterization of the monic orthogonal polynomials

$$\Phi_n(z,\sigma) = \arg\min_{\{a_j\}} \int_{\mathbb{T}} \left|z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0\right|^2 d\sigma,$$

which immediately follows from the orthogonality condition. If $\sigma \in P_{\delta,p}$, then

$$\min_{\{a_j\}} \int_{\mathbb{T}} \left|z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0\right|^2 d\sigma = \min_{\{a_j\}} \int_{\mathbb{T}} \left|1 - \big(a_{n-1}z + \ldots + a_1 z^{n-1} + a_0 z^n\big)\right|^2 d\sigma$$

$$= \delta + \min_{\{a_j\}} \left(\delta \sum_{j=0}^{n-1} |a_j|^2 + \sum_{l=1}^{p} m_l |1 - z_l Q_{n-1}(z_l)|^2\right)$$

where

$$Q_{n-1}(z) = \sum_{j=0}^{n-1} a_j z^j, \quad z_l = e^{i\theta_l}.$$

Therefore, if $\Phi_n(z,\sigma) = z^n + \widehat{Q}_{n-1}(z,\sigma)$ and $\sigma \in P_{\delta,p}$, then

$$\|\widehat{Q}_{n-1}(z,\sigma)\|_2^2 \le \|Q_{n-1}(z)\|_2^2 + \delta^{-1}\sum_{l=1}^{p} m_l |1 - z_l Q_{n-1}(z_l)|^2$$

for every Q_{n-1} and so, by Cauchy-Schwarz,

$$\|\Phi_n(z,\sigma)\|^2_{L^\infty(\mathbb{T})} \lesssim 1 + \|\widehat{Q}_{n-1}(z,\sigma)\|^2_{L^\infty(\mathbb{T})} \le 1 + n\|\widehat{Q}_{n-1}(z,\sigma)\|_2^2 \le$$

$$1 + n\|Q_{n-1}(z)\|_2^2 + n\delta^{-1}\sum_{l=1}^{p} m_l |1 - z_l Q_{n-1}(z_l)|^2$$

for every Q_{n-1}.

Since $\delta \in (0,1)$ is fixed, it is left to show that

$$D_{n,p} = \max_{\{z_l\},\{m_l\},z_l\in\mathbb{T},m_l\geq 0,\|m_l\|_1=1} \min_{Q_n} \left(\|Q_n\|_2^2 + \sum_{l=1}^p m_l |1 - z_l Q_n(z_l)|^2 \right) \lesssim \epsilon_{n,p},\ \epsilon_{n,p} = \min\{\epsilon'_{n,p}, \epsilon''_{n,p}\} \tag{3}$$

with $\epsilon'_{n,p} = p\,\alpha_{n,p}^2/n,\ \epsilon''_{n,p} = p^2/n$. That would follow immediately from

$$\max_{\{z_l\},\{m_l\},z_l\in\mathbb{T},m_l\geq 0,\|m_l\|_1=1} \min_{\|Q_n\|_2\leq C\sqrt{\epsilon_{n,p}}} \left(\sum_{l=1}^p m_l |1 - z_l Q_n(z_l)|^2 \right) \lesssim \epsilon_{n,p}$$

with some C. If

$$I_1(n) = \min_{\deg Q_n \leq n, \|Q_n\|_2 \leq C\sqrt{\epsilon}_{n,p}} \left(\sum_{l=1}^p m_l |1 - z_l Q_n(z_l)|^2 \right),$$

$$I_2(n) = \min_{\deg Q_{n+1} \leq n+1, \|Q_{n+1}\|_2 \leq C\sqrt{\epsilon_{n,p}}} \left(\sum_{l=1}^p m_l |1 - Q_{n+1}(z_l)|^2 \right),$$

then $I_2(n) \leq I_1(n)$. Moreover, if $I_2(n)$ is reached on $a_0 + \ldots + a_{n+1}z^{n+1}$, then $|a_0| \leq C\sqrt{\epsilon}_{n,p}$ and

$$I_2(n) = \sum_{l=1}^p m_l |1 - a_0 - z_l(a_1 + \ldots + a_{n+1}z_l^n)|^2 \geq \sum_{l=1}^p m_l \Big(-|a_0|^2 + \frac{1}{2} \Big| 1 - z_l(a_1 + \ldots + a_{n+1}z_l^n) \Big|^2 \Big)$$

$$\geq -C^2\epsilon_{n,p} + \frac{1}{2}\sum_{l=1}^p m_l |1 - z_l(a_1 + \ldots + a_{n+1}z_l^n)|^2 \geq -C^2\epsilon_{n,p} + \frac{I_1(n)}{2}\,.$$

Since $\epsilon_{n,p} \sim \epsilon_{n+1,p}$, we only need to show that there is some constant C so that

1. If $p \leq \widehat{p}$, then

$$\min_{\|Q_n\|_2 \leq C\sqrt{\epsilon''_{n,p}}} \left(\sum_{l=1}^p m_l |1 - Q_n(z_l)|^2 \right) \lesssim \epsilon''_{n,p} \tag{4}$$

for any choice of $\{z_l\}, \{m_l\}$;

and

2. If $p > \widehat{p}$, then

$$\min_{\|Q_n\|_2 \leq C\sqrt{\epsilon'_{n,p}}} \left(\sum_{l=1}^p m_l |1 - Q_n(z_l)|^2 \right) \lesssim \epsilon'_{n,p} \tag{5}$$

for any choice of $\{z_l\}, \{m_l\}$.

For the first case, we will show that

$$\min_{\|Q_n\|_2 \leq C\sqrt{\epsilon''_{n,p}}} \left(\sum_{l=1}^p m_l |1 - Q_n(z_l)|^2 \right) = 0\,. \tag{6}$$

This will follow from the analysis of a different minimization problem. Consider

$$E_{n,p} = \min_{Q_n : Q_n(z_l)=1, \forall l=1,\ldots,p} \|Q_n\|_2\,. \tag{7}$$

We need to prove that

$$E_{n,p} \leq C\sqrt{\epsilon''_{n,p}}.$$

In [10], the following generalization of the Halász result [8] was obtained. It was shown that

$$\min_{\pi_m:\deg \pi_m\le m,\pi_m(0)=1,\pi_m(1)=0}\|\pi_m\|_{L^\infty(\mathbb{T})}=\cos\left(\frac{\pi}{2(m+1)}\right)^{-(m+1)}.$$

Therefore, taking the product of p polynomials of degree $m=n/p$ (we can assume without loss of generality that p divides n) with zeroes at $\{z_l\}, l=1,\ldots,p$, we construct $P_n, \deg P_n=n$, such that $P_n(z_l)=0$, $l=1,\ldots,p$, $P_n(0)=1$, and

$$\|P_n\|_{L^\infty(\mathbb{T})}\le\left[\cos\left(\frac{\pi p}{2(n+p)}\right)\right]^{-(p+n)}.$$

Letting $Q_n=1-P_n$, we get

$$Q_n(z_l)=1,\ l=1,\ldots,p$$

and

$$\|Q_n\|_2^2=2\pi-2\int \operatorname{Re}P_n d\theta+\int|P_n|^2d\theta=\int|P_n|^2d\theta-2\pi$$

by the Mean Value Theorem for harmonic functions. Thus,

$$\|Q_n\|_2^2\le 2\pi\left(\left[\cos\left(\frac{\pi p}{2(n+p)}\right)\right]^{-2(p+n)}-1\right)$$

and

$$E_{n,p}\le\left(2\pi\left(\left[\cos\left(\frac{\pi p}{2(n+p)}\right)\right]^{-2(p+n)}-1\right)\right)^{1/2}=\zeta_{n,p}\,. \tag{8}$$

Notice that our construction does not guarantee that this estimate is sharp.

If $n\to\infty$ and $p=o(\sqrt{n})$, then

$$\zeta_{n,p}^2\le\widehat{C}\frac{p^2}{n},\quad \widehat{C}>\pi^3/2 \tag{9}$$

for $n>n_0(\widehat{C})$ and (6) is proved.

We are left to consider $p>\widehat{p}$ and prove (5). Recall the following simple Lemma attributed to Havin.

Lemma 2.1. *Let measurable $E\subseteq\mathbb{T}$ and $0<\tau\ll 1$. Then, there is a function $f\in H^\infty(\mathbb{D})$ such that*

$$|f(z)-1|\le\tau,\ z\in E;\quad \|f\|_{H^\infty(\mathbb{D})}\le 2; \tag{10}$$

and

$$\|f\|^2_{H^2(\mathbb{D})}\lesssim|E|(1+|\log\tau|^2)\,.$$

Proof. Take $\phi=\chi_E(\theta)$ and define

$$f(e^{i\theta})=1-\exp\Big((\phi(\theta)+i\widetilde{\phi}(\theta))\log\tau\Big),$$

where $\widetilde{\phi}$ is the harmonic conjugate. Since $\Phi=\phi+i\widetilde{\phi}$ defines the function analytic in $\mathbb{D}$, f is the boundary value of the function analytic in $\mathbb{D}$. The estimates (10) are satisfied by construction. Then,

$$|f|\le 1+\tau,\quad z\in E$$

and

$$|f|=|1-e^{i\widetilde{\phi}\log\tau}|\lesssim\min\{1,|\log\tau|\cdot|\widetilde{\phi}|\},\quad e^{i\theta}\in E^c.$$

Therefore

$$\|f\|_2^2\lesssim|E|+|\log\tau|^2\|\widetilde{\phi}\|_2^2=|E|(1+|\log\tau|^2)$$

since $\|\widetilde{\phi}\|_2 = \|\phi\|_2$. $\square$

$\Big($Had we tried to find the analytic function, not a polynomial, that would give an estimate (5), we would have taken $E = \cup_{j=1}^p [\theta_j - 1/n, \theta_j + 1/n]$ and $\tau = \beta_{n,p}\sqrt{p/n}$. Then, taking f from the Lemma above, we have

$$\|f\|_2^2 \lesssim \frac{p}{n}\Big(1+\log^2\Big(\beta_{n,p}\sqrt{p/n}\Big)\Big) \lesssim \frac{p}{n}\,\left(\log\frac{n}{p}\right)^2 = \beta_{n,p}^2\frac{p}{n} = \tau^2\,,$$

if we let $\beta_{n,p} = \log(n/p)$. This would have allowed us to take κ in the formulation of the Theorem 1.1 equal to 1. However, we need to produce a polynomial of order $\sim n$ and for this purpose we will need to modify the construction a bit.$\Big)$

Take $E = \cup_{j=1}^p I_j$ where $I_j = [\theta_j - \mu, \theta_j + \mu]$, and $\mu > 1/n$ will be adjusted later. Taking $\tau = \alpha_{n,p}\sqrt{p/n}$, we use the Lemma to construct f which satisfies

$$\|f\|_2^2 \lesssim p\mu(1+|\log\tau|^2) \lesssim \frac{p}{n}\,\alpha_{n,p}^2;\ |1-f(z)| \le \tau,\ z\in E$$

provided that

$$\mu \lesssim n^{-1}\left(\log\frac{n}{p}\right)^{2(\kappa-1)}$$

We make the following choice

$$\mu = n^{-1}\left(\log\frac{n}{p}\right)^{2(\kappa-1)}.$$

The function $f\in H^2(\mathbb{D})$ and we take $Q_n = f * \mathcal{F}_n$, where $\mathcal{F}_n$ is defined as follows. Given arbitrary $\gamma\in(0,1)$, consider an even function $g(x)$ such that $\widehat{g}(\omega)$, its Fourier transform, is supported on $(-1,1)$ and (see, e.g., [14])

$$\widehat{g}(0) = 1, \quad |g(x)| \lesssim e^{-|x|^\gamma}, \quad \gamma < 1.$$

Let $g_n(x) = ng(nx)$ and define

$$\mathcal{F}_n(x) = \sum_{j\in\mathbb{Z}} g_n(x - 2j\pi)\,.$$

For this 2π-periodic function, we have $\widehat{\mathcal{F}}_n(l) = \widehat{g}(l/n)$ by the Poisson summation formula and so Q_n is a polynomial of degree at most n. Let us show now that it satisfies the bounds similar to those that hold for f. We trivially have $\|Q_n\|_2 \lesssim \|f\|_2$ and it is left to check that $|Q_n(z_j)-1| \lesssim \tau$, $j = 1,\ldots,p$.

Notice that

$$|\mathcal{F}_n(x)| \lesssim n\sum_j e^{-n^\gamma|x-2\pi j|^\gamma} \lesssim ne^{-n^\gamma} + ne^{-n^\gamma|x|^\gamma}, \quad |x|\ll 1\,.$$

From the Lemma, we get

$$|f(z)| \le 2, \quad z\in\mathbb{T}$$

and

$$|f(z)-1| \le \tau, \quad z\in E. \tag{11}$$

First, notice that

$$\Big|f * \mathcal{F}_n\Big| \lesssim \|\mathcal{F}_n\|_1 \lesssim 1, \quad z\in\mathbb{T}.$$

Second, since $\int_{\mathbb{T}}\mathcal{F}_n(\theta)d\theta = 1$, one can write

$$\left|\int \mathcal{F}_n(\theta_j-\theta)f(\theta)d\theta - 1\right| = \left|\int (f(\theta_j-\theta)-1))\mathcal{F}_n(\theta)d\theta\right|$$

$$\lesssim \left|\int_{|\theta|>\mu} |\mathcal{F}_n(\theta)|d\theta\right| + \left|\int_{|\theta|<\mu} |\mathcal{F}_n(\theta)|\cdot|f(\theta_j-\theta)-1)|d\theta\right| . \tag{12}$$

The last integral is bounded by $C\tau$ due to (11) and the choice of E. The first term in (12) is bounded by

$$ne^{-n^\gamma}+\int_{n\mu}^\infty e^{-\xi^\gamma}d\xi \lesssim e^{-(n\mu)^\gamma}(n\mu)^{1-\gamma} = e^{-(\log(n/p))^{2(\kappa-1)\gamma}}(\log(n/p))^{2(\kappa-1)(1-\gamma)} < C\tau = C\sqrt{\frac{p}{n}}\log^\kappa\left(\frac{n}{p}\right)$$

provided that $2(\kappa-1)\gamma=1$ and C is large. Since γ can be chosen as an arbitrary number smaller than 1, we have $\kappa>1.5$.

□

Remark. One can repeat the argument used for the comparison of I_1 and I_2 to show that

$$D_{n,p}\sim D'_{n+1,p}, \quad D'_{n,p}=\max_{\{z_l\},\{m_l\},z_l\in\mathbb{T},m_l\geq 0,\|m_l\|_1=1}\min_{Q_n}\left(\|Q_n\|_2^2+\sum_{l=1}^p m_l|1-Q_n(z_l)|^2\right) . \tag{13}$$

The following formula attributed to Geronimus [7] needs to be mentioned.

Lemma 2.2. *(Geronimus, [7]) Consider $\sigma_t=(1-t)\sigma+t\delta(\theta)$ where $t\in(0,1)$. Then,*

$$\Phi_n(z,\sigma_t)=\Phi_n(z,\sigma)-t\frac{\Phi_n(1,\sigma)K_{n-1}(1,z,\sigma)}{1-t+tK_{n-1}(1,1,\sigma)} . \tag{14}$$

When $p=1$, this Lemma gives an explicit formula whose analysis confirms the estimate we obtained above. However, if p is large, its recursive application is complicated. Instead, the application of the Theorem 1.1 to this formula gives the following corollary.

Lemma 2.3. *If $\sigma\in P_{\delta,p}$ and $p<n/2$, then*

$$\sup_{z_1,z_2\in\mathbb{T}}\left|\frac{\phi_n(z_1,\sigma)K_{n-1}(z_1,z_2,\sigma)}{1+K_{n-1}(z_1,z_1,\sigma)}\right|\lesssim\min\{\alpha_{n,p}\sqrt{p},p\} .$$

In the case when $p\leq n$ and $\{z_j\}\subseteq\{1^{1/n}\}$, Rakhmanov [13] wrote the exact formula for $\Phi_n(z,\sigma)$ which implies

$$\|\phi_n(z,\sigma)\|_{L^\infty(\mathbb{T})}\lesssim\log n$$

for an arbitrary distribution of masses and this bound is saturated for some $\{m_j\}$ and $\{z_j\}$.

3. The Dual Problem and some inequalities for Exponential Polynomials

Let us start with the following standard Lemma from Linear Algebra.

Lemma 3.1. *Suppose $p<n$ and a linear $A:\mathbb{C}^n\to\mathbb{C}^p$ is surjective. Given fixed $v\in\mathbb{C}^p$, consider the following variational problem: $\min_{At=v}\|t\|_2$. Then, the minimizer t^* (pseudo-inverse) is unique and*

$$\|t^*\|_2=\sup_{x\in\mathbb{C}^p,A^*x\neq 0}\frac{|\langle x,v\rangle|}{\|A^*x\|_2} .$$

Proof. We can assume that $v\neq 0$. Since $\mathbb{C}^n=kerA\oplus ranA^*$, any solution to $At=v$ can be written as $t=t^*+\xi$ where ξ is arbitrary in $\ker A$ and $t^*\in ranA^*$. The Pythagorean theorem then implies that t^* is the pseudo-inverse. If $t^*=A^*\eta$, then

$$AA^*\eta=v$$

gives

$$\|t^*\|_2^2=\langle AA^*\eta,\eta\rangle=\langle v,\eta\rangle .$$

Dividing by $\|A^*\eta\|_2$, we have

$$\|t^*\|_2 = \frac{|\langle \eta, v\rangle|}{\|A^*\eta\|_2} \le \sup_{x\in\mathbb{C}^p, A^*x\neq 0} \frac{|\langle x, v\rangle|}{\|A^*x\|_2} .$$

On the other hand,

$$\frac{|\langle x, v\rangle|}{\|A^*x\|_2} = \frac{|\langle x, At^*\rangle|}{\|A^*x\|_2} = \frac{|\langle A^*x, t^*\rangle|}{\|A^*x\|_2} \le \|t^*\|_2$$

by Cauchy-Schwarz. $\square$

For given points $\{\theta_j\}, j = 1, \ldots, p$, take A as follows:

$$A : t = \{t_l\} \in \mathbb{C}^{n+1} \to \left\{ \sum_{l=1}^{n+1} t_l e^{i\theta_j(l-1)} \right\}_{j=1,\ldots,p} \in \mathbb{C}^p .$$

For $E_{n,p}$, defined in (7), we can write

$$E^2_{n,p} = 2\pi\|t^*\|_2^2$$

where $v = (1, \ldots, 1)$. The inequality (8) and the previous Lemma give

$$\left|\sum_{j=1}^p x_j\right|^2 \le \frac{\zeta^2_{n,p}}{2\pi} \sum_{l=0}^n \left|\sum_{j=1}^p x_j e^{i\theta_j l}\right|^2 .$$

Now, given $\lambda_1 < \ldots < \lambda_p$, define the following exponential polynomial

$$T(x) = \sum_{j=1}^p x_j e^{i\lambda_j x}$$

we get

$$|T(0)|^2 \le \frac{\zeta^2_{n,p}}{2\pi} \sum_{l=0}^n |T(l/n)|^2 ,$$

where $\theta_j = \lambda_j/n$ and n is large enough for $\{\theta_j\} \in \mathbb{T}$. If, for fixed p and $\{\lambda_j\}$, we take $n \to \infty$, then

$$|T(0)|^2 \le \frac{\pi^2}{4} p^2 \|T\|^2_{L^2[0,1]}$$

thanks to (9). Thus, we have

Lemma 3.2. *If $\lambda_1 < \ldots < \lambda_p$ are arbitrary p real numbers and $T(x) = \sum_{j=1}^p x_j e^{i\lambda_j x}$, then*

$$|T(0)| \le \frac{\pi p}{2} \|T\|_{L^2[0,1]} .$$

Many interesting estimates for the exponential sums and its derivatives were recently obtained in [9, 6] (see also [2, 3, 4, 11, 12]). Our estimate has a better constant $\pi/2$ if compared to the bounds in [9, 6]. We believe that the duality argument can be used to replace $\pi/2$ by the optimal constant 1 after the analysis of orthogonal polynomials for measures with Fisher-Hartwig singularities [5].

4. Sharpness of some results

We do not know whether the estimates obtained in Theorem 1.1 are sharp. However, we can discuss other results mentioned in the proofs above.

First, notice that the lemma by Havin is essentially sharp. Indeed, suppose we can find $f \in H^\infty(\mathbb{D})$ such that

$$\int_{\mathbb{T}} |f|^2 \le \epsilon, \quad |1-f| \le \sqrt{\epsilon} \quad \text{on} \quad \theta \in I, \quad |I| \sim \epsilon\,.$$

Take $P = 1 - f^2$. We have

$$P(0) = (2\pi)^{-1} \int_{\mathbb{T}} P d\theta = 1 + O(\epsilon)\,.$$

By the subharmonicity of $\log |P(z)|$, we get

$$O(\epsilon) = \log |P(0)| \le (2\pi)^{-1} \int_{\mathbb{T}} \log |P(\theta)| d\theta = J_1 + J_2$$

$$J_1 = (2\pi)^{-1} \int_{\mathbb{T}} \log^+ |P(\theta)| d\theta,\ J_2 = (2\pi)^{-1} \int_{\mathbb{T}} \log^- |P(\theta)| d\theta$$

Since $\log^+ x \le x - 1, x > 1$, we have by triangle's inequality

$$J_1 \lesssim \int_{|P|>1} (|1-f^2| - 1) d\theta \le \int_{|P|>1} |f|^2 d\theta \lesssim \epsilon$$

For J_2,

$$J_2 \le \int_I \log^- |1 - f^2| d\theta \sim |I| \log \epsilon \sim \epsilon \log \epsilon$$

and this gives contradiction as $\epsilon \to 0$.

Lemma 4.1. *Consider $p < n/2$ and spread p point $\{\theta_j\}, j = 1, \ldots, p$ evenly on the whole interval $[-\pi, \pi]$. Assume that T is a trigonometric polynomial, such that*

$$\deg T = n, \quad |T(\theta_j)| \sim 1, \quad j = 1, \ldots, p\,.$$

Then,

$$\|T\|_2 \gtrsim \sqrt{\frac{p}{n}}\,.$$

Proof. The proof is by contradiction. Assume that

$$\|T\|_2^2 \le \epsilon p/n\,,$$

where ϵ is small. Let I_j be an interval centered at θ_j of length $2\pi/p$ (so $[-\pi, \pi] = \cup_{j=1}^p I_j$ and $\{I_j\}$ are disjoint). We have

$$\sum_{j=1}^p \int_{I_j} |T|^2 d\theta \le \epsilon p/n\,.$$

Therefore, there is a subset $S \subseteq \{1, \ldots, p\}$ so that

$$|S| > p/2,\ \forall j \in S : \int_{I_j} |T|^2 d\theta \le 2\epsilon/n\,.$$

Consider $j \in S$ and assume without loss of generality that $T(\theta_j) = 1$. We have

$$\int_{I_j} |T|^2 d\theta \le 2\epsilon/n\,.$$

Therefore, by a simple contradiction argument, there is a point $\theta_j^* \in I_j$ such that $|T(\theta_j^*)| < 1/2$ and

$$|\theta_j - \theta_j^*| \le 8\epsilon/n\,.$$

Then, the Cauchy-Schwarz inequality yields

$$1/2 < |T(\theta_j) - T(\theta_j^*)| = \left| \int_{\theta_j}^{\theta_j^*} T'(\theta) d\theta \right| \le |\theta_j - \theta_j^*|^{1/2} \left(\int_{I_j} |T'|^2 d\theta \right)^{1/2}$$

and

$$\int_{I_j} |T'|^2 d\theta > \frac{n}{32\epsilon}\,.$$

Summing over S gives

$$\int_{-\pi}^{\pi} |T'|^2 d\theta \ge \sum_{j\in S} \int_{I_j} |T'|^2 d\theta > \frac{n|S|}{32\,\epsilon} \ge \frac{np}{64\,\epsilon}\,.$$

On the other hand, for every trigonometric polynomial of degree n, we have

$$\|T'\|_2^2 \le n^2 \|T\|_2^2\,,$$

which gives

$$\frac{np}{64\epsilon} \le \epsilon pn\,.$$

That leads to contradiction for $\epsilon < 1/8$. □

The last Lemma implies that

$$D_{n,p} > C(\delta)p/n\,.$$

Open problem. Can one improve $\alpha_{n,p}$ in the Theorem 1.1?

In the regime when p is fixed and $n \to \infty$, we can show that the multiple p^2 in the estimate

$$D_{n,p} \lesssim p^2/n\,.$$

is sharp.

Lemma 4.2. *Given any fixed p, we have*

$$\Big(nD_{n,p}\Big) \sim p^2, \quad n > \widehat{n}(p)$$

Proof. We only need to prove one direction. We assume the opposite. By (13), this can be written as

$$\liminf_{n\to\infty} \Big(nD'_{n,p}\Big) \le \Lambda(p), \quad \Lambda(p) = o(p^2), \quad p \to \infty$$

Take $m_j = 1/p$. The points $\{\theta_j\}$ will be chosen later. By our assumption, there is a subsequence $\{n\}$ such that there exists a polynomial Q_n satisfying the properties:

$$\sum_{j=1}^{p} |1 - Q_n(z_j)|^2 < C\frac{p\Lambda(p)}{n}, \quad \|Q_n\|_2^2 < C\frac{\Lambda(p)}{n}$$

for $n > \widehat{n}(p)$. Denote $\delta_j = 1 - Q_n(z_j), j = 1, \ldots, p$ and fix them. Then, we have

$$\min_{\deg r_n \le n, r_n(z_j) = 1+\delta_j} \|r_n\|_2^2 < C\frac{\Lambda(p)}{n}\,.$$

Then, applying the dual formulation, we get inequality

$$\left| \sum_{j=1}^{p} x_j(1-\delta_j) \right|^2 \le C\frac{\Lambda(p)}{n} \sum_{l=0}^{n} \left| \sum_{j=1}^{p} x_j e^{i\theta_j l} \right|^2\,.$$

Now, let $\theta_j = \lambda_j/n$ where $\lambda_1 < \ldots < \lambda_p$ are arbitrary. Fix $\{x_j\}$ and $\{\lambda_j\}$ and send $n \to \infty$. Since $\lim_{n\to\infty} \delta_j = 0$ for each $j = 1, \ldots, p$, we get

$$|T(0)|^2 < C\Lambda(p)\|T\|^2_{L^2[0,1]}, \text{ for every } T(x) = \sum_{j=1}^{p} x_j e^{i\lambda_j x}\,.$$

However, it was proved in [6] (see also the remark after Theorem 7.71.1 in [16]) that

$$\sup_{\deg T \le p} \frac{|T(0)|}{\|T\|_{L^2[0,1]}} \ge Cp$$

and this gives a contradiction for large p. □

5. Acknowledgement.

The author is grateful to F. Nazarov who gave him references to the results by Havin and Halasz and explained some of them. The author thanks S. Kupin for interesting discussion and T. Erdélyi for sharing the preprint [6] with him and for many useful remarks. The research of S.D. was supported by NSF grant DMS-1067413, RScF-14-21-00025, and by the IdEx Bordeaux Visiting Professor scholarship (2014).

References

[1] A. Aptekarev, S. Denisov, D. Tulyakov, "On a problem by Steklov", submitted.
[2] P. Borwein, T. Erdélyi, Nikolskii-type inequalities for shift invariant function spaces. Proc. Amer. Math. Soc. 134 (2006), no. 11, 3243–3246.
[3] P. Borwein, T. Erdélyi, Polynomials and polynomial inequalities, Springer, New-York, 1995.
[4] P. Borwein, T. Erdélyi, A sharp Bernstein-type inequality for the exponential sums, J. Reine Angew. Math. 476, 1996, 121–141.
[5] P. Deift, A. Its, I. Krasovsky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities. Ann. of Math. (2) 174 (2011), no. 2, 1243–1299.
[6] T. Erdélyi, Inequalities for exponential sums, preprint.
[7] Ya. L. Geronimus, Polynomials orthogonal on the circle and on the interval, GIFML, Moscow, 1958 (in Russian); English translation: International Series of Monographs on Pure and Applied Mathematics, Vol. 18 Pergamon Press, New York-Oxford-London-Paris, 1960.
[8] G. Halász, On the first and second main theorem in Turán's theory of power sums, Studies in Pure Mathematics, 259–269, Birkhauser, Basel, 1983.
[9] G. Kós, Two Turán type inequalities. Acta Math. Hungar. 119 (2008), no. 3, 219–226.
[10] M. Lachance, E. Saff, R. Varga, Inequalities for polynomials with a prescribed zero. Math. Z. 168 (1979), no. 2, 105–116.
[11] D. Lubinsky, L^p Markov-Bernstein inequalities on arcs of the circle. J. Approx. Theory 108 (2001), no. 1, 1–17.
[12] B. Nagy, V. Totik, Bernstein's inequality for algebraic polynomials on circular arcs. Constr. Approx. 37 (2013), no. 2, 223–232.
[13] E. A. Rahmanov, On Steklov's conjecture in the theory of orthogonal polynomials, Matem. Sb., 1979, 108(150), 581–608; English translation in: Math. USSR, Sb., 1980, 36, 549–575.
[14] W. Rudin, Real and Complex Analysis, McGraw-Hill, 1987.
[15] B. Simon, Orthogonal polynomials on the unit circle, volumes 1 and 2, AMS 2005.
[16] G. Szegő, Orthogonal polynomials, Vol. 23, Amer. Math. Soc. Colloq. Publ., Providence, Rhode Island, 1975.

University of Wisconsin–Madison
Mathematics Department
480 Lincoln Dr., Madison, WI, 53706, USA
denissov@math.wisc.edu