REMARK ON ITÔ'S DIFFUSION IN MULTIDIMENSIONAL SCATTERING WITH SIGN-INDEFINITE POTENTIALS.

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Abstract. This paper extends some results of [2] to the case of sign–indefinite potentials by applying methods developed in [3]. This enables us to prove the presence of a.c. spectrum for the generic coupling constant.

Introduction

In this paper, we consider the Schrödinger operator

$$H_\lambda = -\Delta + \lambda V \tag{0.1}$$

on Euclidean space $\mathbb{R}^3$ where V is real–valued potential and λ is a real parameter usually called a coupling constant. We will study the dependence of the absolutely continuous spectrum of H on the behavior of potential V by blending the methods of two papers [2] and [3]. This question attracted much attention recently which resulted in many publications (see, e.g. [2]-[6], [9]-[10], [14]-[15]) due to new very fruitful ideas from one–dimensional theory finding the way in the multidimensional scattering problems.

In [2], we introduced the stochastic differential equation that has random trajectories as its solutions. These trajectories are natural for describing the scattering properties of (0.1) provided that $\lambda V \geq 0$. The reason why this requirement was made is rooted in the method itself. Indeed, as $\lambda V \geq 0$, we have $\sigma(H_\lambda) \subseteq [0,\infty)$ for the spectrum of H_λ. Moreover, the Green's function $L(x, y, k) = (-\Delta + \lambda V - k^2)^{-1}(x, y, k)$ can be analytically continued in k to the whole upper half–plane $\mathbb{C}^+$ and methods of the complex function theory can be used then. If V is sign–indefinite, the negative spectrum might occur, which is hard to control, and this approach breaks down. The paper [3] (see also [15]) however develops new technique which allows to overcome this difficulty by complexifying the coupling constant and considering the hyperbolic pencil

$$P_\lambda(k) = -\Delta + k\lambda V - k^2$$

instead. Then, provided that the spatial asymptotics for the new Green's kernel $P_\lambda^{-1}(k)(x, y, k)$ is established, we can conclude that the a.c. spectrum of H_λ contains $[0,\infty)$ for a.e. λ. In the next section, we state the main result of [2] and explain how it can be generalized to the sign-indefinite case.

We are going to use the foll owing notation. Let $\omega_R(r)$ be infinitely smooth function on $\mathbb{R}^+$ such that $\omega_R(r) = 1$ for $r < R-1$, $\omega_R(r) = 0$ for $r > R+1$,

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and $0 \le \omega_R(r) \le 1$. The function

$$L^0(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \tag{0.2}$$

denotes the Green's function of the free 3d Schrödinger operator, i.e., the kernel of $R_0(k^2) = (-\Delta - k^2)^{-1}$ when $\operatorname{Im} k > 0$. The standard symbol B_t stands for the 3–dimensional Brownian motion and $\mathbb{S}^2$ denotes the two-dimensional unit sphere.

We consider only the three–dimensional case because the formula (0.2) for L^0 is easier then and because some of the results from [3] that we use are written up for $d = 3$. The method, however, is good for any $d > 1$. We will often suppress the dependence on λ unless we want to emphasize it.

1. Main result

We start with stating some results from [2]. Consider the Lipschitz vector field

$$p(x) = \left(\frac{I'_\nu(|x|)}{I_\nu(|x|)} - \nu|x|^{-1}\right) \cdot \frac{x}{|x|}, \quad \nu = 1/2,$$

where I_ν denotes the modified Bessel function [1, Sect. 9.6] and the formula for $I_{1/2}(r)$ is remarkably simple:

$$I_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sinh(r)$$

Then, fix any point $x^0 \in \mathbb{R}^3$ and consider the following stochastic process

$$dX_t = p(X_t)dt + dB_t, \quad X_0 = x^0 \tag{1.1}$$

with the drift given by p. The solution to this diffusion process exists and all trajectories are continuous and escape to infinity almost surely in the ballistic way: $|X_t| = t(1 + \bar{o}(1))$. The transition probability for X_t and its asymptotics can be easily computed (see [2], formula (1.9)).

One of the main results in [2] states (assume here that $\lambda = 1$)

Theorem 1.1 ([2]). *Let V be any continuous nonnegative function. Assume that $f \in L^2(\mathbb{R}^3)$ is nonnegative and has a compact support. Let σ_f be the spectral measure of f with respect to H_V and σ'_f be the density of its a.c. part. Then we have*

$$\exp\left[\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk\right] \ge C_f \int f(x^0) \mathbb{E}_{x^0}\left[\exp\left(-\frac{1}{2}\int_0^\infty V(X_\tau)d\tau\right)\right] dx^0 \tag{1.2}$$

where the constant $C_f > 0$ does not depend on V.

The natural corollary of this theorem is a statement that the a.c. spectrum of H contains $\mathbb{R}^+$ provided that the potential V is summable along the trajectory X_t with positive probability (which is the same as saying that there are "sufficiently many" paths over which the potential is summable).

We are going to prove the following

Theorem 1.2. *Assume that V is bounded and continuous on $\mathbb{R}^3$ and*

$$\int_0^\infty |V(X_t)|dt < \infty \tag{1.3}$$

with positive probability. Then the a.c. spectrum of H_λ contains $\mathbb{R}^+$ for a.e. λ.

Remark. The conditions of continuity and boundedness of potential are assumed for simplicity only and can probably be relaxed. The condition of λ being generic is perhaps also redundant but this method does not yield any result for a particular value of $\lambda \neq 0$. Under the conditions of the theorem, the a.c. spectrum can be larger than the positive half–line as can be easily seen upon taking $V = -1$ on half–space and $V = 0$ on the complement.

We need to start with some preliminary results. They will be mostly concerned with the study of the kernel of the operator $P_R^{-1}(k) = (-\Delta + k\lambda V_R + k^2)^{-1}$ where $V_R(x) = V(x) \cdot \omega_R(|x|)$. The existence of $P_R^{-1}(k)$ for $\operatorname{Im} k > 0$ as a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ was proved in [3]. Denote the kernel of $P_R^{-1}(x, y, k)$ by $K_R(x, y, k)$ and compare it to the free Green's kernel $L^0(x, y, k)$ in the following way. For $k = i$, we introduce the amplitude

$$b_R(\theta) = \lim_{r\to\infty} \frac{K_R(0, r\theta, i)}{L^0(0, r, i)}$$

where $\theta \in \mathbb{S}^2$. If $L_R(x, y, k)$ is the Green's function for $(-\Delta + |V_R| - k^2)^{-1}$, then similarly

$$a_R(\theta) = \lim_{r\to\infty} \frac{L_R(0, r\theta, i)}{L^0(0, r, i)}, \quad a(\theta) = \lim_{R\to\infty} a_R(\theta)$$

In [2], the following formulas were proved ($C_{1(2)}$ below are some absolute constants)

$$\int\limits_{\mathbb{S}^2} a(\theta)d\theta = C_1 \mathbb{E}_{X_0=0}\left[\exp\left(-\frac{1}{2}\int\limits_0^\infty |V(X_\tau)|d\tau\right)\right] \tag{1.4}$$

For $\theta \in \mathbb{S}^2$, let

$$dG_t = \theta dt + dB_t \tag{1.5}$$

Then

$$a(\theta) = C_2 \mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int\limits_0^\infty |V(G_\tau)|d\tau\right)\right] \tag{1.6}$$

The condition (1.3) yields

$$\int\limits_{\mathbb{S}^2} a(\theta)d\theta > 0$$

and thus $a(\theta) > 0$ for $\theta \in \Omega \subseteq \mathbb{S}^2$ and $|\Omega| > 0$. In particular, that means

$$\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V(G_t)|dt\right)\right] > 0 \tag{1.7}$$

for any $\theta \in \Omega$. The bound (1.7) is exactly what we are going to use in this paper.

The first step is to prove an analog of the formula (1.6) for the function $b_R(\theta)$. The lemma below holds for any λ so we take $\lambda = 1$. We need the following proposition first.

Proposition 1.1. *Let Σ_r and B_r denote the sphere and the closed ball of radius r both centered at the origin. If V is continuous and real–valued in B_r and $F(x) \in C(B_r)$, then $\phi(x)$, the solution to*

$$\frac{1}{2}\Delta\phi + \phi_{x_1} - \frac{i}{2}V\phi = -F, \quad \phi|_{\Sigma_r} = 0$$

admits the following representation

$$\phi(x) = \mathbb{E}_{G_0=x}\left[\int_0^T \exp\left(-\frac{i}{2}\int_0^t V(X(G_\tau))d\tau\right) F(G_t)dt\right] \tag{1.8}$$

where $G_t = t(1,0,0) + B_t, G_0 = x$ and T is the exit time.

Notice that the solution ϕ always exists as the boundary problem can be reduced to inverting the operator $-\Delta + 1 + iV$ with Dirichlet boundary condition. This invertibility is a simple corollary of the spectral theory for hyperbolic pencils and it was proved in [3] in the context of the operators on the whole space. The existence of the expectation in the right hand side of (1.8) is guaranteed because V is real–valued and all trajectories $\{G_t\}$ are continuous almost surely with the exit time distribution having a small tail.

Proof. (proposition 1.1) This proof is quite standard for negative potentials (see [7], p.145 and [11], lemma 7.3.2) but we present it here for the reader's convenience. Take any $x, |x| < r$ and consider

$$\Xi_t = \phi(G_t)\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right), \quad G_0 = x, \quad t < T$$

By Itô's calculus,

$$d\Xi_t = \exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right)d\phi(G_t) - \frac{i}{2}\phi(G_t)V(G_t)\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right)dt$$

where

$$d\phi(G_t) = \left(\phi_{x_1}(G_t) + \frac{\Delta}{2}\phi(G_t)\right)dt + \phi_{x_1}(G_t)dB_t$$

and we used

$$d\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right) = -\frac{i}{2}V(G_t)\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right)dt$$

Since $\Xi_0 = \phi(0)$, we have

$$\mathbb{E}_x(\Xi_T) = \phi(x) + \mathbb{E}_x\left[\int_0^T -F(G_t)\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right)dt\right]$$
$$+\mathbb{E}_x\left[\int_0^T \phi_{x_1}(G_t)\exp\left(-\frac{i}{2}\int_0^t V(G_\tau)d\tau\right)dB_t\right]$$

and the last term is equal to zero. As the left hand side is equal to zero as well due to the Dirichlet boundary conditions imposed, we have the statement of the proposition. $\square$

This proposition immediately implies the following lemma.

Lemma 1.1. *If G_t is defined by (1.5), then*

$$b_R(\theta) = C_2 \mathbb{E}_{G_0=0}\left[\exp\left(-\frac{i}{2}\int\limits_0^\infty V_R(G_\tau)d\tau\right)\right] \tag{1.9}$$

for any $R > 0$.

Proof. The proof repeats the proof of the formula (2.8) (theorem 2.1, [2]) word for word. □

Assume that $\theta \in \Omega$ so (1.7) holds. Consider truncations $V^{(\rho)}(x) = V(x) \cdot (1 - \omega_\rho(|x|))$.

Lemma 1.2. *If $\theta \in \Omega$, then*

$$\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V^{(\rho)}(G_t)|dt\right)\right] \to 1 \tag{1.10}$$

as $\rho \to \infty$.

Proof. Take any $0 < R_1 < R_2$ and introduce t_1, the random time of hitting the sphere $|x| = R_1$ for the first time. Then, denoting by $\widetilde{G}(t)$ the solution to (1.5) with initial condition G_{t_1}, we have the elementary inequality

(1.11)

$$\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^{t_1} |V_{R_1/2}(G_t)|dt\right)\mathbb{E}_{G_{t_1}}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V^{(R_2)}(\widetilde{G}_t)|dt\right)\right]\right]$$
$$\geq \mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V(G_t)|dt\right)\right]$$

The trajectory G_t is a linear drift plus Browning motion thus for fixed R_1 we have decoupling

$$\lim_{R_2\to\infty}\left(\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^{t_1} |V_{R_1/2}(G_t)|dt\right) \times\right.\right. \tag{1.12}$$
$$\left.\left.\mathbb{E}_{G_{t_1}}\left[\exp\left(-\frac{1}{2}\int\limits_0^\infty |V^{(R_2)}(\widetilde{G}_t)|dt\right)\right]\right]\right) = \gamma \mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int\limits_0^{t_1} |V_{R_1/2}(G_t)|dt\right)\right]$$

with

$$\gamma = \lim_{R_2\to\infty}\left(\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V^{(R_2)}(G_t)|dt\right)\right]\right)$$

On the other hand,

$$\mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^{t_1} |V_{R_1/2}(G_t)|dt\right)\right] \to \mathbb{E}_{G_0=0}\left[\exp\left(-\frac{1}{2}\int_0^\infty |V(G_t)|dt\right)\right],$$
$$R_1 \to \infty$$

which along with (1.11) and (1.12) implies $\gamma = 1$. □

Remark. Let $a^{(\rho)}$ denote an amplitude for the potential $V^{(\rho)}$. The lemma then says that $a^{(\rho)}(\theta) \to 1$ as $\rho \to \infty$ for any $\theta \in \Omega$. Notice also that the lemma is wrong in general if the trajectory G_t is replaced by X_t as can be easily seen by letting $V = 1$ on the half-space and $V = 0$ on the complement.

Now we are ready for the proof of theorem 1.2.

Proof. (theorem 1.2)

Notice first that the standard trace-class perturbation argument [12] implies that $\sigma_{ac}(-\Delta+V_\lambda)=\sigma_{ac}(-\Delta+V_\lambda^{(\rho)})$ for any ρ. Fix some large $c>0$ and take $\lambda\in[-c,c]$. Then, by lemma 1.2, we can make ρ large enough so that for any $\theta\in\Omega_1\subseteq\Omega$, we have

$$\mathbb{E}\left[\exp\left(-\frac{c}{2}\int_0^\infty |V^{(\rho)}(G_t)|dt\right)\right]>0.99$$

Then, due to (1.9),

$$|b_R^{(\rho)}(\theta)|>1/2 \tag{1.13}$$

for any $\theta\in\Omega_1$ and any $R>\rho$. Now, we need to recall several results from [3] and repeat a couple of arguments from this paper. Let $f=\chi_{|x|<1}$. Denote the spectral measure of f with respect to $-\Delta+\lambda V_R^{(\rho)}$ by $\sigma_{\rho,R}(E,\lambda)$. For $k\in\mathbb{C}^+$, consider

$$J_{\rho,R}(k,\theta,\lambda)=\lim_{r\to\infty}\frac{\left((-\Delta+\lambda kV_R^{(\rho)}+k^2)^{-1}f\right)(r\theta)}{r^{-1}e^{ikr}}$$

Then, the formula (38) from [3] says

$$\sigma'_{\rho,R}(k^2,k\lambda)=k\pi^{-1}\|J_{\rho,R}(k,\theta,\lambda)\|^2_{L^2(\mathbb{S}^2)} \tag{1.14}$$

where $k\neq 0$ is real and in the right hand side the limiting value of $B_{\rho,R}$ as $\operatorname{Im}k\to+0$ is taken. This function $J_{\rho,R}(k,\theta,\lambda)$ is continuous on $\overline{\mathbb{C}^+}\backslash\{0\}$ as seen from the absorption principle ([13], chapter 13, section 8 or [3]). Around zero, we have an estimate

$$|J_{\rho,R}(k,\theta,\lambda)|<C(\rho,R)|k|^{-1} \tag{1.15}$$

that can be deduced from the representation

$$P^{-1}(k)f=R_0(k^2)f-k\lambda R_0(k^2)V_R^{(\rho)}P^{-1}(k)f \quad k\in\mathbb{C}^+$$

and an estimate

$$\|P^{-1}(k)\|\le(\operatorname{Im}k)^{-2}$$

(see (37), [3]).

For large $|k|$, we have the following uniform estimate

$$\int_{\mathbb{S}^2}|J_{\rho,R}(k,\theta,\lambda)|^2d\theta<C\frac{1+|k|\operatorname{Im}k}{[\operatorname{Im}k]^4}\|f(x)\|_2\|f(x)e^{2\operatorname{Im}k|x|}\|_2 \tag{1.16}$$

(take $r\to\infty$ in (48), [3]).

Now, consider the function

$$g(k)=\ln\|ke^{2ik}J_{\rho,R}(k,\theta,\lambda)\|_{L^2(\mathbb{S}^2)}$$

This function is subharmonic in $\mathbb{C}^+$ and the estimates (1.15) and (1.16) enable us to apply the mean–value inequality with the reference point $k=i$.

This, along with the identity (1.14), gives

$$\int_{\mathbb{R}}\frac{\ln\sigma'_{\rho,R}(k^2,k\lambda)}{k^2+1}dk>C_1+C_2\ln\int_{\mathbb{S}^2}|J_{\rho,R}(i,\theta,\lambda)|^2d\theta,\quad C_2>0$$

Now, notice that the choice of f guarantees that

$$|J_{\rho,R}(i,\theta,\lambda)|\sim|b_R^{(\rho)}(i,\theta,\lambda)|$$

and (1.13) implies that

$$C_1 + C_2 \ln \int_{\mathbb{S}^2} |J_{\rho,R}(i,\theta,\lambda)|^2 d\theta > C_3$$

uniformly in R and $\lambda \in [-c,c]$. Thus, we have an estimate

$$\int_{\mathbb{R}} \frac{\ln \sigma'_{\rho,R}(k^2, k\lambda)}{k^2+1} dk > -C$$

uniformly in $R > \rho$ and $\lambda \in [-c,c]$ with any fixed c. Now, taking any interval $(a,b) \subset (0,\infty)$, we have

$$\int_a^b dE \int_{-c}^{c} \ln \sigma'_{\rho,R}(E,\lambda) d\lambda > -C$$

uniformly in R so taking $R \to \infty$ and using the lower semicontinuity of the entropy (see [8] and [3], p.21 and Lemma 3.4), we get

$$\int_a^b dE \int_{-c}^{c} \ln \sigma'_{\rho}(E,\lambda) d\lambda > -C$$

The Fubini-Tonelli theorem now gives

$$\int_a^b \ln \sigma'_{\rho}(E,\lambda) dE > -\infty$$

for a.e. λ so $[a,b] \subseteq \sigma_{ac}(H_\lambda^{(\rho)})$ for a.e. λ. As was mentioned already, the a.c. spectrum is stable under changing the potential on any compact set. Thus, $[a,b] \subseteq \sigma_{ac}(H_\lambda)$ for a.e. λ. Since $[a,b]$ was taking arbitrarily, we have the statement of the theorem. □

Remark. The inequality (1.3) is not always so easy to check. In the paper [2], however, we gave some conditions for (1.3) to hold true. One of them is

$$\int_{\mathbb{R}^3} \frac{|V(x)|}{|x|^2+1} dx < \infty$$

which is a weighted L^1 condition (see corollary 2.1 in [2]). In [2] section 4, we also considered the case when the nonnegative potential V is supported on some set E (a good example to think about is a countable collection of balls) of any geometric complexity. The special modified capacity and the harmonic measure were introduced and studied which allowed the effective estimation of probabilities in the natural geometric terms (e.g., theorem 4.1). The same results, e.g., the estimates in terms of the anisotropic Hausdorff content and the size of the spherical projection, are true in the current setting when the potential is not assumed to be positive. However, we prove them only for generic λ.

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