

ON SCHUR PARAMETERS IN STEKLOV'S PROBLEM

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ABSTRACT. We study the recursion (aka Schur) parameters for monic polynomials orthogonal on the unit circle with respect to a weight which provides negative answer to the conjecture of Steklov.

1. INTRODUCTION

Given a probability measure $d\sigma$ on the unit circle $\mathbb{T} = \partial\mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with infinitely many growth points, we define the monic polynomials $\{\Phi_n\}$ orthogonal in $L^2(d\sigma)$ by beginning with the set $\{1, z, z^2, \dots\}$ and applying the Gram-Schmidt orthogonalization procedure, i.e.,

$$\Phi_n(z, \sigma) = z^n + a_{n-1,n}z^{n-1} + \dots + a_{0,n}, \quad \int_{\mathbb{T}} \Phi_n(e^{i\theta}, \sigma) e^{-ij\theta} d\sigma(\theta) = 0, \quad j = 0, \dots, n-1.$$

Consider the so-called $*$ -operation (of order n) defined as

$$P_n^*(z) = z^n \overline{P_n(\bar{z}^{-1})}, \quad z \in \mathbb{C},$$

which gives

$$P_n^*(z) = z^n \overline{P_n(z)}, \quad z \in \mathbb{T}$$

for any polynomial P_n of degree at most n . Given $\{\Phi_n\}$, one can define the orthonormal polynomials by the formula

$$\phi_n(z, \sigma) = \frac{\Phi_n(z, \sigma)}{\|\Phi_n\|_{2,\sigma}}, \quad \|\Phi_n\|_{2,\sigma} = \left(\int_{\mathbb{T}} |\Phi_n|^2 d\sigma \right)^{1/2}. \quad (1)$$

The polynomials of the second kind will be denoted by $\{\Psi_n\}$ (monic) and $\Psi_n^*(z) = z^n \overline{\Psi_n(\bar{z}^{-1})}$, $z \in \mathbb{C}$, $\{\psi_n\}$ (orthonormal), and $\{\psi_n^*\}$. The pair (Φ_n, Φ_n^*) satisfies the Szegő recurrence:

$$\begin{cases} \Phi_{n+1}(z, \sigma) = z\Phi_n(z, \sigma) - \bar{\gamma}_n \Phi_n^*(z, \sigma) \\ \Phi_{n+1}^*(z, \sigma) = \Phi_n^*(z, \sigma) - \gamma_n z \Phi_n(z, \sigma) \end{cases} \quad (2)$$

The pair (Ψ_j, Ψ_j^*) satisfies the same recurrence except that the parameters are $\{-\gamma_n\}$. We refer the reader to [20, 21] for the basic theory (our $\gamma_j = \alpha_j$ from [21]). The recursion parameters $\{\gamma_n\} \subseteq \mathbb{D}^\infty$ are sometimes called the Schur parameters due to their relationship with Schur functions and the Schur algorithm. Two of the key identities in the theory are

$$\begin{aligned} \|\Phi_n\|_{2,\sigma} &= \left(\prod_{j=0}^{n-1} (1 - |\gamma_j|^2) \right)^{1/2}, \\ \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(2\pi\sigma'(\theta)) d\theta\right) &= \prod_{j \geq 0} \rho_j, \quad \rho_j = (1 - |\gamma_j|^2)^{1/2}. \end{aligned} \quad (3)$$

and the measure σ is said to satisfy the Szegő condition if the left hand side in the last formula is positive (which is equivalent to $\{\gamma_j\} \in \ell^2(\mathbb{Z}^+)$). Due to (1), we have

$$|\phi_n(z)| \sim |\Phi_n(z)| \quad (4)$$

uniformly in n and z provided that σ is Szegő measure.

The condition on the measure to be a probability one controls only the size of the orthonormal polynomials, not the monic ones. Indeed,

$$\Phi_j(z, \alpha\sigma) = \Phi_j(z, \sigma), \quad \phi_j(z, \alpha\sigma) = \alpha^{-1/2}\phi_j(z, \sigma). \quad (5)$$

In 1921, Steklov conjectured in [22] that, for a positive weight p given on the interval $[a, b]$ of the real line \mathbb{R} , the corresponding sequence of orthonormal polynomials $\{P_n(x, p)\}$ is bounded in n for every fixed $x \in (a, b)$. This conjecture attracted considerable interest (see, e.g., [10]-[14]) and was answered negatively by Rakhmanov in the series of papers [18, 19]. The proofs first dealt with polynomials orthogonal on the unit circle and then the obtained results were recast to handle $\{P_n\}$ by making use of a formula due to Geronimus. In this paper, we will only be focusing on the case of orthogonality on the unit circle. We define the Steklov class of measures as

$$S_\delta = \left\{ \sigma : \int d\sigma = 1, \quad \sigma' \geq \delta/(2\pi), \quad \text{a.e. } \theta \in [0, 2\pi) \right\},$$

where $\delta \in (0, 1)$. The following variational problem was considered in [18, 19]

$$M_{n,\delta} = \sup_{\sigma \in S_\delta} \|\phi_n(z, \sigma)\|_{L^\infty(\mathbb{T})}.$$

In [19], the estimates

$$\left(\frac{n}{\ln^3 n} \right)^{1/2} <_\delta M_{n,\delta} <_\delta n^{1/2}$$

were established (see formula (9) below for the definition of $<_\delta$) and [2] improved it to

$$M_{n,\delta} \sim_\delta n^{1/2}.$$

The paper [2] contained a method that presents both the measure $\sigma^{*(n)} \in S_\delta$ and the polynomial $\phi_n(z, \sigma^{*(n)})$, which satisfies $\|\phi_n(z, \sigma^{*(n)})\|_\infty \sim_\delta n^{1/2}$, explicitly. However, the parameters $\{\gamma_j^{(n)}\}$ that correspond to this construction were defined only implicitly. Neither did methods of Rakhmanov provide much information about the behavior of $\{\gamma_j^{(n)}\}$. In the current paper, we study the recursion parameters that give a negative answer to the conjecture of Steklov. That opens new, difference-equation perspective to the problem and addresses a question raised in [2]. The main result of the current paper is

Theorem 1.1. *Fix $\epsilon \in (0, \epsilon_0]$ where ϵ_0 is sufficiently small. Then, there is $n_0(\epsilon)$ such that for every $n > n_0(\epsilon)$, there is a weight $\tilde{w}^{(n)}$ such that*

$$\tilde{w}^{(n)} \text{ satisfies the uniform Steklov condition : } \left\| \frac{1}{\tilde{w}^{(n)}} \right\|_{L^\infty(\mathbb{T})} \lesssim 1, \quad (6)$$

$$\|\phi_{2n+1}(z, \tilde{w}^{(n)})\|_{L^\infty(\mathbb{T})} >_\epsilon \ln n, \quad (7)$$

and the asymptotics for $\gamma_j^{(n)}$ is given by

$$\gamma_j^{(n)} = - \begin{cases} \frac{j^{j+1}}{j+1} \sum_{v=\pm 1} v^{j+1} (2(j+1))^{\frac{2i\epsilon v}{\pi}} \frac{\Gamma(1 - \frac{i\epsilon v}{\pi})}{\Gamma(\frac{i\epsilon v}{\pi})} + r_{j,\epsilon}, & 0 \leq j \leq n-1 \\ -\frac{j^{j'+1}}{j'+1} \sum_{v=\pm 1} v^{j'+1} (2(j'+1))^{\frac{2i\epsilon v}{\pi}} \frac{\Gamma(1 - \frac{i\epsilon v}{\pi})}{\Gamma(\frac{i\epsilon v}{\pi})} + r_{j',\epsilon}, & n \leq j \leq 2n-1 \\ 0, & j = 2n \\ \frac{(-1)^j j^{j-2n}}{j-2n} \sum_{v=\pm 1} v^{j-2n} (2(j-2n))^{\frac{2i\epsilon v}{\pi}} \frac{\Gamma(1 - \frac{i\epsilon v}{\pi})}{\Gamma(\frac{i\epsilon v}{\pi})} + r_{j-2n,\epsilon}, & 2n+1 \leq j \leq 3n \\ 0, & j > 3n \end{cases} \quad (8)$$

where $j' = 2n - 1 - j$ and $|r_{j,\epsilon}| < C_\epsilon(j+1)^{-2}$.

Remark. The existence of the measure satisfying these conditions is not a new result [1] and the logarithmic growth is not optimal [6]. However, the construction and the asymptotics of the Schur parameters are new and might be interesting. The polynomial constructed by our method has a structure similar to the one from [18, 7].

Remark. The bulk of the technical work done in this paper is based on the recent progress in understanding the asymptotics of the orthogonal polynomials given by the so-called Fisher-Hartwig weights (see, e.g., [4, 8, 16]).

Remark. The careful analysis of (8) shows that the main terms are real-valued and converge to 0 as $\epsilon \rightarrow 0$ for every fixed j . The results in [4] allow one to control the dependence of C_ϵ on ϵ when it converges to zero and we conjecture that $\lim_{\epsilon \rightarrow 0} C_\epsilon = 0$.

We will use the following notation. If $f_{1(2)}(x)$ are two positive functions for which $f_1 < C f_2$ with some absolute constant C , uniformly in the argument, we will write $f_1 \lesssim f_2$. If $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$, then $f_1 \sim f_2$. If

$$\sup_x \frac{f_1}{f_2} < C(\epsilon), \quad (9)$$

where ϵ is a parameter, then we will write $f_1 <_\epsilon f_2$. Relations $f_1 \sim_\epsilon f_2, f_1 >_\epsilon f_2$ are defined similarly.

The symbol Γ denotes the Gamma function. We will call a function F Carathéodory if it is analytic in \mathbb{D} and its real part is positive. Given a measure σ , we will denote the Carathéodory function given by the Schwarz transform of σ as

$$F(z) = \mathfrak{S}(\sigma) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta).$$

If σ is in Szegő class, Π will denote the outer function in \mathbb{D} that satisfies

$$|\Pi(z)|^{-2} = 2\pi\sigma', \quad \text{a.e. } z \in \mathbb{T}, \quad \Pi(0) > 0.$$

The structure of the paper is as follows: the second section contains the proofs of auxiliary Lemmas and the main Theorem. The third section is an Appendix with the proof of a Lemma in the main text.

2. PROOF OF THEOREM 1.1

The measures considered below will be symmetric with respect to \mathbb{R} and the related Schur parameters will be real. We will need the following simple Lemma.

Lemma 2.1. *Suppose $\Phi_k, \Psi_k, \Phi_k^*, \Psi_k^*$ are the polynomials that correspond to **real** Schur parameters $\{\alpha_j\}_{j=0}^{k-1}$. Then, the polynomials associated to the sequence of Schur parameters $\{\gamma_j\}$ given by*

$$\gamma_j = \begin{cases} \alpha_j & : 0 \leq j \leq k-1 \\ -\alpha_{2k-1-j} & : k \leq j \leq 2k-1 \end{cases}$$

satisfy

$$\begin{aligned} 2\Phi_{2k} &= \Phi_k^2 + \Phi_k \Psi_k - z^{-1}(\Phi_k^*)^2 + z^{-1}\Phi_k^* \Psi_k^*, \\ 2\Phi_{2k}^* &= (\Phi_k^*)^2 + \Phi_k^* \Psi_k^* - z\Phi_k^2 + z\Phi_k \Psi_k. \end{aligned}$$

Proof. It is known that the pair (Ψ_j, Ψ_j^*) satisfies the Szegő recurrence with parameters $\{-\alpha_j\}$. If

$$A = \prod_{j=k-1}^0 \begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (10)$$

then

$$\begin{bmatrix} \Phi_k & \Psi_k \\ \Phi_k^* & -\Psi_k^* \end{bmatrix} = A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Thus,

$$a = \frac{\Phi_k + \Psi_k}{2}, \quad b = \frac{\Phi_k - \Psi_k}{2}, \quad c = \frac{\Phi_k^* - \Psi_k^*}{2}, \quad d = \frac{\Phi_k^* + \Psi_k^*}{2}.$$

First we reverse the dynamics. We are interested in

$$\prod_{j=0}^{k-1} \begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix}.$$

We see that

$$\begin{aligned} A^T &= \prod_{j=0}^{k-1} \begin{bmatrix} z & -z\gamma_j \\ -\gamma_j & 1 \end{bmatrix} = \prod_{j=0}^{k-1} \left(\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \prod_{j=0}^{k-1} \left(\begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}. \end{aligned}$$

Therefore,

$$\prod_{j=0}^{k-1} \begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} A^T \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z & -\gamma_j \\ -z\gamma_j & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} z & \gamma_j \\ z\gamma_j & 1 \end{bmatrix}.$$

Therefore, (10) implies

$$\prod_{j=k-1}^0 \begin{bmatrix} z & \gamma_j \\ z\gamma_j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Combining the above results, we get

$$\prod_{j=0}^{k-1} \begin{bmatrix} z & \gamma_j \\ z\gamma_j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -z \end{bmatrix} A^T \begin{bmatrix} 1 & 0 \\ 0 & -z^{-1} \end{bmatrix},$$

and so

$$\begin{aligned} \begin{bmatrix} \Phi_{2k} & \Psi_{2k} \\ \Phi_{2k}^* & -\Psi_{2k}^* \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -z \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -z^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a(a+b) - z^{-1}c(c+d) & a(a-b) + z^{-1}c(d-c) \\ d(c+d) - zb(a+b) & d(c-d) + zb(b-a) \end{bmatrix}. \end{aligned}$$

Thus,

$$2\Phi_{2k} = \Phi_k^2 + \Phi_k \Psi_k - z^{-1}(\Phi_k^*)^2 + z^{-1}\Phi_k^* \Psi_k^*$$

and

$$2\Phi_{2k}^* = (\Phi_k^*)^2 + \Phi_k^* \Psi_k^* - z\Phi_k^2 + z\Phi_k \Psi_k.$$

□

The following result is well-known.

Lemma 2.2. *If $\{\alpha_j\}$ are the Schur parameters for the measure $\sigma(\theta)$, then parameters $\{\alpha_j^{(\beta)}\}$ for the translated measure $\sigma^{(\beta)} = \sigma(\theta - \beta)$ are given by*

$$\alpha_j^{(\beta)} = e^{-i(j+1)\beta} \alpha_j, \quad j = 0, 1, \dots$$

Proof. The proof follows from making the following two observations:

$$\Phi_n(z, \sigma^{(\beta)}) = e^{in\beta} \Phi_n(ze^{-i\beta})$$

(which follows from the definition) and, take $z = 0$ in (2), $\Phi_{j+1}(0, \sigma^{(\beta)}) = -\bar{\alpha}_j^{(\beta)}$. \square

We will need the following decoupling Lemma. Its proof is contained in [2, 7]. However, following [7], we include the sketch for the reader's convenience.

Lemma 2.3. *Suppose we are given a polynomial ϕ_n of degree n and Carathéodory function \tilde{F} which satisfy the following properties*

1. $\phi_n^*(z)$ has no roots in $\bar{\mathbb{D}}$.
2. Normalization on the size and "rotation"

$$\int_{\mathbb{T}} |\phi_n^*(z)|^{-2} d\theta = 2\pi, \quad \phi_n^*(0) > 0. \quad (11)$$

3. $\tilde{F} \in C^\infty(\mathbb{T})$, $\operatorname{Re} \tilde{F} > 0$ on \mathbb{T} , and

$$\frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{Re} \tilde{F}(e^{i\theta}) d\theta = 1. \quad (12)$$

Denote the Schur parameters given by the probability measures μ_n and $\tilde{\sigma}$

$$d\mu_n = \frac{d\theta}{2\pi |\phi_n^*(e^{i\theta})|^2}, \quad d\tilde{\sigma} = \tilde{\sigma}' d\theta = \frac{\operatorname{Re} \tilde{F}(e^{i\theta})}{2\pi} d\theta,$$

as $\{\gamma_j\}$ and $\{\tilde{\gamma}_j\}$, respectively. Then, the probability measure σ , corresponding to Schur coefficients

$$\gamma_0, \dots, \gamma_{n-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \dots$$

is purely absolutely continuous with the weight given by

$$\sigma' = \frac{4\tilde{\sigma}'}{|\phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n)|^2} = \frac{2 \operatorname{Re} \tilde{F}}{\pi |\phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n)|^2}. \quad (13)$$

The polynomial ϕ_n is the n -th orthonormal polynomial for σ .

Proof. First, notice that $\{\tilde{\gamma}_j\} \in \ell^1$ by Baxter's Theorem (see, e.g., [21], Vol.1, Chapter 5). Therefore, σ is purely absolutely continuous by the same Baxter's criterion. Define the orthonormal polynomials of the first/second kind corresponding to measure $\tilde{\sigma}$ by $\{\tilde{\phi}_j\}, \{\tilde{\psi}_j\}$. Similarly, let $\{\phi_j\}, \{\psi_j\}$ be orthonormal polynomials for σ . Since, by construction, μ_n and σ have identical first n Schur parameters, ϕ_n is n -th orthonormal polynomial for σ .

Let us compute the polynomials ϕ_j and ψ_j , orthonormal with respect to σ , for the indexes $j > n$. The recursion can be rewritten in the following matrix form

$$\begin{pmatrix} \phi_{n+m} & \psi_{n+m} \\ \phi_{n+m}^* & -\psi_{n+m}^* \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} \phi_n & \psi_n \\ \phi_n^* & -\psi_n^* \end{pmatrix}, \quad (14)$$

where $\mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{D}_m$ satisfy

$$\begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} = \frac{1}{\tilde{\rho}_0 \cdots \tilde{\rho}_{m-1}} \begin{pmatrix} z & -\tilde{\gamma}_{m-1} \\ -z\tilde{\gamma}_{m-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} z & -\tilde{\gamma}_0 \\ -z\tilde{\gamma}_0 & 1 \end{pmatrix}$$

and thus depend only on $\tilde{\gamma}_0, \dots, \tilde{\gamma}_{m-1}$. Moreover, we have

$$\begin{pmatrix} \tilde{\phi}_m & \tilde{\psi}_m \\ \tilde{\phi}_m^* & -\tilde{\psi}_m^* \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m & \mathcal{B}_m \\ \mathcal{C}_m & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus, $\mathcal{A}_m = (\tilde{\phi}_m + \tilde{\psi}_m)/2$, $\mathcal{B}_m = (\tilde{\phi}_m - \tilde{\psi}_m)/2$, $\mathcal{C}_m = (\tilde{\phi}_m^* - \tilde{\psi}_m^*)/2$, $\mathcal{D}_m = (\tilde{\phi}_m^* + \tilde{\psi}_m^*)/2$ and their substitution into (14) yields

$$2\phi_{n+m}^* = \phi_n(\tilde{\phi}_m^* - \tilde{\psi}_m^*) + \phi_n^*(\tilde{\phi}_m^* + \tilde{\psi}_m^*) = \tilde{\phi}_m^* \left(\phi_n + \phi_n^* + \tilde{F}_m(\phi_n^* - \phi_n) \right) \quad (15)$$

where

$$\tilde{F}_m(z) = \frac{\tilde{\psi}_m^*(z)}{\tilde{\phi}_m^*(z)}.$$

Since $\{\tilde{\gamma}_n\} \in \ell^1$ and $\{\gamma_n\} \in \ell^1$, we have ([21], p. 225)

$$\tilde{F}_m \rightarrow \tilde{F} \text{ as } m \rightarrow \infty \text{ and } \phi_j^* \rightarrow \Pi, \tilde{\phi}_j^* \rightarrow \tilde{\Pi} \text{ as } j \rightarrow \infty.$$

uniformly on $\overline{\mathbb{D}}$. The functions Π and $\tilde{\Pi}$ are related to σ and $\tilde{\sigma}$ as follows: they are the outer functions in \mathbb{D} that satisfy

$$|\Pi|^{-2} = 2\pi\sigma', \quad |\tilde{\Pi}|^{-2} = 2\pi\tilde{\sigma}', \quad \Pi(0) > 0, \quad \tilde{\Pi}(0) > 0 \quad (16)$$

on \mathbb{T} . In (15), send $m \rightarrow \infty$ to get

$$2\Pi = \tilde{\Pi} \left(\phi_n + \phi_n^* + \tilde{F}(\phi_n^* - \phi_n) \right) \quad (17)$$

and we have (13) after taking the square of absolute values and using (16). \square

The next result has to do with the recent analysis of the asymptotics of the polynomials orthogonal with the respect to the so-called Fisher-Hartwig weights. We will use [4] as the main reference.

Consider the weight on \mathbb{T} given by

$$f(z) = e^\epsilon g_{i, -\frac{i\epsilon}{\pi}}(z) g_{-i, \frac{i\epsilon}{\pi}}(z), \quad (18)$$

where $z = e^{i\theta}$, $\theta \in [0, 2\pi)$ and $\epsilon \in (0, \epsilon_0]$ with ϵ_0 to be chosen sufficiently small. For $z_j = e^{i\theta_j}$,

$$g_{z_j, \beta_j} = \begin{cases} e^{i\pi\beta_j} & : 0 \leq \arg z < \theta_j \\ e^{-i\pi\beta_j} & : \theta_j \leq \arg z < 2\pi \end{cases}.$$

That is, f is a weight with two jumps, one from e^ϵ to $e^{-\epsilon}$ around i (in counterclockwise direction), and one from $e^{-\epsilon}$ back to e^ϵ around $-i$ (again in counterclockwise direction). It does not define a probability measure but this will not influence the polynomials much due to (4),(5). Notice that f is symmetric with respect to \mathbb{R} so all its recursion parameters are real.

We will need the following two Lemmas the proofs of which (essentially contained in [4]) will be discussed in the Appendix.

Lemma 2.4. *The Schur parameters $\{\gamma_j\}$ associated to the f above satisfy:*

$$\gamma_j = -(j+1)^{-1} i^{j+1} \left((2(j+1))^{\frac{2i\epsilon}{\pi}} \frac{\Gamma(1 - \frac{i\epsilon}{\pi})}{\Gamma(\frac{i\epsilon}{\pi})} + (-1)^{j+1} (2(j+1))^{-\frac{2i\epsilon}{\pi}} \frac{\Gamma(1 + \frac{i\epsilon}{\pi})}{\Gamma(\frac{-i\epsilon}{\pi})} \right) + r_{j,\epsilon},$$

$$|r_{j,\epsilon}| < C_\epsilon (j+1)^{-2}.$$

The next Lemma will be needed in the proof of Theorem 1.1.

Lemma 2.5. *Let $\epsilon \in (0, \epsilon_0]$ and $n > n_0(\epsilon)$. Then, for the weight f given by (18), the n -th associated monic polynomials of the first and second kinds Φ_n and Ψ_n respectively, and their $*$ -polynomials Ψ_n^* and Φ_n^* satisfy the following estimates:*

$$|\Phi_n^*(z)| \sim 1, \quad z \in \mathbb{T}, \quad (19)$$

$$\|\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n\|_{L^\infty(\mathbb{T})} >_\epsilon \ln n, \quad z \in \mathbb{T}, \quad (20)$$

$$\left| \frac{\Psi_n^*(z)}{\Phi_n^*(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} \right| \lesssim 1, \quad z \in \mathbb{T}. \quad (21)$$

Remark. Denote the Schur parameters that correspond to measure f in (18) by $\{\alpha_j\}$. The following general identity is immediate from Szegő recursion by taking the determinant

$$\det \begin{bmatrix} \Phi_n & \Psi_n \\ \Phi_n^* & -\Psi_n^* \end{bmatrix} = -2z^n \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)$$

so

$$\overline{\Phi_n^* \Psi_n^*} + \Phi_n^* \overline{\Psi_n^*} = 2 \prod_{j=0}^{n-1} (1 - |\alpha_j|^2).$$

Then, (19) gives

$$\left| \frac{\Psi_n^*}{\Phi_n^*} + \overline{\left(\frac{\Psi_n^*}{\Phi_n^*} \right)} \right| \lesssim 1 \quad (22)$$

uniformly over \mathbb{T} if the polynomials correspond to the weight (18).

Now we are in position to give a proof of Theorem 1.1.

Proof. (of Theorem 1.1). Given $\epsilon > 0$ and large n , we consider f defined by (18). Recall that the corresponding Schur parameters are denoted by $\{\alpha_j\}$. We consider the weight $w^{(n)}$ given by Schur parameters $\{\gamma_j^{(n)}\}$ defined as

$$\gamma_j^{(n)} = \begin{cases} \alpha_j, & j \leq n-1 \\ -\alpha_{j'}, & j' = 2n-1-j, n \leq j \leq 2n-1 \\ 0, & j = 2n \\ (-1)^{j-2n} \alpha_{j-2n-1}, & 2n+1 \leq j \leq 3n \end{cases}.$$

Now, (8) follows immediately from Lemma 2.4 and we need to show (6) and (7).

Let us prove (7). Notice first that $|\phi_j| \sim |\Phi_j|$ by (4) and (5) so it is sufficient to consider Φ_{2n+1} . We apply Lemma 2.1 to say

$$\begin{aligned} 2\Phi_{2n} &= \Phi_n^2 + \Phi_n \Psi_n - z^{-1} \Phi_n^{*2} + z^{-1} \Phi_n^* \Psi_n^*, \\ 2\Phi_{2n}^* &= \Phi_n^{*2} + \Phi_n^* \Psi_n^* - z \Phi_n^2 + z \Phi_n \Psi_n. \end{aligned}$$

Since the $\gamma_{2n}^{(n)} = 0$, we get $\Phi_{2n+1} = z \Phi_{2n}$, $\Psi_{2n+1} = z \Psi_{2n}$, $\Phi_{2n+1}^* = \Phi_{2n}^*$, and $\Psi_{2n+1}^* = \Psi_{2n}^*$ so

$$2\Phi_{2n+1} = z \Phi_n^2 + z \Phi_n \Psi_n - \Phi_n^{*2} + \Phi_n^* \Psi_n^*, \quad 2\Phi_{2n+1}^* = \Phi_n^{*2} + \Phi_n^* \Psi_n^* - z \Phi_n^2 + z \Phi_n \Psi_n. \quad (23)$$

We have

$$2\|\Phi_{2n+1}\|_\infty \geq \|\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n\|_\infty - 2\|\Phi_n\|_\infty^2 >_\epsilon \ln n$$

by Lemma 2.5.

To show (6), we will use Lemma 2.3. We choose Carathéodory function for the decoupled problem as

$$\tilde{F}(z) = \frac{\psi_n^*(-z)}{\phi_n^*(-z)} = \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} = -\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + O(1) \quad \text{by (21)}. \quad (24)$$

We have (see [21], Theorem 3.2.4)

$$\operatorname{Re} \tilde{F}(e^{i\theta}) = |\phi_n^*(e^{i(\theta+\pi)})|^{-2}$$

and \tilde{F} is Carathéodory function of the Bernstein-Szegő weight $(2\pi)^{-1}|\phi_n^*(e^{i(\theta+\pi)})|^{-2}$ having the Schur parameters

$$\begin{cases} (-1)^{j+1}\alpha_j, & j < n \\ 0, & j \geq n \end{cases}$$

as follows from Lemma 2.2. Since our Schur parameters $\gamma_j^{(n)} = 0, j > 3n$, we have

$$|\tilde{\Pi}(-z)| = |\phi_n^*(-z, f)| \sim 1, \quad \text{by (19).}$$

So, by the identity (17), we only need to show

$$|\Phi_{2n+1} + \Phi_{2n+1}^* + \tilde{F}(\Phi_{2n+1}^* - \Phi_{2n+1})| \lesssim 1$$

uniformly on \mathbb{T} . Recall that $\tilde{F} = -\frac{\Psi_n^*}{\Phi_n^*} + O(1)$ by (21). We introduce auxiliary

$$D = \frac{\Phi_n^* \Psi_n^*}{2}, \quad A = -\frac{\Phi_n^{*2}}{2}.$$

Notice that

$$|A| \sim 1, \quad \frac{D}{A} = -\frac{\Psi_n^*}{\Phi_n^*} = -\overline{\left(\frac{D}{A}\right)} + O(1) \quad (25)$$

by (19) and (22). We can now rewrite (23) as

$$\Phi_{2n+1} = -A^* + D^* + A + D, \quad \Phi_{2n+1}^* = -A + D + A^* + D^*.$$

(where the (*)-operations in these identities are of order $2n+1$).

Then, by (24),

$$\begin{aligned} \Phi_{2n+1} + \Phi_{2n+1}^* + \tilde{F}(\Phi_{2n+1}^* - \Phi_{2n+1}) &= 2 \left((D + D^*) + \frac{D}{A}(A^* - A) \right) + O(1) = \\ &= 2 \left(D^* + \frac{D}{A}A^* \right) + O(1) = \frac{2z^{2n}}{A} \left(\frac{D}{A} + \overline{\left(\frac{D}{A}\right)} \right) = O(1) \end{aligned}$$

by (25). □

3. APPENDIX: PROPERTIES OF POLYNOMIALS AND SCHUR PARAMETERS, PROOFS OF LEMMA 2.5 AND LEMMA 2.4

3.1. Setup. We wish to analyze the asymptotics of the orthogonal polynomials with respect to the weight

$$f(z) = e^\epsilon g_{i, -\frac{i\epsilon}{\pi}}(z) g_{-i, \frac{i\epsilon}{\pi}}(z),$$

where $z = |z|e^{i\theta}$, $\theta \in [0, 2\pi)$. For $z_j = e^{i\theta_j}$,

$$g_{z_j, \beta_j} = \begin{cases} e^{i\pi\beta_j} & : 0 \leq \arg z < \theta_j \\ e^{-i\pi\beta_j} & : \theta_j \leq \arg z < 2\pi \end{cases}$$

as defined in the main text.

This f belongs to the class of Fisher-Hartwig weights considered in [4] (see also [8] for the weight on the real line), so much of this Appendix consists of examining the results of [4], which performs asymptotic analysis of a Riemann-Hilbert problem involving the orthogonal polynomials. Recall the three properties of these polynomials we need:

(1)

$$|\Phi_n^*(z)| \sim 1,$$

(2)

$$\|\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n\|_{L^\infty(\mathbb{T})} > \epsilon \ln n,$$

(3)

$$\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} = O(1),$$

for $z \in \mathbb{T}, n > n_0(\epsilon)$.

As was noted in [3] and [9], the orthogonal polynomials of the first and second kinds satisfy a particular Riemann-Hilbert problem with contour $C = \mathbb{T}$. If Y is defined by

$$Y(z) = \begin{pmatrix} \Phi_n(z) & \int_{\mathbb{T}} \frac{\Phi_n(\xi) f(\xi) d\xi}{\xi - z} \frac{1}{2\pi i \xi^n} \\ -\Phi_{n-1}^*(z) & - \int_{\mathbb{T}} \frac{\overline{\Phi_{n-1}(\xi) f(\xi) d\xi}}{\xi - z} \frac{1}{2\pi i \xi} \end{pmatrix}, \quad (26)$$

then it satisfies the following Riemann-Hilbert problem:

- $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{T}$.
- For $z \in \mathbb{T} \setminus \{i, -i\}$, Y has continuous boundary values $Y_+(z)$ as z approaches \mathbb{T} from the inside, and $Y_-(z)$ from the outside, related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}.$$

- $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

- As $z \rightarrow \pm i, z \in \mathbb{C} \setminus \mathbb{T}$,

$$Y(z) = \begin{pmatrix} O(1) & O(\ln |z - \pm i|) \\ O(1) & O(\ln |z - \pm i|) \end{pmatrix}.$$

In what follows, we consider n to be a sufficiently large but fixed parameter.

The analysis of Riemann-Hilbert problems of this type proceeds by: enclosing the singularity points z_j by small but fixed disks U_{z_j} ; enclosing the curve $C = \mathbb{T}$ by lenses meeting in these disks; solving the Riemann-Hilbert problems induced in these regions; finally stitching them back together. For Fisher-Hartwig singularities this was performed in [4]. Since two of our estimates must be uniform over $z \in \mathbb{T}$, we will be concerned with the formulas that result in the regions enclosed by U_{z_j} as well as the the regions outside U_{z_j} but inside the lenses.

Before we begin the analysis of Riemann-Hilbert problem, we list a few useful identities and notations. Following [4], our complex logarithm will be cut at the negative real axis unless otherwise noted.

The measure $d\mu = \frac{f}{2\pi \cosh \epsilon} d\theta$ is a probability one. Therefore, one has (see [21], equation (3.2.53)):

$$\begin{aligned} \mathcal{S}(\mu) \Phi_{n-1}^*(z) - \Psi_{n-1}^*(z) &= z^{n-1} \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \overline{\Phi_{n-1}(e^{i\theta})} d\mu \\ &= z^{n-1} \int_{\mathbb{T}} \frac{e^{i\theta} - z + 2z}{e^{i\theta} - z} \overline{\Phi_{n-1}(e^{i\theta})} d\mu = 0 + 2z^n \int_{\mathbb{T}} \frac{\overline{\Phi_{n-1}(\xi)}}{\xi - z} \frac{f(\xi) d\xi}{i2\pi \xi \cosh \epsilon} \end{aligned}$$

so

$$-Y_{22}(z) = \int_{\mathbb{T}} \frac{\overline{\Phi_{n-1}(\xi)} f(\xi) d\xi}{\xi - z} \frac{1}{2\pi i \xi} = \frac{1}{2z^n} (F(z) \Phi_{n-1}^*(z) - (\cosh \epsilon) \Psi_{n-1}^*(z)), \quad (27)$$

where F is the Carathéodory function associated to $f(\theta)\frac{d\theta}{2\pi}$ (i.e. $F = \mathcal{S}(f/2\pi)$), and Ψ_n is the second kind polynomial associated to Φ_n .

The exact expression for F is easy to compute:

$$\begin{aligned} F(z) &= \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} f(\theta) \frac{d\theta}{2\pi} \\ &= -\frac{i(e^\epsilon - e^{-\epsilon})}{\pi} \ln \left(\frac{i-z}{i+z} \right) + \frac{1}{2}(e^{-\epsilon} + e^\epsilon). \end{aligned} \quad (28)$$

We will use the following notation from [4]:

$$\begin{aligned} \beta_1 &= -i\epsilon/\pi, & \alpha_1 &= 0, & z_1 &= i \\ \beta_2 &= i\epsilon/\pi, & \alpha_2 &= 0, & z_2 &= -i \end{aligned}$$

Before we discuss the results obtained in [4], we need to introduce confluent hypergeometric function $\psi(a, b, \zeta)$ which plays the key role in the analysis of the Riemann-Hilbert problem. We will have to use many facts about ψ . For this purpose we refer the reader to the National Institute of Standards and Technology's Digital Library of Mathematical Functions [24] and the appendix of [15].

The function $\psi(a, b, \zeta)$ is the confluent hypergeometric function of the second kind, often written as $U(a, b, \zeta)$. It is defined as the unique solution to Kummer's equation

$$\zeta \frac{d^2 w}{d\zeta^2} + (b - \zeta) \frac{dw}{d\zeta} - aw = 0,$$

satisfying $w(a, b, \zeta) \sim \zeta^{-a}$ as $\zeta \rightarrow \infty$. We will be interested in the following choices of the parameters: $b = 1$ and $a = \{\beta_j, 1 + \beta_j\}, j = 1, 2$. The function ψ is analytic in ζ on the universal cover of $\mathbb{C} \setminus 0$ and can be represented by the series (formula 13.2.9 in [24]):

$$\psi(a, 1, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!^2} \zeta^k \left(\ln \zeta + \frac{\Gamma'(a+k)}{\Gamma(a+k)} - \frac{2\Gamma'(k+1)}{\Gamma(k+1)} \right), \quad (29)$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol. This allows us to write ψ as

$$\psi(\zeta) = g(\zeta) \ln \zeta + h(\zeta), \quad (30)$$

where g and h are entire and single-valued. In particular, we have

$$\psi(a, 1, \zeta) = -\frac{1}{\Gamma(a)} \left(\ln \zeta + \frac{\Gamma'(a)}{\Gamma(a)} - \frac{2\Gamma'(1)}{\Gamma(1)} \right) + O(\zeta \ln \zeta) \quad (31)$$

for $\zeta : |\zeta| < 1$. The precise asymptotics of ψ as $\zeta \in \mathbb{C} \rightarrow \infty, -3\pi/2 < \arg \zeta < 3\pi/2$ for fixed a is (formula (7.2) in [15])

$$\psi(a, 1, \zeta) = \zeta^{-a} [1 - a^2 \zeta^{-1} + O(\zeta^{-2})] \quad (32)$$

This asymptotics is a consequence of the following integral representation of ψ (formula (7.3), [15]):

$$\psi(a, 1, \zeta) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{-i\alpha}} t^{a-1} (1+t)^{-a} e^{-\zeta t} dt, \quad -\pi < \alpha < \pi, \quad -\pi/2 + \alpha < \arg \zeta < \pi/2 + \alpha.$$

That representation, in particular, implies that

$$\sup_{u \in i\mathbb{R}, |u| > 1} |\psi(\pm i\epsilon, 1, u)| \rightarrow 1, \quad \epsilon \rightarrow 0 \quad (33)$$

and

$$\sup_{u \in i\mathbb{R}, |u| > 1} |\psi(1 \pm i\epsilon, 1, u) - \psi(1, 1, u)| \rightarrow 0, \quad \epsilon \rightarrow 0. \quad (34)$$

The crucial property of ψ which makes it indispensable for the Riemann-Hilbert analysis is the following transformation formula (formula (7.30) in [15])

$$\psi(a, c, e^{-2\pi i} \zeta) = e^{2\pi i a} \psi(a, c, \zeta) - \frac{2\pi i}{\Gamma(a)\Gamma(a-c+1)} e^{i\pi a} e^{\zeta} \psi(c-a, c, e^{-i\pi} \zeta).$$

Following [4], we will use the convention that, unless otherwise mentioned, ζ always satisfies $0 \leq \arg \zeta < 2\pi$.

Concerning the logarithmic derivative of the Gamma function (the digamma function) which appears above, we will have occasion to use its reflection formula (equation 5.15.6, [24])

$$\frac{\Gamma'(1-z)}{\Gamma(1-z)} - \frac{\Gamma'(z)}{\Gamma(z)} = \pi \cot(\pi z). \quad (35)$$

Now, we need to discuss another function which will be important below. Consider

$$\mathcal{D}(z) = \exp \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln f(s)}{s-z} ds \right).$$

This function is analytic away from \mathbb{T} with a.e. boundary values satisfying $\mathcal{D}_+(z) = \mathcal{D}_-(z)f(z)$, $|\mathcal{D}_+(z)|^2 = f(z)$ (compare with Π defined in (16)). Notice this definition a priori differs from that in [21], section 2.4. However due to the normalization $\int_{\mathbb{T}} \ln(f(\theta))d\theta = 0$, in fact these two agree. We may compute \mathcal{D} explicitly:

$$\mathcal{D}(z) = \exp \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln f(s)}{s-z} ds \right) = \exp \left(\frac{\epsilon}{\pi i} \ln \left(\frac{i-z}{i+z} \right) \right). \quad (36)$$

3.2. Asymptotic formulas for solution of the Riemann-Hilbert problem. In the next two subsections, we recall how asymptotics of Y on \mathbb{T} is obtained through solving Riemann-Hilbert problem. Then we will apply this asymptotics to prove Lemma 2.5.

In [4], the Riemann-Hilbert problem undergoes various transformations until it is in a form for which explicit solutions can be written. The singularity points i and $-i$, along with the artificially introduced point $z = 1$ (though the analysis reduces to triviality here and we drop this case) are all enclosed by the small disks U_i, U_{-i} of fixed radius $\delta > 0$. The remainder of the unit circle is enclosed in “lenses” (see Figure 1).

We trace through the various transformations of the RH problem for z away from the points of singularity i and $-i$. These reductions are:

$$Y \rightarrow T \rightarrow S \rightarrow R. \quad (37)$$

We will explain each of these transformations below. However, our analysis will be limited to the case when $\epsilon \in (0, \epsilon_0], n > n_0(\epsilon), |z| \leq 1$ and z belongs to one of the lenses. This choice is motivated by our goal to control Y only on the unit circle \mathbb{T} itself so we will only need to take $|z| = 1$ later on. In these domains, some of the transformations in (37) are trivial, e.g., $T = Y$ (formula (4.1) in [4]).

Then, (formula (4.3), [4]), S is related to T by

$$S(z) = T(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1}z^n & 1 \end{pmatrix}.$$

Now, the original Riemann-Hilbert problem for Y can be written in terms of S and its solution proceeds by first choosing various parametrices (approximate solutions) in each of the domains. The parametrices outside of $\cup_j U_{z_j}$ and inside of each U_{z_j} will be denoted by N and P_{z_j} , respectively. Our final transformation is to R , which satisfies the Riemann-Hilbert problem of very special form.

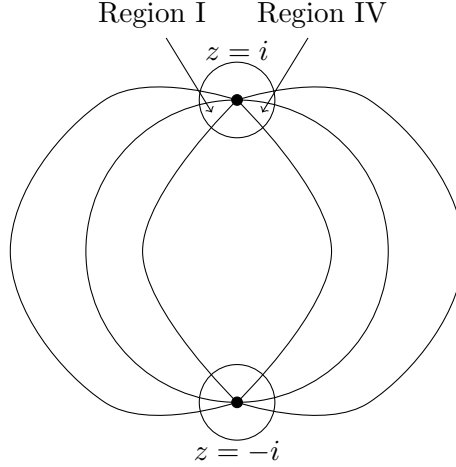


FIGURE 1. Setup of Riemann-Hilbert problem on \mathbb{T}

In fact, the correct choices of parametrices N and P_{z_j} makes it possible to say that each jump in the Riemann-Hilbert problem for R is of the form $I + O(n^{-1})$ when $n \rightarrow \infty$ on each of the contours involved (see (4.57)–(4.59)) and an asymptotics of R at infinity is $I + O(1/z)$. Then, the standard argument (see, e.g., [5]) implies that

$$R = I + O(n^{-1}) \quad (38)$$

uniformly over $z \in \mathbb{C}$. It is clear now that the main asymptotics of Y is captured by N and P_{z_j} . Below we will discuss these parametrices in detail.

3.2.1. *Case 1. Parametrix N , z outside of U_{z_j} .* For $z : |z| < 1$, we write (formula (4.7), [4])

$$N(z) = \begin{pmatrix} \mathcal{D}(z) & 0 \\ 0 & \mathcal{D}(z)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R(z) = S(z)N^{-1}(z)$$

and

$$R(z) = I + O\left(\frac{1}{n}\right)$$

by equations (4.61) and (4.65-71) in [4]. Since we are away from the singularities of the weight, all terms are uniformly bounded. Collecting $O\left(\frac{1}{n}\right)$ errors, we have

$$S(z) = N(z) + O\left(\frac{1}{n}\right).$$

Reversing these transformations

$$\begin{aligned} Y(z) = T(z) &= S(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1}z^n & 1 \end{pmatrix}^{-1} = \left(N(z) + O\left(\frac{1}{n}\right)\right) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{pmatrix} \\ &= \left(\left(\begin{pmatrix} \mathcal{D}(z) & 0 \\ 0 & \mathcal{D}(z)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O\left(\frac{1}{n}\right)\right) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{pmatrix}\right). \end{aligned}$$

So,

$$Y(z) = \begin{pmatrix} z^n f^{-1} \mathcal{D}(z) & \mathcal{D}(z) \\ -\mathcal{D}(z)^{-1} & 0 \end{pmatrix} + O\left(\frac{1}{n}\right), \quad (39)$$

when $|z| < 1$ and $z \notin U_{z_j}$. Now, to get asymptotical behavior of $Y(z)$ (and thus of $\Phi_n^*(z)$ and $\Psi_n^*(z)$ by formula (27)) on \mathbb{T} but outside U_{z_j} , we need to take $|z| \rightarrow 1$ in (39) and this gives

$$Y(z) = \begin{pmatrix} z^n f^{-1} \mathcal{D}_+(z) & \mathcal{D}_+(z) \\ -\mathcal{D}_+(z)^{-1} & 0 \end{pmatrix} + O\left(\frac{1}{n}\right) \quad (40)$$

uniformly over $z \in \mathbb{T}, z \notin U_{z_j}$.

3.2.2. Case 2: Parametrix P_{z_j} , $z \in U_{z_j}$. The nature of the singularities of the weight f at points z_1 and z_2 is the same so we will discuss only asymptotics in U_{z_1} in detail. By formula (4.23) in [4], we write

$$P_{z_j}(z) = E(z) \Psi^{(j)}(z) \begin{pmatrix} z^{n/2} & 0 \\ 0 & z^{-n/2} \end{pmatrix},$$

where $E(z)$ is chosen so that P_{z_j} and N approximately match across ∂U_{z_j} . For example, we choose region I on Figure 1 (we can also take the similar domain around z_2). This corresponds to $|z| < 1, 0 < \arg \frac{z}{z_j} < \pi/2$. By introducing $\zeta = n \ln \frac{z}{z_j}$ we get $\arg \zeta \in (\pi/2, \pi)$ and the choice of E and $\Psi^{(j)}$ are made by (formula (4.32), [4])

$$\begin{aligned} \Psi_j^{(I)}(\zeta) &= \begin{pmatrix} \psi(\beta_j, 1, \zeta) e^{2i\pi\beta_j} \left(\frac{z}{z_j}\right)^{-n/2} & -e^{i\pi\beta_j} \psi(1 - \beta_j, 1, e^{-i\pi\zeta}) \left(\frac{z}{z_j}\right)^{n/2} \frac{\Gamma(1-\beta_j)}{\Gamma(\beta_j)} \\ -e^{i\pi\beta_j} \psi(1 + \beta_j, 1, \zeta) \left(\frac{z}{z_j}\right)^{-n/2} \frac{\Gamma(1+\beta_j)}{\Gamma(-\beta_j)} & \psi(-\beta_j, 1, e^{-i\pi\zeta}) \left(\frac{z}{z_j}\right)^{n/2} \end{pmatrix} \\ &:= \begin{pmatrix} e^{2i\pi\beta_j} \left(\frac{z}{z_j}\right)^{-n/2} \psi_1^{(j)} & -e^{i\pi\beta_j} \left(\frac{z}{z_j}\right)^{n/2} c_2^{(j)} \psi_3^{(j)} \\ -e^{i\pi\beta_j} \left(\frac{z}{z_j}\right)^{-n/2} c_1^{(j)} \psi_2^{(j)} & \left(\frac{z}{z_j}\right)^{n/2} \psi_4^{(j)} \end{pmatrix}, \quad (\text{this defines } \psi_{1,2,3,4}^{(j)}, c_{1,2}^{(j)}) \end{aligned}$$

and (formula (4.47))

$$E(z) = N(z) \begin{pmatrix} \zeta^{\beta_j} & 0 \\ 0 & \zeta^{-\beta_j} \end{pmatrix} \begin{pmatrix} z_j^{-n/2} & 0 \\ 0 & z_j^{n/2} \end{pmatrix} \begin{pmatrix} e^{-2\pi i\beta_j} & 0 \\ 0 & e^{\pi i\beta_j} \end{pmatrix}.$$

Multiplying matrices, we find

$$P_{z_j}(z) = \begin{pmatrix} 0 & \mathcal{D}(z) \\ -\mathcal{D}(z)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \zeta^{\beta_j} \psi_1^{(j)} & -e^{-\pi i\beta_j} c_2^{(j)} \zeta^{\beta_j} z_j^{-n} \psi_3^{(j)} \\ -e^{2\pi i\beta_j} c_1^{(j)} z_j^n \zeta^{-\beta_j} \psi_2^{(j)} & e^{\pi i\beta_j} \zeta^{-\beta_j} \psi_4^{(j)} \end{pmatrix}.$$

Restricting our attention to $j = 1$, and using the notation $Y^{(1,I)}$ to identify this as the solution Y in region I around point $z_1 = i$:

$$Y^{(1,I)} = T = RP_{z_1} \begin{pmatrix} 1 & 0 \\ f(z)^{-1} z^n & 1 \end{pmatrix}.$$

We have again

$$R(z) = S(z) P_{z_1}^{-1}(z)$$

and

$$R(z) = I + O\left(\frac{1}{n}\right)$$

as follows from (38). However, since we have singularities in our expressions, we will leave R in this form for now. This yields

$$Y^{(1,I)}(z) = \left(I + O\left(\frac{1}{n}\right) \right) P_{z_1}(z) \begin{pmatrix} 1 & 0 \\ f^{-1} z^n & 1 \end{pmatrix} \quad (41)$$

$$\begin{aligned}
&= \left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} 0 & \mathcal{D}(z) \\ -\mathcal{D}(z)^{-1} & 0 \end{pmatrix} \\
&\times \begin{pmatrix} \zeta^{-i\epsilon/\pi} \psi_1^{(1)} & -e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} \\ -e^{2\epsilon} c_1^{(1)} i^n \zeta^{i\epsilon/\pi} \psi_2^{(1)} & e^\epsilon \zeta^{i\epsilon/\pi} \psi_4^{(1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^{-1} z^n & 1 \end{pmatrix} \\
&= \left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} 0 & \mathcal{D}(z) \\ -\mathcal{D}(z)^{-1} & 0 \end{pmatrix} \\
&\times \begin{pmatrix} \zeta^{-i\epsilon/\pi} \psi_1^{(1)} - e^{-\epsilon} f^{-1} z^n c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} & -e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} \\ -e^{2\epsilon} c_1^{(1)} i^n \zeta^{i\epsilon/\pi} \psi_2^{(1)} + e^\epsilon f^{-1} z^n \zeta^{i\epsilon/\pi} \psi_4^{(1)} & e^\epsilon \zeta^{i\epsilon/\pi} \psi_4^{(1)} \end{pmatrix} \\
&= \left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} \mathcal{D}(z)(-e^{2\epsilon} c_1^{(1)} i^n \zeta^{i\epsilon/\pi} \psi_2^{(1)} + e^\epsilon f^{-1} z^n \zeta^{i\epsilon/\pi} \psi_4^{(1)}) & \mathcal{D}(z) e^\epsilon \zeta^{i\epsilon/\pi} \psi_4^{(1)} \\ -\mathcal{D}(z)^{-1} (\zeta^{-i\epsilon/\pi} \psi_1^{(1)} - e^{-\epsilon} f^{-1} z^n c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)}) & \mathcal{D}(z)^{-1} e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} \end{pmatrix}.
\end{aligned}$$

In U_{z_2} , the calculations are exactly the same, but with $z_1 = i$ replaced by $z_2 = -i$ and β_1 replaced by $\beta_2 = -\beta_1$. Therefore, we have

$$\begin{aligned}
&Y^{(2,I)}(z) = \\
&\left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} \mathcal{D}(z)(-e^{-2\epsilon} c_1^{(2)} (-i)^n \zeta^{-i\epsilon/\pi} \psi_2^{(2)} + e^{-\epsilon} f^{-1} z^n \zeta^{i\epsilon/\pi} \psi_4^{(2)}) & \mathcal{D}(z) e^{-\epsilon} \zeta^{-i\epsilon/\pi} \psi_4^{(2)} \\ -\mathcal{D}(z)^{-1} (\zeta^{i\epsilon/\pi} \psi_1^{(2)} - e^\epsilon f^{-1} z^n c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)}) & \mathcal{D}(z)^{-1} e^\epsilon c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)} \end{pmatrix}.
\end{aligned}$$

Recalling that

$$f(z) = \begin{cases} e^\epsilon, & -\pi/2 \leq \arg z < \pi/2 \\ e^{-\epsilon}, & \text{otherwise} \end{cases},$$

we see that $f = e^{-\epsilon}$ in U_{z_1} , region I, and $f = e^\epsilon$ in region I of U_{z_2} . Thus, we get

$$\begin{aligned}
&Y^{(1,I)}(z) = \\
&\left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} e^{2\epsilon} \mathcal{D}(z)(-c_1^{(1)} \zeta^{i\epsilon/\pi} i^n \psi_2^{(1)} + z^n \zeta^{i\epsilon/\pi} \psi_4^{(1)}) & \mathcal{D}(z) e^\epsilon \zeta^{i\epsilon/\pi} \psi_4^{(1)} \\ -\mathcal{D}(z)^{-1} (\zeta^{-i\epsilon/\pi} \psi_1^{(1)} - z^n c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)}) & \mathcal{D}(z)^{-1} e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&Y^{(2,I)}(z) = \\
&\left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} e^{-2\epsilon} \mathcal{D}(z)(-c_1^{(2)} \zeta^{-i\epsilon/\pi} (-i)^n \psi_2^{(2)} + z^n \zeta^{-i\epsilon/\pi} \psi_4^{(2)}) & \mathcal{D}(z) e^{-\epsilon} \zeta^{-i\epsilon/\pi} \psi_4^{(2)} \\ -\mathcal{D}(z)^{-1} (\zeta^{i\epsilon/\pi} \psi_1^{(2)} - z^n c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)}) & \mathcal{D}(z)^{-1} e^\epsilon c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)} \end{pmatrix}.
\end{aligned}$$

Because of the singularities involved, we must take care in performing this last multiplication. Denote by \tilde{Y}_{z_j} the right-hand matrix in the above equation. That is

$$\tilde{Y}_i^{(I)}(z) := \begin{pmatrix} e^{2\epsilon} \mathcal{D}(z)(-c_1^{(1)} \zeta^{i\epsilon/\pi} i^n \psi_2^{(1)} + z^n \zeta^{i\epsilon/\pi} \psi_4^{(1)}) & \mathcal{D}(z) e^\epsilon \zeta^{i\epsilon/\pi} \psi_4^{(1)} \\ -\mathcal{D}(z)^{-1} (\zeta^{-i\epsilon/\pi} \psi_1^{(1)} - z^n c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)}) & \mathcal{D}(z)^{-1} e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} \end{pmatrix}, \quad (42)$$

$$\tilde{Y}_{-i}^{(I)}(z) := \begin{pmatrix} e^{-2\epsilon} \mathcal{D}(z)(-c_1^{(2)} \zeta^{-i\epsilon/\pi} (-i)^n \psi_2^{(2)} + z^n \zeta^{-i\epsilon/\pi} \psi_4^{(2)}) & \mathcal{D}(z) e^{-\epsilon} \zeta^{-i\epsilon/\pi} \psi_4^{(2)} \\ -\mathcal{D}(z)^{-1} (\zeta^{i\epsilon/\pi} \psi_1^{(2)} - z^n c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)}) & \mathcal{D}(z)^{-1} e^\epsilon c_2^{(2)} \zeta^{i\epsilon/\pi} (-i)^{-n} \psi_3^{(2)} \end{pmatrix}. \quad (43)$$

We start with proving Lemma 2.4.

Proof. (of Lemma 2.4). We only need to use formula (1.23) from [4]. This equation shows, in the notations introduced above,

$$-\bar{\gamma}_{k-1} = \Phi_k(0) = k^{-2\beta_1-1} z_1^k 2^{2\beta_2} \frac{\Gamma(1+\beta_1)}{\Gamma(-\beta_1)} + k^{-2\beta_2-1} z_2^k 2^{2\beta_1} \frac{\Gamma(1+\beta_2)}{\Gamma(-\beta_1)} + r_{k,\epsilon} \quad (44)$$

and

$$|r_{k,\epsilon}| < C_\epsilon (k+1)^{-2}. \quad (45)$$

□

Remark. The estimates (44), (45), and (3) imply that the recursion coefficients for the weight f satisfy

$$\|\{\gamma_k\}\|_{\ell^2} \lesssim \sqrt{\epsilon}, \quad |\gamma_k| <_\epsilon (k+1)^{-1}. \quad (46)$$

Recall that Ψ_n and Ψ_n^* satisfy recursion

$$\begin{cases} \Psi_{n+1} = z\Psi_n + \bar{\gamma}_n \Psi_n^* \\ \Psi_{n+1}^* = \Psi_n^* + \gamma_n z \Psi_n \end{cases} \quad (47)$$

and we have

$$|\Psi_{n+1}| \leq |\Psi_n|(1 + |\gamma_n|), \quad |\Psi_0| = 1$$

Iterating this formula and using (46) we get a rough upper bound

$$\|\Psi_n\|_{L^\infty(\mathbb{T})} <_\epsilon n^{C_\epsilon}. \quad (48)$$

This estimate can be substantially improved by Riemann-Hilbert analysis but (48) will be good enough for our purposes.

Now we are ready to verify Lemma 2.5.

Proof. (of Lemma 2.5).

3.3. $|\Phi_n^*(z)| \sim 1, z \in \mathbb{T}$ for $\epsilon \in (0, \epsilon_0]$ and $n > n_0(\epsilon)$.

We consider two cases.

3.3.1. z outside U_{z_j} . By equation (39), $\Phi_n^*(z) \rightarrow \mathcal{D}_+(z)^{-1}$ uniformly in this region. Since $|\mathcal{D}_+| = f^{1/2}$ a.e. on \mathbb{T} , this trivially implies our desired estimate for z outside of $U_{\pm i}$.

3.3.2. z inside U_{z_j} . We consider the boundary values as $|z| \rightarrow 1$ in the asymptotics for Y . Notice that, since $|\mathcal{D}_+|^2 = f$, we have $|\mathcal{D}_+| = e^{-\epsilon/2}$ in region I around $z = i$. We will focus on region I where $\zeta = iu, u > 0$. In the other regions, analysis is the same. Recall that $\zeta = n \ln \frac{z}{i}$. So, if $z = e^{i(\pi/2+\tau)}, \tau > 0$, we have

$$\zeta^{-i\epsilon/\pi} = \left(n \ln \frac{z}{i} \right)^{-i\epsilon/\pi} = (ni\tau)^{-i\epsilon/\pi} = e^{\epsilon/2} e^{-\frac{i\epsilon}{\pi} \ln(n\tau)}.$$

Therefore, in region I, in which $\arg \frac{z}{i} > 0$, one has

$$\left| \frac{1}{\mathcal{D}(z)\zeta^{i\epsilon/\pi}} \right| = e^\epsilon. \quad (49)$$

Similarly, in region IV, the other side of i , $|\mathcal{D}(z)| = e^{\epsilon/2}$ and $|\zeta^{-i\epsilon/\pi}| = |e^{-i\epsilon\pi^{-1} \ln \zeta}| = e^{3\epsilon/2}$ since $\arg \zeta = 3\pi/2$. We again obtain $\left| \frac{1}{\mathcal{D}(z)\zeta^{i\epsilon/\pi}} \right| = e^\epsilon$.

Consider the expressions involving the ψ in the first column of $\tilde{Y}_i^{(I)}(z)$. We focus on $\tilde{Y}_{i,21}^{(I)}$, the bottom left corner of the \tilde{Y} matrix. Due to (49) and definition of ζ ,

$$\left| \tilde{Y}_{i,21}^{(I)}(z) \right| = e^\epsilon \left| \psi_1^{(1)} - \left(\frac{z}{i}\right)^n c_2^{(1)} \psi_3^{(1)} \right| = e^\epsilon \left| \psi \left(-\frac{i\epsilon}{\pi}, 1, \zeta \right) - e^\zeta \frac{\Gamma(1 + \frac{i\epsilon}{\pi})}{\Gamma(-\frac{i\epsilon}{\pi})} \psi \left(1 + \frac{i\epsilon}{\pi}, 1, e^{-i\pi} \zeta \right) \right|.$$

Consider

$$\Omega(\zeta, \epsilon) = \psi \left(-\frac{i\epsilon}{\pi}, 1, \zeta \right) - e^\zeta \frac{\Gamma(1 + \frac{i\epsilon}{\pi})}{\Gamma(-\frac{i\epsilon}{\pi})} \psi \left(1 + \frac{i\epsilon}{\pi}, 1, e^{-i\pi} \zeta \right).$$

It is the analysis of this function which concerns us. We want to show that

$$\max_{\zeta=iu, u \in \mathbb{R}} \left| |\Omega(\zeta, \epsilon)| - 1 \right| \rightarrow 0, \quad \epsilon \rightarrow 0.$$

We do this in two steps. The estimates (32),(33), and (34) imply that

$$\max_{\zeta=iu, |u|>1} \left| |\Omega(\zeta, \epsilon)| - 1 \right| \rightarrow 0, \quad \epsilon \rightarrow 0.$$

For $|\zeta| < 1$, we will use series (29) for ψ . We want to show

$$\max_{\zeta=iu, |u|<1} \left| |\Omega(\zeta, \epsilon)| - 1 \right| \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Recall that $\Gamma(\zeta)$ has a pole at 0 so $\lim_{\epsilon \rightarrow 0} \Gamma^{-1}(\pm i\epsilon) = 0$. From (30), we get

$$\begin{aligned} \Omega(\zeta, \epsilon) &= (\ln \zeta)g(-i\epsilon/\pi, \zeta) + h(-i\epsilon/\pi, \zeta) \\ &\quad - e^\zeta \frac{\Gamma(1 + i\epsilon/\pi)}{\Gamma(-i\epsilon/\pi)} \left((\ln(e^{-i\pi} \zeta))g(1 + i\epsilon/\pi, e^{-i\pi} \zeta) + h(1 + i\epsilon/\pi, e^{-i\pi} \zeta) \right), \end{aligned}$$

where

$$g(a, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!^2} \zeta^k, \quad (50)$$

$$h(a, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!^2} \zeta^k \left(\frac{\Gamma'(a+k)}{\Gamma(a+k)} - \frac{2\Gamma'(k+1)}{\Gamma(k+1)} \right). \quad (51)$$

These expansions converge uniformly in $\zeta : |\zeta| < 1$ and the coefficients depend on ϵ explicitly. In Ω , the logarithmic singularities cancel each other as follows (recall that $\zeta = iu, u > 0$)

$$-\frac{\ln \zeta}{\Gamma(-i\epsilon/\pi)} + \frac{\ln(e^{-i\pi} \zeta)}{\Gamma(-i\epsilon/\pi)} = \frac{-i\pi}{\Gamma(-i\epsilon/\pi)} \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where we accounted for the first terms in the series (50) and (51) only since for the other terms we can use

$$|\zeta \ln \zeta| \lesssim 1, \quad |\zeta| < 1.$$

Therefore, the required asymptotics of Ω will follow from

$$-\lim_{\epsilon \rightarrow 0} \frac{\Gamma'(-i\epsilon/\pi)}{\Gamma^2(-i\epsilon/\pi)} = 1.$$

Now, recall that (26),(43),(41) yield

$$-\Phi_{n-1}^*(z) = O(n^{-1})\tilde{Y}_{i,11}^{(I)}(z) + (1 + O(n^{-1}))\tilde{Y}_{i,21}^{(I)}(z).$$

The analysis to show that $\tilde{Y}_{i,11}^{(I)}(z) = O(1)$ is nearly identical to that showing $|\tilde{Y}_{i,21}^{(I)}(z)| \sim 1$ except that it may be performed with less care, since only an upper bound is needed. The estimates we obtained prove (19) in Lemma 2.5.

3.4. $\|\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n\|_{L^\infty(\mathbb{T})} >_\epsilon \ln n$. We will investigate $\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n$ for $z = e^{i\theta}$, $\theta \in (\pi/2 + n^{-0.5}, \pi/2 + 2n^{-0.5})$. Since $\zeta = n \ln \frac{z}{i} = iu$, $u \sim \sqrt{n}$, this puts us in the $\zeta \rightarrow \infty$ regime when using the parametrix P_{z_1} . This also allows us to easily perform the final multiplication in (41), since all elements in the \tilde{Y} matrix are $O(1)$ when $|\zeta| > 1$. Recall equation (27):

$$2z^n Y_{22}(z) = (\cosh \epsilon) \Psi_{n-1}^*(z) - F(z) \Phi_{n-1}^*(z). \quad (52)$$

Performing the multiplication and noting the error, (42) gives

$$Y_{22}(z) = \mathcal{D}(z)^{-1} e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} + O\left(\frac{1}{n}\right).$$

By (32), $\mathcal{D}(z)^{-1} e^{-\epsilon} c_2^{(1)} \zeta^{-i\epsilon/\pi} i^{-n} \psi_3^{(1)} = O(n^{-1/2})$ and $Y_{22}(z) = O(n^{-1/2})$. Similarly, by equation (42),

$$\Phi_n^* = \mathcal{D}^{-1} + O\left(\frac{1}{\sqrt{n}}\right) \quad (53)$$

for $\zeta = iu$, $u \sim \sqrt{n}$. Therefore, we finally have

$$|\Psi_n^*(z)| \sim \epsilon \ln n \quad (54)$$

due to (26). Recall that $z^n \overline{\Phi_n^*} = \Phi_n$. So, we can write

$$\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n = \Phi_n^* \Psi_n^* + z^{2n+1} \overline{\Phi_n^* \Psi_n^*} = \Phi_n^* \Psi_n^* \left(1 + z^{2n+1} \frac{\overline{\Phi_n^* \Psi_n^*}}{\Phi_n^* \Psi_n^*}\right).$$

Inserting (52), (53), and using $|\mathcal{D}| \sim 1$, we have

$$\frac{\overline{\Phi_n^* \Psi_n^*}}{\Phi_n^* \Psi_n^*} = \frac{\overline{F(z)(\mathcal{D}(z)^{-2} + O(1/\sqrt{n}))}}{F(z)(\mathcal{D}(z)^{-2} + O(1/\sqrt{n}))} = \frac{\overline{F}}{F} \cdot \left(\frac{\overline{\mathcal{D}}}{\mathcal{D}}\right)^{-2} (1 + O(n^{-0.5})). \quad (55)$$

From (28) and (36) we can compute

$$\frac{\overline{F}}{F} = -1 + o(1), \quad n \rightarrow \infty$$

and

$$\frac{\overline{\mathcal{D}}}{\mathcal{D}} = \exp\left(\frac{2i\epsilon}{\pi} \ln(\theta/2 - \pi/4)\right) (1 + o(1)), \quad n \rightarrow \infty$$

in our range of $z = e^{i\theta}$. Therefore, substitution into (55) shows that there is some $\theta_0 : \theta_0 \in (\pi/2 + n^{-0.5}, \pi/2 + 2n^{-0.5})$ for which

$$\left|1 + z_0^{2n+1} \frac{\overline{\Phi_n^*(z_0) \Psi_n^*(z_0)}}{\Phi_n^*(z_0) \Psi_n^*(z_0)}\right| \sim 1$$

where $z_0 = e^{i\theta_0}$.

Since, by (54), we have $|\Psi_n^*(z_0)| = O(\ln n)$, and $|\Phi_n^*| \sim 1$, we get

$$\|\Phi_n^* \Psi_n^* + z \Phi_n \Psi_n\|_{L^\infty(\mathbb{T})} >_\epsilon \ln n$$

as desired.

3.5. $\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} = O(1)$, $z \in \mathbb{T}$, $n > n_0(\epsilon)$. Outside of $U_{z_{1(2)}}$ this statement is trivial by (39), so we only consider z inside $U_{z_{1(2)}}$. Further, since the calculations are exactly similar in U_i and U_{-i} , we let $z \in U_i$.

Before we proceed with the analysis of Riemann-Hilbert problem, let us make two remarks. Firstly, since $|\Phi_n^*| \sim 1$, $n > n_0(\epsilon)$, we only need to show that U_{2n} , defined by $U_{2n}(z) = \Psi_n^*(z)\Phi_n^*(z) + \Psi_n^*(-z)\Phi_n^*(-z)$, satisfies

$$\|U_{2n}\|_{L^\infty(\mathbb{T})} \lesssim 1,$$

Secondly, U_{2n} is a polynomial of degree at most $2n$ and

$$\|U_{2n}\|_{L^\infty(\mathbb{T})} <_\epsilon n^{C_\epsilon}$$

by (48). The Bernstein inequality gives us

$$\|U'_{2n}\|_{L^\infty(\mathbb{T})} <_\epsilon n^{1+C_\epsilon}$$

Thus, to prove $\|U_{2n}\|_{L^\infty(\mathbb{T})} = O(1)$, we only need

$$|U_{2n}(e^{i\theta})| \lesssim 1 \tag{56}$$

for $\theta : \theta \in (\pi/2 + e^{-\sqrt{n}}, \pi/2 + \delta_1)$ and the parameter δ_1 here is of the same size as the radius of U_i .

Recall that $Y^{(1(2),I)}$ denotes the Y matrix in the region I that corresponds to point $z_{1(2)}$, respectively. By (27),

$$2z^n \frac{Y_{22}^{(1,I)}(z)}{Y_{21}^{(1,I)}(z)} + 2(-z)^n \frac{Y_{22}^{(2,I)}(-z)}{Y_{21}^{(2,I)}(-z)} = \left(F(z) + F(-z)\right) - (\cosh \epsilon) \left(\frac{\Psi_{n-1}^*(z)}{\Phi_{n-1}^*(z)} + \frac{\Psi_{n-1}^*(-z)}{\Phi_{n-1}^*(-z)}\right),$$

so we want to show

$$\left(F(z) + F(-z)\right) - 2 \left(z^n \frac{Y_{22}^{(1,I)}(z)}{Y_{21}^{(1,I)}(z)} + (-z)^n \frac{Y_{22}^{(2,I)}(-z)}{Y_{21}^{(2,I)}(-z)}\right) = O(1)$$

uniformly on \mathbb{T} provided that n is large enough.

The formula (28) implies

$$F(z) + F(-z) = -\frac{i(e^\epsilon - e^{-\epsilon})}{\pi} \left(\ln\left(\frac{i-z}{i+z}\right) + \ln\left(\frac{i+z}{i-z}\right)\right) + e^{-\epsilon} + e^\epsilon = e^\epsilon + e^{-\epsilon}.$$

Due to this cancelation, we may focus on

$$z^n \frac{Y_{22}^{(1)}(z)}{Y_{21}^{(1)}(z)} + (-z)^n \frac{Y_{22}^{(2)}(-z)}{Y_{21}^{(2)}(-z)}$$

where $z = e^{i\theta}$, $\theta \in (\pi/2, \pi/2 + \delta_1)$. In fact, since we know that $|\Phi_n^*| \sim 1$ uniformly on \mathbb{T} , we may multiply out the denominators and only examine

$$z^n Y_{22}^{(1)}(z) Y_{21}^{(2)}(-z) + (-z)^n Y_{22}^{(2)}(-z) Y_{21}^{(1)}(z).$$

We must take care in performing the final multiplication in the Riemann-Hilbert problem. We have previously seen that $Y_{21}(z) = \tilde{Y}_{21}(z) + O(\frac{1}{n}) \sim O(1)$ uniformly $z \in \mathbb{T}$. Therefore,

$$z^n Y_{22}^{(1)}(z) Y_{21}^{(2)}(-z) + (-z)^n Y_{22}^{(2)}(-z) Y_{21}^{(1)}(z) = \tag{57}$$

$$z^n \tilde{Y}_{22}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{22}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) + O\left(\max_{j=1,2} \frac{|\tilde{Y}_{12}^{(j)}(z)| + |\tilde{Y}_{22}^{(j)}(z)|}{n}\right).$$

Since the final term has a logarithmic singularity at $z = i$, we will handle it away from this point. In the range $z = e^{i\theta}, \pi/2 + e^{-\sqrt{n}} < \theta < \pi/2 + \delta_1$, we have (by (42) and estimates on ψ)

$$\max_{j=1,2} \frac{|\tilde{Y}_{12}^{(j)}(z)| + |\tilde{Y}_{12}^{(j)}(z)|}{n} = O\left(\frac{|\ln \zeta|}{n}\right) = O(\sqrt{n}/n) = O(n^{-1/2}).$$

So, it suffices to show

$$z^n \tilde{Y}_{22}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{22}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) = O(1). \quad (58)$$

By (42) and (43), we have

$$\begin{aligned} & z^n \tilde{Y}_{22}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{22}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) = \\ & \frac{1}{e^\epsilon \mathcal{D}(z) \zeta^{i\epsilon/\pi}} \left(\frac{z}{i}\right)^n c_2^{(1)} \psi_3^{(1)} \left(\frac{-\zeta^{i\epsilon/\pi}}{\mathcal{D}(-z)} \left(\psi_1^{(2)} - \left(\frac{-z}{-i}\right)^n c_2^{(2)} \psi_3^{(2)}\right)\right) \\ & + \frac{e^\epsilon \zeta^{i\epsilon/\pi}}{\mathcal{D}(-z)} \left(\frac{-z}{-i}\right)^n c_2^{(2)} \psi_3^{(2)} \left(\frac{-1}{\mathcal{D}(z) \zeta^{i\epsilon/\pi}} \left(\psi_1^{(1)} - \left(\frac{z}{i}\right)^n c_2^{(1)} \psi_3^{(1)}\right)\right). \end{aligned}$$

Notice it is not ambiguous to leave the arguments of the ψ functions unidentified, as $\zeta = n \ln \frac{z}{z_j}$ and $\frac{z}{i} = \frac{-z}{-i}$. Taking absolute values, we have

$$\begin{aligned} & \left| z^n \tilde{Y}_{22}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{22}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) \right| = \\ & |\mathcal{D}(z) \mathcal{D}(-z)|^{-1} \left| e^{-\epsilon} \left(c_2^{(1)} \psi_3^{(1)} \psi_1^{(2)} - \left(\frac{z}{i}\right)^n c_2^{(1)} c_2^{(2)} \psi_3^{(2)} \psi_3^{(1)} \right) + e^\epsilon \left(c_2^{(2)} \psi_1^{(1)} \psi_3^{(2)} - \left(\frac{z}{i}\right)^n c_2^{(1)} c_2^{(2)} \psi_3^{(2)} \psi_3^{(1)} \right) \right| \\ & \lesssim \left| e^{-\epsilon} \frac{\Gamma(1 + \frac{i\epsilon}{\pi})}{\Gamma(-\frac{i\epsilon}{\pi})} \psi\left(1 + \frac{i\epsilon}{\pi}, 1, e^{-i\pi\zeta}\right) \psi\left(\frac{i\epsilon}{\pi}, 1, \zeta\right) + e^\epsilon \frac{\Gamma(1 - \frac{i\epsilon}{\pi})}{\Gamma(\frac{i\epsilon}{\pi})} \psi\left(1 - \frac{i\epsilon}{\pi}, 1, e^{-i\pi\zeta}\right) \psi\left(-\frac{i\epsilon}{\pi}, 1, \zeta\right) \right. \\ & \quad \left. - (e^\epsilon + e^{-\epsilon}) \frac{\Gamma(1 + \frac{i\epsilon}{\pi})}{\Gamma(-\frac{i\epsilon}{\pi})} \frac{\Gamma(1 - \frac{i\epsilon}{\pi})}{\Gamma(\frac{i\epsilon}{\pi})} \left(\frac{z}{i}\right)^n \psi\left(1 + \frac{i\epsilon}{\pi}, 1, e^{-i\pi\zeta}\right) \psi\left(1 - \frac{i\epsilon}{\pi}, 1, e^{-i\pi\zeta}\right) \right|. \end{aligned}$$

By (32),(33),(34), these expressions are uniformly bounded in $\zeta, |\zeta| > 1, \epsilon < \epsilon_0, n > n_0(\epsilon)$. Thus, we only need to consider the case $\zeta : |\zeta| < 1$. On that interval, $(z/i)^n = e^\zeta = 1 + O(\zeta)$. We are concerned with the logarithmic singularities and the constant terms in the series expansions for ψ .

We isolate these terms and denote their sum c_0 . Using the notation $d(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ for the digamma function, we get

$$\begin{aligned} c_0 &= e^{-\epsilon} \frac{1}{\Gamma(-\frac{i\epsilon}{\pi}) \Gamma(\frac{i\epsilon}{\pi})} \left(\ln(e^{-i\pi\zeta}) + d\left(1 + \frac{i\epsilon}{\pi}\right) - 2d(1) \right) \left(\ln \zeta + d\left(\frac{i\epsilon}{\pi}\right) - 2d(1) \right) \\ & - (e^\epsilon + e^{-\epsilon}) \frac{1}{\Gamma(\frac{i\epsilon}{\pi}) \Gamma(-\frac{i\epsilon}{\pi})} \left(\ln(e^{-i\pi\zeta}) + d\left(1 + \frac{i\epsilon}{\pi}\right) - 2d(1) \right) \left(\ln(e^{-i\pi\zeta}) + d\left(1 - \frac{i\epsilon}{\pi}\right) - 2d(1) \right) \\ & + e^\epsilon \frac{1}{\Gamma(\frac{i\epsilon}{\pi}) \Gamma(-\frac{i\epsilon}{\pi})} \left(\ln(e^{-i\pi\zeta}) + d\left(1 - \frac{i\epsilon}{\pi}\right) - 2d(1) \right) \left(\ln \zeta + d\left(\frac{-i\epsilon}{\pi}\right) - 2d(1) \right). \end{aligned}$$

Performing these multiplications, writing $\ln(iu) = \ln u + i\pi/2$, $\ln(e^{-i\pi}iu) = \ln u - i\pi/2$, and pulling out the common factor yields

$$c_0 = \frac{\ln u}{\Gamma(\frac{i\epsilon}{\pi}) \Gamma(-\frac{i\epsilon}{\pi})} \left(e^{-\epsilon} \left(d\left(\frac{i\epsilon}{\pi}\right) - d\left(1 - \frac{i\epsilon}{\pi}\right) + i\pi \right) + e^\epsilon \left(d\left(\frac{-i\epsilon}{\pi}\right) + d\left(1 + \frac{i\epsilon}{\pi}\right) + i\pi \right) \right) + O(1),$$

where $\zeta = iu$. Using the reflection formula (35) gives

$$\begin{aligned} & \frac{\ln u}{\Gamma(\frac{i\epsilon}{\pi})\Gamma(-\frac{i\epsilon}{\pi})} (e^\epsilon(-\pi \cot(-i\epsilon)) + e^\epsilon i\pi + e^{-\epsilon}(-\pi \cot(i\epsilon)) + e^{-\epsilon} i\pi) \\ &= \frac{\pi \ln u}{\Gamma(\frac{i\epsilon}{\pi})\Gamma(-\frac{i\epsilon}{\pi})} (i(e^\epsilon + e^{-\epsilon}) - e^\epsilon \cot(-i\epsilon) - e^{-\epsilon} \cot(i\epsilon)) = 0, \end{aligned}$$

because $\cot z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$. Therefore,

$$z^n \tilde{Y}_{22}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{22}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) = O(1)$$

and

$$\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} = O(1)$$

uniformly in $z \in \mathbb{T}$ for n large enough. This finishes the proof of Lemma 2.5. \square

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