MULTIDIMENSIONAL L^2 CONJECTURE: A SURVEY

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To the 70-th birthday of N.K. Nikolskii

ABSTRACT. In this survey, we will give a short overview of the recent progress on the multidimensional L^2 conjecture. It can also serve as a quick introduction to the subject.

1. INTRODUCTION

The one-dimensional scattering theory for Schrödinger and Dirac operators is fairly well-understood by now with only a few very difficult problems left. This progress is mostly due to applying the tools of complex function theory and harmonic analysis. In multidimensional situation, very little is known.

The multidimensional L^2 conjecture was suggested by Barry Simon [28] and it is as follows.

Conjecture. Consider
$$H = -\Delta + V, \quad x \in \mathbb{R}^d$$
where V is real valued potential which satisfies
$$\int_{\mathbb{R}^d} \frac{V^2(x)}{1 + |x|^{d-1}} dx < \infty \qquad (1)$$
Is it true that $\sigma_{ac}(H)$ contains the positive half-line and it is of infinite multiplicity there (here and below the symbol $\sigma_{ac}(H)$ denotes the absolutely continuous spectrum of the operator H)?

One might have to require more local regularity from the potential to have H well-defined [2], e.g., assuming $V \in L^\infty(\mathbb{R}^d)$ is already good enough.

This conjecture was completely solved only for $d = 1$ [3]. Even for the case $|V(x)| \lesssim (1 + |x|)^{-\gamma}, \gamma = (1-)$ nothing is known. Below, we will discuss some cases in which the progress was made. We will also briefly explain the methods and suggest some open problems. The recent survey [20] contains somewhat different perspective on the subject.

2. CAYLEY TREE

The material in this section is taken from [4, 5, 29]. Assume that the Cayley tree $\mathbb{B}$ is rooted with the root (the origin) denoted by O, O has two neighbors and other vertices have one ascendant and two descendants (the actual number of

descendants is not important but it should be the same for all points $X \neq O$). The set of vertices of the tree is denoted by $\mathbb{V}(\mathbb{B})$. For an $f \in \ell^2(\mathbb{V}(\mathbb{B}))$, define the free Laplacian by

$$(H_0 f)_n = \sum_{\text{dist}(i,n)=1} f_i, \quad n \in \mathbb{V}(\mathbb{B})$$

One can show rather easily that the spectrum of H_0 is purely a.c. on $[-2\sqrt{2}, 2\sqrt{2}]$. Assume now that V is a bounded potential on $\mathbb{V}(\mathbb{B})$ so that

$$H = H_0 + V$$

is well-defined. Denote the spectral measure related to delta function at O by σ_O; the density of its absolutely continuous part is σ'_O. Take $w(\lambda) = (4\pi)^{-1}(8 - \lambda^2)^{1/2}$ and let $\rho_O(\lambda) = \sigma'_O(\lambda) w^{-1}(\lambda)$.

Consider also the probability space on the set of nonintersecting paths in $\mathbb{B}$ that go from the origin to infinity. This space is constructed by assigning the Bernoulli random variable to each vertex and the outcome of Bernoulli trial (0 or 1) then corresponds to whether the path (stemming from the origin) goes to the "left" or to the "right" descendant at the next step. Notice also that (discarding a set of Lebesgue measure zero) each path is in one-to-one correspondence with a point on the interval $[0, 1]$ by the binary decomposition of reals. In this way, the "infinity" for $\mathbb{B}$ can be identified with $[0, 1]$. For any $t \in [0, 1]$, we can then define the function ϕ as

$$\phi(t) = \sum_{n=1}^{\infty} V^2(x_n)$$

where the path $\{x_n\} \subset \mathbb{V}(\mathbb{B})$ corresponds to t. This function does not have to be finite at any point t but it is well-defined and is Lebesgue measurable. See [4] for

Theorem 2.1. *For any bounded V,*

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} w(\lambda) \log \rho_O(\lambda) d\lambda \geq \log \mathbb{E} \left\{ \exp \left[-\frac{1}{4} \sum_{n=1}^{\infty} V^2(x_n) \right] \right\}$$

$$= \log \int_0^1 \exp \left(-\frac{\phi(t)}{4} \right) dt$$

where the expectation is taken with respect to all paths $\{x_n\}$ and the probability space defined above. In particular, if the right hand side is finite, then $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$.

The proof of the theorem is based on the adjusted form of sum rules in the spirit of Killip-Simon [14]. Higher order sum rules are applied to different classes of potentials in Kupin [15].

Notice that ϕ is always nonnegative, therefore the right hand side is bounded away from $-\infty$ iff $V \in \ell^2$ with a positive probability. This is the true multi-dimensional L^2 condition. The simple application of Jensen's inequality then immediately implies that the estimate

$$\int \phi(t) dt = \sum_{n=0}^{\infty} 2^{-n} \sum_{\text{dist}(X,O)=n} V^2(X) < \infty$$

guarantees $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$. The last condition is precisely the analogue of (1) for the Cayley tree. Indeed, the factor 2^n is the "area" of the sphere of radius n in $\mathbb{B}$ and is exactly the counterpart of $|x|^{d-1}$ in the same formula.

3. Slowly decaying oscillating potentials

There are two different methods to handle this case.

1. Asymptotics of Green's function for the complex values of spectral parameter.

For simplicity, take $d = 3$ and assume that V is supported on the ball of radius ρ around the origin. Consider the resolvent $R_z = (H - z)^{-1}, z \in \mathbb{C}^+$ and denote its integral kernel by $G_z(x, y)$. The approach suggested in [6] requires the careful analysis of the asymptotical behavior of $G_z(x, y)$ when $z \in \mathbb{C}^+$, $y \in \mathbb{R}^3$ are fixed and $x \to \infty$ in some direction. To be more precise, we compare $G_z(x, y)$ to the unperturbed Green's function

$$G_z^0(x, y) = \frac{\exp(ik|x - y|)}{4\pi|x - y|}, \quad z = k^2$$

in the following way. Take any $f(x) \in L^2(\mathbb{R}^3)$ with a compact support and define $u = R_z f$. As V is compactly supported, we have

$$
\begin{aligned}
u(x, k) = \quad & \frac{\exp(ikr)}{r} \left(A(k, \theta) + \bar{o}(1) \right), \\
\frac{\partial u(x, k)}{\partial r} = \quad & ik \frac{\exp(ikr)}{r} \left(A(k, \theta) + \bar{o}(1) \right), \quad \text{(Sommerfeld's radiation conditions)} \\
& r = |x|, \, \theta = \frac{x}{|x|}, |x| \to \infty
\end{aligned}
\tag{2}
$$

Let us call A an amplitude. Clearly, its analysis boils down to computing the asymptotics for $G_z(x, y)$. The amplitude $A(k, \theta)$ has the following properties (see [6])

1. $A(k, \theta)$ is a vector-valued function analytic in $k \in \{\text{Im } k > 0, \text{Re } k > 0\}$.
2. The absorption principle holds, i.e. $A(k, \theta)$ has a continuous extension to the positive half-line.
3. For the boundary value of the resolvent, we have
$\text{Im}(R_{k^2}^+ f, f) = k \|A(k, \theta)\|_{L^2(\Sigma)}^2, k > 0$. Therefore,

$$\sigma_f'(E) = k\pi^{-1} \|A(k, \theta)\|_{L^2(\Sigma)}^2, E = k^2 \tag{3}$$

where $\sigma_f(E)$ is the spectral measure of f.

The last formula is the crucial one. The key observation made in [6] is that the function $\log \|A(k, \theta)\|_{L^2(\Sigma)}$ is subharmonic in $k \in \{\text{Im } k \geq 0, \text{Re } k > 0\}$. Thus, provided that some rough estimates (uniform in ρ) are available for $\|A(k, \theta)\|_{L^2(\Sigma)}$ away from real axis, one can use the mean-value formula to get

$$\int_I \log \sigma_f'(E) dE > C \tag{4}$$

for any interval $I \subset \mathbb{R}^+$. Then, as C is ρ–independent, one can extend this estimate to the class of potentials that are not necessarily compactly supported. This requires using the lower-semicontinuity of the entropy [14]. Estimate (4) yields $\sigma_f'(E) > 0$ for a.e. $E > 0$ and the statement on the a.c. spectrum easily follows.

The technical part is to obtain the estimates on the amplitude A. This can be done by the perturbation theory technique. The typical result [6] one can obtain this way is

Theorem 3.1. *Let $Q(x)$ be a $C^1(\mathbb{R}^3)$ vector-field in $\mathbb{R}^3$ and*

$$|Q(x)| < \frac{C}{1 + |x|^{0.5+\varepsilon}}, \ |div \ Q(x)| < \frac{C}{1 + |x|^{0.5+\varepsilon}}, \varepsilon > 0$$

Then, $H = -\Delta + div \ Q$ has an a.c. spectrum that fills $\mathbb{R}^+$.

The decay of potential here is nearly optimal but an additional oscillatory behavior is also needed. If one does not assume some sort of oscillation, then the whole method breaks down. We will say more about that later.

2. Method of Laptev-Naboko-Safronov.

This very elegant approach was suggested in [17] and was later developed in subsequent publications [20, 21, 22, 23, 24, 13, 25, 16]. We will again give only a sketch of the idea. Rewriting the operator in the spherical coordinates we have

$$H \sim -\frac{d^2}{dr^2} - \frac{B}{r^2} + V(r, \theta)$$

where B is Laplace-Beltrami on the unit sphere. Now, let us treat this as a one-dimensional operator with operator-valued potential

$$Q = -\frac{B}{r^2} + V(r, \theta)$$

Denote the projection to the first spherical harmonic by P_0. The idea of [17] is to write $P_0(H - z)^{-1}P_0$ as

$$-\frac{d^2}{dr^2} + Q(z)$$

with nonlocal potential $Q(z)$ and then apply the one-dimensional technique to this operator. For example, the following result can be obtained this way

Theorem 3.2. *Let $d \geq 3$ and $V(x)$ be such that*

1. *$\lim_{|x|\to\infty} V(x) = 0$*
2. *$V \in L^{d+1}(\mathbb{R}^{d+1})$*
3. *The Fourier transform of V is well-defined around the origin as L^2_{loc} function, i.e.*

$$\int_{|\xi|<\delta} |\hat{V}(\xi)|^2 d\xi < \infty$$

   for some $\delta > 0$.

Then, $\sigma_{ac}(H) = \mathbb{R}^+$.

Notice that the second condition on V is satisfied for, e.g., $|V(x)| < C(|x| + 1)^{-\gamma}, \gamma > d/(d+1)$. However, the third condition implies that V either decays fast or oscillates. The substantial problem with this method is that one needs good bounds on the discrete negative spectrum, e.g. Lieb-Thirring estimates. However, the needed estimates can be obtained only under rather strict assumptions on the decay of potential. In section 6, we will explain how this difficulty can be overcome.

4. Nontrivial WKB correction

The method of asymptotical analysis of Green's function explained in the previous section can also be used in two different situations: when V_θ is short-range and when V is sparse.

In [19], the following result was obtained.

Theorem 4.1. *Let $d = 3$ and*

$$|V(x)| + |x| \cdot |\nabla' V(x)| \lesssim (1 + |x|)^{-0.5-}$$

where $\nabla' V = \nabla V \cdot (x/|x|)^\perp$ is the angular part of the gradient of V. Then, $\sigma_{ac}(H) = \mathbb{R}^+$.

The method employed is essentially the same as the one used in [6] with one exception: the Green's function asymptotics contains the well-known WKB-type correction:

$$G_z(x, y) \sim \frac{1}{4\pi |x - y|} \exp\left(ik|x - y| + \frac{1}{2ik} \int_0^{|x|} V(\hat{x}s) ds \right) \tag{5}$$

as $|x| \to \infty$ and $\hat{x} = x/|x|$. This correction becomes a unimodular factor when $k \in \mathbb{R}$ and so the main arguments of [6] go through.

In the paper [7], quite a different situation was considered. Take the sequence R_n to be very sparse (e.g., $R_{n+1} > e^{2R_n}, \forall n$) and consider the potential V supported on the concentric three-dimensional shells Σ_n with radii R_n and width ~ 1.

Theorem 4.2. *Assume that $|V(x)| < v_n$ if $x \in \Sigma_n$ and $v_n \in \ell^2(\mathbb{N})$, then $\sigma_{ac}(H) = \mathbb{R}^+$.*

The method is again based on the calculation of the asymptotics of Green's function for the fixed complex k. This asymptotics involves new and nontrivial WKB factor which is not unimodular for real k but it is sufficiently regular to apply the same technique.

The WKB correction obtained so far in the literature was always a quite explicit multiplier. That, however, does not seem to be the case in general. First of all, in spite of many attempts, no results on the asymptotics of the Green's function was ever obtained for complex k as long as the only condition assumed of V is the slow decay: $|V(x)| \lesssim (|x| + 1)^{-1+}$. The possible explanation is the following. Consider $G_{k^2}(x, 0)$ for real k as a function of $\hat{x} = x/|x|$ by going to the spherical coordinates. As $|x| \to \infty$, the contribution to the L^2–norm of this function coming from the higher angular modes (i.e. the modes of order $|x|^\alpha, \alpha > 0.5$) is likely to be more and more pronounced. If one makes k complex, this phenomenon can hardly disappear but the waves corresponding to different angular frequencies have different Lyapunov exponents even in the free case so their contributions are all mixed up in the Fourier sum for $G_{k^2}(x, 0)$ making establishing any asymptotics nearly impossible. That questions the applicability of the method of [6] and perhaps the Green function analysis for real k is needed. The analysis for real k performed in [1, 18], however, was never sufficient to handle the optimal case $V \in L^2(\mathbb{R}^+)$ and so the prognosis for the resolution of the full L^2 conjecture is rather negative.

What is the WKB correction to Green's function for real k if there is any asymptotics at all? We do not know yet but there is one special case when the correction

to the asymptotics of the evolution group e^{itH} was computed. That was done in [8].

Assume that $d = 3$ and V satisfies the following conditions

Conditions A:

$$|V| \lesssim r^{-\gamma}, \left|\frac{\partial V}{\partial r}\right| \lesssim r^{-1-\gamma}, \left|\frac{\partial^2 V}{\partial r^2}\right| \lesssim r^{-1-2\gamma}, \quad V(x) \in C^2(\mathbb{R}^3), \quad r = |x|$$

and

$$1/2 < \gamma < 1 \tag{6}$$

The standard by now Mourre estimates immediately show that the spectrum is purely a.c. on the positive half-line. The question is what is the long-time asymptotics of e^{itH}? Well, the answer is not easy as we will see and it requires quite a bit of notation. Let, again, B be the Laplace-Beltrami operator on the unit sphere Σ. Consider the following evolution equation:

$$iky_\tau(\tau, \theta) = \frac{(By)(\tau, \theta)}{\tau^2} + V(\tau, \theta)y(\tau, \theta), \quad \tau > 0 \tag{7}$$

where $k \in \mathbb{R} \backslash \{0\}$, $V(\tau, \theta) = V(\tau \cdot \theta)$ is the potential written in spherical coordinates, and the function $y(\tau, \theta) \in L^2(\Sigma)$ for any $\tau > 0$. We introduce $U(k, \tau_0, \tau)f$, the solution of (7) satisfying an initial condition $U(k, \tau_0, \tau_0)f = f$ where $\tau, \tau_0 > 0$ and $f \in L^2(\Sigma)$. For any $f \in L^2(\Sigma)$, denote

$$W(k, \tau)f = U(k, 1, \tau)f \tag{8}$$

and consider the following operator

$$[\mathcal{E}(t)f](x) = (2it)^{-3/2} \exp\left(i|x|^2/(4t)\right) \cdot W(|x|/t, |x|) \left[\hat{f}(|x|/(2t)\theta)\right](\hat{x})$$

$f \in L^2(\mathbb{R}^3)$. In this formula, the last factor is defined by (8) where $k = |x|/t$, $\tau = |x|$, and the function on the sphere is given by the restriction of the Fourier transform of f to the sphere of radius $|x|/(2t)$.

Theorem 4.3. *Assume that V satisfies Conditions A. Then, for any $f \in L^2(\mathbb{R}^3)$, the following limits (modified wave operators) exist*

$$\mathcal{W}_\pm f = \lim_{t \to \pm \infty} \exp(iHt)\mathcal{E}(t)f$$

If V is short-range then W can be dropped in the definition of $\mathcal{E}$ and the statement of the theorem will still be correct. If V is long-range but the gradient is short-range, then one can show that W has the standard multiplicative WKB correction in large τ asymptotics similar to the one present in (5). In general, the Wf factor can not be simplified much and so the WKB correction happens to be given by a very complicated evolution equation.

One should expect that in the case when $|V(x)| < C(1 + |x|)^{-\gamma}, \gamma \in (0.5, 1)$ even more complicated evolution equation appears both in the spatial asymptotics of the Green's function and in the long-time asymptotics of e^{itH}. Meanwhile, proving this seems to be a monumental task as the statement like that even in one-dimensional case holds not for all $k \neq 0$ but rather for Lebesgue a.e. k.

5. Ito's stochastic equation and modified Harmonic measure

The resolution of L^2 conjecture for the Cayley tree suggests that may be condition (1) can be relaxed. Although the above discussion might have somewhat sobering effect on the reader, one can hope to at least try to exploit the idea of introducing the right space of paths. One step in this direction was made in [22] where the Laptev-Naboko-Safronov method was adjusted to the case when potential is small in the cone. Further progress was made in the paper [9]. Recall that for the Cayley tree one can very naturally introduce the probability space of paths escaping to infinity. Then, one knows that if the potential is small (ℓ^2 or is just zero) with positive probability, then the a.c. spectrum is present. What is an analog of this probability space in the Euclidean case?

Consider the Lipschitz vector field

$$p(x) = \left(\frac{I_\nu'(|x|)}{I_\nu(|x|)} - \nu|x|^{-1} \right) \cdot \frac{x}{|x|}, \quad \nu = (d-2)/2 \tag{9}$$

where I_ν denotes the modified Bessel function. Then, fix any point $x^0 \in \mathbb{R}^d$ and define the following stochastic process

$$dX_t = p(X_t)dt + dB_t, \quad X_0 = x^0 \tag{10}$$

with the drift given by p. The solution to this diffusion process exists and all trajectories are continuous and escape to infinity almost surely. The next result is from [9].

Theorem 5.1. *Assume that V is continuous nonnegative bounded function and*

$$\mathbb{E}_{x^0} \left[\exp \left(-\int\limits_0^\infty V(X_\tau)d\tau \right) \right] > 0$$

for some x_0. Then, $\mathbb{R}^+ \subseteq \sigma_{ac}(H)$.

The positivity of the expectation means that with positive probability we have $V(X_t) \in L^1(\mathbb{R}^+)$. The application of Jensen's inequality immediately yields that

$$\int_{\mathbb{R}} \frac{V(x)}{|x|^{d-1}+1} dx < \infty$$

implies the preservation of the a.c. spectrum. The method used to prove theorem 5.1 is again more or less rephrasing of the one from [4] but the language is probabilistic.

The question now is how to compute those probabilities. The reasonable simplification here is to assume that V is supported on some complicated set (say, a countable collection of balls) and then study when is the probability to hit this set smaller than one. This problem was addressed by introducing the suitable potential theory and by proving the estimates on the modified Harmonic measure. The interesting aspect of this analysis is in relating the geometric properties of support of V to the scattering properties of the medium. For more details, the reader is referred to [9].

6. Hyperbolic Schrödinger pencils

Consider the Schrödinger operator

$$H_\lambda = -\Delta + \lambda V$$

with the coupling constant λ and decaying V. The study of its resolvent

$$R_z = (H_\lambda - k^2)^{-1}, \quad z = k^2$$

is often complicated by the presence of the negative discrete spectrum for H_λ. For example, proving the sum rules requires the Lieb-Thirring bounds.

In [10], the following idea was suggested. Instead of inverting the operator $H - k^2$, let us try to invert $P(k) = -\Delta + k\mu V - k^2$, where μ is a fixed constant. In other words, we make the coupling constant momentum-dependent: $\lambda = k\mu$. The operator $P(k)$ is a hyperbolic pencil and $P(k)$ is invertible for all $k \in \mathbb{C}^+$. The function $P^{-1}(k)$ is analytic there and so one has no problem with poles. The study of the Green's function for $P^{-1}(k)$ is more straightforward and this Green function (call it $M_k(x, y, \mu)$) agrees with the Schrödinger Green's function as long as the potential V is compactly supported:

$$G_{k^2}(x, y, k\mu) = M_k(x, y, \mu)$$

The Fubini theorem then allows to translate results obtained for the Schrödinger pencil to the results for the original Schrödinger operator. This idea greatly expands the class of potentials that can be treated but the results hold only for generic coupling constant. Below we list three theorems that can be obtained this way. The first two are taken from [10] and they assume $d = 3$, the third one is from [11].

Theorem 6.1. *Assume*

$$V(x) = div\, Q(x)$$

where the smooth vector field $Q(x)$ satisfies

$$Q(x), |DQ(x)| \in L^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \frac{|Q(x)|^2}{|x|^2 + 1} dx < \infty$$

Then for a.e. λ, $\mathbb{R}^+ \subseteq \sigma_{ac}(H_\lambda)$.

This theorem, in contrast to theorem 3.1, does not assume the pointwise decay of V and the result obtained is sharp in terms of the decay of Q.

Theorem 6.2. *Assume $V(x)$ is bounded and*

$$\int\limits_1^\infty r|v(r)|^2 dr < \infty$$

for $v(r) = \sup_{|x|=r} |V(x)|$. Then for a.e. λ, $\sigma_{ac}(H_\lambda) = \mathbb{R}^+$.

This result is interesting in that it covers the case when the nontrivial WKB correction to the Green's function asymptotics can be present. However, there is no any need to establish it over specific direction as only the estimate on the angular average of Green function

$$\int_{|x|=r} |M_k(x, 0, \mu)|^2 d\sigma_x \sim e^{2\,\mathrm{Im}\,kr}$$

is used. In this case, the oscillation of $M_k(x, 0, \mu)$ in the angular variable is rather weak and this is what makes the analysis possible.

Yet another result can illustrate the power of this technique

Theorem 6.3. *Assume that V is continuous bounded function and*

$$\mathbb{E}_{x^0}\left[\exp\left(-\int\limits_0^\infty |V(X_\tau)|d\tau\right)\right] > 0$$

for some x_0. Then, $\mathbb{R}^+ \subseteq \sigma_{ac}(H_\lambda)$ for a.e. λ.

As one can see a rather unnatural requirement of V to be nonnegative present in theorem 5.1 is now removed.

In several recent publication, the idea of making the coupling constant momentum dependent was applied in combination with other interesting techniques, see e.g. [26, 27].

Remark. It is well-known that the analysis of the one-dimensional Schrödinger operator is more technically involved than the analysis of, say, Dirac operator or the Krein system. In fact, the one-dimensional differential equation for the Schrödinger pencil considered above happens to be identical to the Dirac operator (and automatically to the Krein system). So, it is not so unexpected that in multidimensional case this trick makes analysis simpler.

The following question is quite natural in view of the results listed above.

Question. Assume V decays in some way. How does the a.c. spectrum of H_λ as a set depend on the value of λ when $\lambda \neq 0$. That boils down to studying the dependence of

$$F_\lambda(z) = \operatorname{Im}\langle(H_\lambda - z)^{-1}f, f\rangle$$

on λ around the regular points $z \in \mathbb{R}^+$. For the one-dimensional case, the analysis in [1] reveals that for $V \in L^p, p < 2$ there is a λ-independent set of energies of the full Lebesgue measure which supports the a.c. spectrum of H_λ.

7. Possible directions

The L^2 conjecture the way it is stated does not say much about, say, Schrödinger dynamics so it is possible that there are some soft analysis arguments that can nail it. The good example of the soft analysis is the so-called sum rules for the Jacobi matrices [14]. More in-depth approach, in our opinion, is to try to control some quantities relevant for scattering: the Green's function, evolution group, etc. That, most likely, will require application of hard analysis methods. In this section, we will try to explain what kind of technical difficulties one stumbles upon when trying to address these questions.

Some special evolution equations seem to provide an adequate model for understanding of what is going on and one very important example of these equations is (for $d = 2$)

$$iu_t(t, \theta, k) = k\frac{\partial^2_{\theta\theta}u(t, \theta, k)}{(t+1)^2} + V(t, \theta)u(t, \theta, k), \quad u(0, \theta, k) = u_0(\theta) \qquad (11)$$

where $|V(t, \theta)| \lesssim (1+t)^{-\gamma}, \gamma \in (0.5, 1)$ and k and V are real. This equation appears (check (7)) in the WKB correction for the dynamics of e^{itH} and it is likely to be the right correction for the spatial asymptotics of the Green's function.

What can be said about the solution $u(t, \theta, k)$ as $t \to \infty$? The L^2-norm $\|u\|_{L^2(\mathbb{T})}$ is preserved in time but how about the growth of Sobolev norms? The conjecture

stated in [12] is that generically in k we should have

$$\|u(t,\theta,k)\|_{H^1(\mathbb{T})} \lesssim t, \quad t \to \infty$$

thus the transfer of the L^2 norm to higher modes happens in a controlled way and that prevents the resonance formation.

Another important quantity to study is how concentrated the function $u(t,\theta,k)$ can be in the θ variable. That can be controlled by quantities like

$$I_1 = \|u(t,\theta,k)\|_{L^p(\mathbb{T})}, p > 2;$$

or

$$I_2 = \int_{\mathbb{T}} \log|u(t,\theta,k)|d\theta$$

or

$$I_3(\delta) = \inf_{|\Omega|>\delta} \int_{\Omega} |u(t,\theta,k)|^2 d\theta$$

One might guess that for typical k one has: I_1 is bounded in t and/or $I_2 > -C$ and/or $I_3(\delta) > C\delta$ as long as $\delta > 0$.

What is the technical difficulty in the analysis of (11)? Notice first that by going on the Fourier side in θ one gets the infinite system of ODE's coupled to each other through $\hat{V}$. The differential operator will become the diagonal one and the gaps between the eigenvalues will decrease in t. This deterioration of the gaps is the key signature of the multidimensional case. It necessitates handling increasing number of frequencies at once and this is the hardest part of the analysis. One should notice that the evolution equation with only two frequencies interacting with each other, e.g.

$$iX_t = \begin{bmatrix} 0 & V(t) \\ V(t) & k \end{bmatrix} X, \quad X(0,k) = I$$

can be handled by Harmonic analysis methods developed in [1, 18]. Anyhow, the equation (11) is poorly understood and its analysis is very complicated. Some progress was made in [12] but there is clearly a long way to go.

Conclusion. We hope that this survey makes a good point that the wave propagation through the medium with slowly decaying potential is an interesting physical phenomenon with WKB correction given by evolution equation. The phenomenon of resonances appearing for some energies becomes far more complicated in multidimensional case and the oscillatory behavior of Green's function is just another manifestation of that.

Based on the literature published in the last five years, it appears that the group of people actively working on the problem is rather small so hopefully this review will attract more interest to this beautiful subject.

8. Acknowledgment

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