SPATIAL ASYMPTOTICS OF GREEN'S FUNCTION AND APPLICATIONS

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Abstract. We study the spatial asymptotics of Green's function for the 1d Schrödinger operator with operator-valued decaying potential. The bounds on the entropy of the spectral measures are obtained. They are used to establish the presence of a.c. spectrum.

1. Introduction and the main result

In this note, we revisit the spectral theory of Schrödinger operators with long-range potentials. In dimension one, the quest for the minimal assumptions on the decay of potential that guarantee the preservation of absolutely continuous spectrum resulted in the theorem (Deift-Killip [1], see also [13]), which says:

If $V \in L^2(\mathbb{R}^+)$, then $\sigma_{\rm ac}(-\partial^2_{rr}+V)=[0,\infty)$ where $\sigma_{\rm ac}$ denotes a.c. spectrum of the operator with Dirichlet boundary condition at zero.

In the case of the Dirac equation, an analogous result was obtained by M. Krein already in 1955 (see [15] and [5]). L^2–condition is sharp: it is known [14] that $V \in L^p(\mathbb{R}^+), p>2$ can lead to an empty a.c. spectrum. In higher dimension, one again is interested in finding the minimal assumptions on the decay of V in $-\Delta+V$, $x\in\mathbb{R}^d$, $d\geqslant 2$ that guarantee "scattering" which can be understood either in the sense of preservation of a.c. spectrum or as existence of wave operators in Schrödinger dynamics. Some results were obtained for decaying potentials that oscillate (see [2, 17] for their surveys). However, if the oscillation condition is dropped and no additional smoothness (see, e.g., [16, 18] for various classes of potentials) is assumed then the identity $\sigma_{\rm ac}(-\Delta+V)=[0,\infty)$ is not known even for V obeying fairly strong constraints, such as $|V(x)|\leqslant C(1+|x|)^{-1+\epsilon}, 0<\epsilon\ll 1$. Notice that the last assumption is only slightly weaker than the short-range condition of the classical scattering theory [11]. In this paper, we make progress on a related problem.

Consider the Hilbert space $\mathcal{H}:=\oplus_{n=1}^\infty L^2(\mathbb{R}^+)$ with the inner product defined by

$$\langle F,G\rangle_{\mathcal{H}}=\int_0^\infty \langle F,G\rangle dr=\sum_{n=1}^\infty\int_0^\infty f_n\overline{g_n}dr\,,$$

where $F=(f_1,f_2,\ldots),G=(g_1,g_2,\ldots)$. We define the 1-d Schrödinger operators

$$H=-\partial^2_{rr}+V,\quad H^{(0)}=-\partial^2_{rr},\quad x\geqslant 0 \tag{1.1}$$

with Dirichlet boundary condition at the origin and operator-valued potential V. It satisfies $V(r)=V^*(r)$ for a.e. $r>0$ and $\|V\|\in L^\infty[0,\infty)$. By the general theory of symmetric operators, H defines the self-adjoint operator with the domain $\mathcal{D}(H)=\mathcal{D}(H^{(0)})=\oplus_{n=1}^\infty\mathcal{H}^2_0(\mathbb{R}^+)$, where $\mathcal{H}^2_0(\mathbb{R}^+):=\{f: f,f''\in L^2(\mathbb{R}^+), f(0)=0\}$ is the standard $\mathcal{H}^2(\mathbb{R}^+)$ Sobolev space of functions vanishing at the origin. Denote the Green's function of H by $G(r,\rho,z)$, i.e.,

$$R_zF=(H-z)^{-1}F=\int_{\mathbb{R}^+}G(r,\rho,z)F(\rho)d\rho\,,\ F\in\mathcal{H}\,.$$

We let $z\in\mathbb{C}^+$ and $k=\sqrt{z}\in\{k\in\mathbb{C}^+,\operatorname{Im}k>0,\operatorname{Re}k>0\}$. The Green's function of unperturbed operator will be called $G^{(0)}$. Notice that

$$G^{(0)}(r,\rho,k^2)=\frac{i}{2k}\Big(e^{ik|r-\rho|}-e^{ik(r+\rho)}\Big)\,. \tag{1.2}$$

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Let $u := R_{k^2}F$, $\psi := e^{-ikr}u$. We have $-u'' + Vu = k^2u + F$, $u(0,k) = 0$ and

$$-\psi'' - 2ik\psi' + V\psi = Fe^{-ikr}\,. \tag{1.3}$$

In this note, we develop the perturbative theory which partially controls the spatial asymptotics of u when F has compact support, $r \to +\infty$ and $z \in \mathbb{C}^+$ is taken close to $\mathbb{R}^+$. Our analysis allows the direct study of $G(r,\rho,z)$ when ρ is fixed and $r \to \infty$ but u has a better regularity and we will work with it instead. The following theorem showcases the typical application of our analysis to the study of spectral type.

Theorem 1.1. *Suppose $\gamma > \frac{2}{3}, \lambda > 0$, and $\|V\| \leqslant \lambda(1+r)^{-\gamma}$. Then, $\mathbb{R}^+ \subseteq \sigma_{\rm ac}(H)$.*

Later in the text, we can assume that γ is fixed in the range $\gamma \in (\frac{2}{3}, 1)$. Many constants the reader encounters in this text depend on γ and λ but we might not explicitly mention that.

Remark. The proof of the theorem employs elementary properties of subharmonic functions and a few apriori integral estimates obtained directly from the equation itself. We avoid ODE asymptotical methods so this technique can potentially be applied to study elliptic partial differential equations and difference operators on graphs.

The connection between σ_F, the spectral measure of $F \in \mathcal{H}$, and the asymptotics of u at infinity is revealed in the following lemma.

Lemma 1.1. *Suppose $T > 1$, $\operatorname{supp} V \subset [0,T]$, $F \in \mathcal{H}$, and $\operatorname{supp} F \subset [0,1]$. Then, σ_F is absolutely continuous on $\mathbb{R}^+$ and*

$$\sigma_F'(k^2) = k\pi^{-1}\|\psi(\infty,k)\|^2 \tag{1.4}$$

for $k \in \mathbb{R}^+$.

Proof. Under the assumption of the lemma, the so-called absorption principle holds (see, e.g., [6–8] for the Weyl-Titchmarsh theory of operator-valued Schrödinger operator). In particular, for every interval $I \subset (0,\infty)$ and every positive r, the function $u(r,k) = (R_{k^2}F)(r)$ has continuous extension in k from $R_{I,1} := I \times (0,1)$ to the interval I and this u satisfies $-u'' + Vu = k^2u + F$, $u(0,k) = 0$ for $k \in \overline{R_{I,1}}$. Thus, $\psi(r,k) = e^{-ikr}u(r,k)$ is defined as well for $k \in I$ and $\psi(r,k) = \psi(\infty,k)$ if $r > T$. That explains why the right-hand side in (1.4) is well-defined. The absorption principle also implies that σ_F is purely a.c. on $\mathbb{R}^+$. Next, we take $k \in R_{I,1}$ and write $-u'' + Vu = k^2u + F$. Take inner product with u and integrate over $[0,T]$. Subtracting the resulting identity from its conjugate gives us

$$\langle u'(T,k), u(T,k)\rangle - \langle u(T,k), u'(T,k)\rangle = (\bar{k}^2 - k^2)\int_0^T \|u\|^2 d\rho + \langle R_{k^2}F, F\rangle - \langle F, R_{k^2}F\rangle\,.$$

Due to absorption principle, we can take $\operatorname{Im} k \to 0$ in the last formula. This gives (1.4) after we take into account that $u(r,k) = e^{ikr}\psi(\infty,k)$ for $r > T$.

□

Remark. One of the key ideas in the proof of Theorem 1.1 is based on the following observation. Taking the logarithm of the both sides in (1.4) gives $\log \sigma_F'(k^2) = \log(k\pi^{-1}) + 2\log\|\psi(\infty,k)\|$. The function $\log\|\psi(\infty,k)\|$ is subharmonic in $R_{I,1} = I \times (0,1)$ for every closed interval $I \subset (0,\infty)$. Thus, rough bounds for $\log\|\psi(\infty,k)\|$ in $R_{I,1}$ can provide the lower bounds for the entropy $\int_{I'} \log \sigma_F'(k^2)dk, I' \subset I$ by application of mean-value inequality for subharmonic functions. The uniform control over the logarithmic integral implies the a.c. spectral type by the standard argument. A serious obstacle we will face is that the good control of $\|\psi(\infty,k)\|$ is only possible when $\operatorname{Im} k$ is very small. The development of strategy that overcomes this difficulty was the main motivation to write this note.

Some previous results. In [20], the reader can find an overview of one-dimensional results related to the topic. The survey papers [2,17] discuss the higher-dimensional case. See also [3,18,19] for more recent advances. The one-dimensional Schrödinger with operator-valued potential was extensively studied in the past and a thorough account of the literature can be found in [6–8]. The a.c. spectrum of operator-valued Schrödinger with decaying potential was studied in the context of hyperbolic pencils in [4]. In particular, it was established that $\mathbb{R}^+ \subseteq \sigma_{\rm ac}(-\partial^2_{rr} + tV)$ for a.e. $t \in \mathbb{R}$, provided that the operator-valued potential V satisfies $\|V\| \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$.

Motivation. To relate (1.1) to multidimensional problems, consider the three-dimensional Schrödinger operator $-\Delta + V, x \in \mathbb{R}^3$ which allows the representation

$$-\partial^2_{rr} - \frac{B}{r^2} + V(r,\theta) \tag{1.5}$$

in the spherical coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^2$. Here, B stands for Laplace-Beltrami operator on $\mathbb{S}^2$ and the Dirichlet boundary condition is assumed at the origin. If the higher spherical modes can be neglected, one considers

$$H = -\partial^2_{rr} - \frac{P_{\leqslant r^\kappa} B}{r^2} + V(r,\theta) \tag{1.6}$$

instead of (1.5), where $P_{\leqslant r^\kappa}$ is an orthogonal projection to the first $[r^\kappa]$ spherical harmonics. Assuming $|V(x)| \leqslant C(1+|x|)^{-\gamma}$ with $\gamma > \frac{2}{3}$ and choosing κ in a suitable way, we reduce (1.6) to the form (1.1).

Structure of the paper. The second section contains some apriori estimates for the solutions to equation (1.3). In the third section, we give the proof of Theorem 1.1. Some useful estimates on subharmonic functions are collected in Appendix 1. The second Appendix contains general bounds on Green's function.

Notation

- If I is a closed interval on $\mathbb{R}$, c_I denotes its center and $|I|$ denotes its length. I_r stands for the interval centered at zero with radius r. $\mathbb{R}^+ = (0,\infty)$.
- If ψ is a vector in Hilbert space $\ell^2(\mathbb{N})$, then $\|\psi\|$ denotes its norm. If V is a bounded linear operator acting in $\ell^2(\mathbb{N})$, then $\|V\|$ denotes its operator norm.
- If I is an closed interval in $\mathbb{R}^+$ and $\delta > 0$, then $R_{I,\delta} := I \times (0,\delta)$.
- If $\varphi, \psi \in \ell^2(\mathbb{N})$, then $\langle \varphi, \psi \rangle$ refers to the inner product in $\ell^2(\mathbb{N})$.
- For $a > 0$, we define $\log_+ a = \max\{0, \log a\}$, $\log_- a = \min\{0, \log a\}$.
- The symbol C_α will indicate a positive constant whose dependence on a parameter α we want to emphasize. The actual value of this constant can change from one formula to another.
- For two non-negative functions $f_{1(2)}$, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that $f_1 \leqslant C f_2$ for all values of the arguments of $f_{1(2)}$. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. If the constant C depends on parameter α, we might write $f_1 \lesssim_\alpha f_2$.
- For the set $\Delta \subset \mathbb{R}$, we denote $\Delta^2 = \{E^2 : E \in \Delta\}$.

2. Two simple estimates obtained from the equation

In this section, we consider the case when $\operatorname{supp} V \subset [0,T]$ and $\|V(r)\| < \lambda(r+1)^{-\gamma}, \gamma \in (\frac{2}{3}, 1)$. In later discussion, we will be taking $T = 2^n, n \geqslant n_0 \gg 1$. Let, e.g., F be such that

$$F = (f, 0, \ldots), \quad \|f\|_{L^2(\mathbb{R}^+)} = 1, \quad \operatorname{supp} f \subset [0,1], \quad f \not\equiv 0\,. \tag{2.1}$$

Let σ_F be the spectral measure of F, i.e.,

$$\langle R_z F, F \rangle_{\mathcal{H}} = \int \frac{d\sigma_F(E)}{E - z}\,,\ z \in \mathbb{C}\backslash\mathbb{R}\,.$$

Recall that σ_F is a probability measure and that $u = R_z F$. Rewrite equation (1.3) for ψ as

$$\psi' = i\frac{\psi''}{2k} - i\frac{V\psi}{2k}\,, \quad r > 1\,. \tag{2.2}$$

Lemma 2.1. *If I is any closed interval in $\mathbb{R}^+$, $\alpha \in (0,1)$ and $k \in R_{I,T^{-\alpha}}$, then*

$$\sup_{r>0} \|\psi(r,k)\| \leqslant C_{I,\alpha} \exp\left(2(\operatorname{Im} k)^{-(1-\alpha)/\alpha}\right)\,.$$

Proof. Since $V(r) = 0$ for $r > T$, $\psi(r,k) = \psi(T,k)$ if $r > T$ and we can assume that $r \leqslant T$. Because $\|u\|_{L^2[0,\infty)} \leqslant C_I (\operatorname{Im} k)^{-1}$ we have $\|u''\|_{L^2[0,\infty)} \leqslant C_I(\operatorname{Im} k)^{-1}$ from equation $-u'' + Vu = k^2 u + F$. Then, $\|u\|_{L^\infty[0,\infty)} \leqslant C_I(\operatorname{Im} k)^{-1}$ as follows from the standard Sobolev's embedding. Since $\|\psi(r,k)\| = e^{(\operatorname{Im} k)r}\|u(r,k)\|$, this gives us the statement of the lemma because

$$(\operatorname{Im} k) r \leqslant (\operatorname{Im} k) T \leqslant (\operatorname{Im} k)^{-(1-\alpha)/\alpha}$$

and

$$(\operatorname{Im} k)^{-1}\exp((\operatorname{Im} k)^{-(1-\alpha)/\alpha}) \leqslant C_{\alpha,I}\exp(2(\operatorname{Im} k)^{-(1-\alpha)/\alpha})$$

□

Remark. Notice that this lemma only requires that $\|V\| \in L^\infty(\mathbb{R}^+)$ and $\operatorname{supp} V \subset [0,T]$.

Next, we will study $\psi(r,k)$ when $r \in [T/2,T]$. In particular, we will be interested in how $\|\psi(r,k)\|$ deviates from $\|\psi(T/2,k)\|$ when $r > T/2$, $k \in R_{I,1}$, and $\operatorname{Im} k$ is small. Our basic tool is the following integral identity.

Lemma 2.2. *Let* $1 < a < b$ *and* $\operatorname{Re} k > 0, \operatorname{Im} k > 0$, *then*

$$\|\psi(b,k)\|^2 + \frac{\operatorname{Im} k}{|k|^2}\int_a^b \|\psi'\|^2 d\rho = \|\psi(a,k)\|^2 + Q_1 - Q_2 - \frac{\operatorname{Im} k}{|k|^2}\int_a^b \langle V\psi,\psi\rangle d\rho \tag{2.3}$$

where

$$Q_1 := \frac{i}{2k}\langle \psi'(b,k),\psi(b,k)\rangle - \frac{i}{2\bar{k}}\langle \psi(b,k),\psi'(b,k)\rangle$$

and

$$Q_2 := \frac{i}{2k}\langle \psi'(a,k),\psi(a,k)\rangle - \frac{i}{2\bar{k}}\langle \psi(a,k),\psi'(a,k)\rangle\,.$$

Proof. Take inner product of both sides in (2.2) with ψ and integrate from a to b. Then, take the real part of the resulting identity. We get

$$\|\psi(b,k)\|^2 = \|\psi(a,k)\|^2 + \frac{i}{2k}\int_a^b \langle \psi'',\psi\rangle d\rho - \frac{i}{2\bar{k}}\int_a^b \langle \psi,\psi''\rangle d\rho - \frac{\operatorname{Im} k}{|k|^2}\int_a^b \langle V\psi,\psi\rangle d\rho\,.$$

Integration by parts gives the statement of the lemma. □

Remark. In (2.3), an additional condition $V \geqslant 0$ immediately provides apriori estimate on $\int_1^\infty \|\psi'\|^2 d\rho$ with essentially no assumptions on the decay of V.

The following lemma is straightforward.

Lemma 2.3. *Let* Y *and* A *be two* $\ell^2(\mathbb{N})$*-valued functions defined on* $[a,\infty)$ *that satisfy* $\|Y\|, \|Y'\|, \|A\| \in L^2[a,\infty)$ *and*

$$Y = \frac{i}{2k}Y' + A, \quad \operatorname{Im} k > 0\,.$$

Then,

$$\|Y\|_{L^\infty[a,\infty)} \lesssim \frac{|k|\|A\|_{L^2[a,\infty)}}{\sqrt{\operatorname{Im} k}}, \quad \|Y\|_{L^2[a,\infty)} \lesssim \frac{|k|\|A\|_{L^2[a,\infty)}}{\operatorname{Im} k}. \tag{2.4}$$

Proof. We have $Y' = -2ikY + 2ikA$. If Ψ is defined by $\Psi := e^{2ikr}Y$, then $\Psi = -2ik\int_r^\infty A(s)e^{2iks}ds$. In the end, one has

$$Y = -2ike^{-2ikr}\int_r^\infty A(s)e^{2kis}ds\,.$$

Applying the convolution bounds, we get our lemma. □

If $T > 1$, we arrange for two positive numbers $\mathcal{L}_T$ and ℓ_T such that $\ell_T < \mathcal{L}_T$, $\ell_T := T^{1-2\gamma+2\delta_1}$ and $\mathcal{L}_T := T^{\gamma-1-\delta_1}$, where δ_1 is a positive parameter (e.g., take $\delta_1 = \frac{\gamma}{2} - \frac{1}{3}$). Its choice is possible since $\gamma \in (\frac{2}{3},1)$. Given any closed interval $I \subset \mathbb{R}^+$, define the set

$$PC_{I,T} := R_{I,1} \cap \{k : \ell_T \leqslant \operatorname{Im} k \leqslant \mathcal{L}_T\}\,.$$

We will refer to $PC_{I,T}$ as the *zone of perfect control.* The reader will see that this name is justified from the next two results.

Lemma 2.4. *For* $k \in PC_{I,T/2}$, *we have*

$$\|\psi(T,k)\|^2 = \|\psi(T/2,k)\|^2(1+\epsilon_T), \quad \epsilon_T \leqslant C_I T^{-\delta_1} \tag{2.5}$$

where $\delta_1 > 0$.

Proof. We introduce $M := \sup_{r>T/2} \|\psi(r,k)\|$. Let $k \in R_{I,1}$. Applying Lemma 2.3 to (2.2) on the interval $[T/2,\infty)$, one has

$$\|\psi'\|_{L^\infty[T/2,\infty)} \leqslant C_I M \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}}\,.$$

Hence,

$$\sup_{a,b>T/2} \|Q_{1(2)}\| \leqslant C_I M^2 \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}}\,.$$

By the same Lemma 2.3,

$$\|\psi'\|_{L^2[T/2,\infty)} \leqslant C_I M \frac{T^{0.5-\gamma}}{\operatorname{Im} k}\,.$$

Taking supremum in $b \geqslant T/2$ in (2.3) and letting $a = T/2$, we get

$$|M^2 - \|\psi(T/2,k)\|^2| \leqslant C_I \Big((\operatorname{Im} k) T^{1-\gamma} + \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}} + \frac{T^{1-2\gamma}}{\operatorname{Im} k} \Big) M^2\,.$$

Thus, one has

$$M^2 = \|\psi(T/2,k)\|^2(1+\epsilon_T), \quad \epsilon_T \leqslant C_I \Big((\operatorname{Im} k) T^{1-\gamma} + \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}} + \frac{T^{1-2\gamma}}{\operatorname{Im} k} \Big) \leqslant C_I T^{-\delta_1} \tag{2.6}$$

for given k. Now, we can take $b = T, a = T/2$ in (2.3) and use the bound on M to get the desired statement. □

We just saw that the $\|\psi(r,k)\|$ does not change much in r when $r \in [T/2,T]$ and k is fixed in *the zone of perfect control*. Next, we set up the iteration scheme which will play the key role in the proof of the main result. Suppose $T_n = 2^n, n \geqslant n_0$ where n_0 is a large parameter which will be fixed later. Given $V : \|V\| \leqslant \lambda(1+r)^{-\gamma}, \gamma > \frac{2}{3}$, we let

$$V_{(n)} := V \cdot \chi_{[0,T_n]}, \quad H_{(n)} := H^{(0)} + V_{(n)}, \quad \psi_n := e^{-ikr} R_{(n),k^2} F\,, \tag{2.7}$$

where function F has been chosen in the beginning of this section and $R_{(n),z} := (H_{(n)} - z)^{-1}$. The next lemma estimates $\psi_n(\infty,k)$ in the $(n-1)$-th *zone of perfect control*.

Lemma 2.5. *Let I be an closed interval in $\mathbb{R}^+$. If $k \in PC_{I,T_{n-1}}$, then*

$$\|\psi_n(\infty,k)\| = \|\psi_{n-1}(\infty,k)\|(1+\epsilon_n'), \quad |\epsilon_n'| \leqslant C_I T_n^{-\delta_2}$$

where δ_2 is a positive parameter.

Proof. Recall that $\psi_j(T_j,k) = \psi_j(\infty,k)$ for every j. By the previous lemma, it is enough to show that

$$\|\psi_n(T_n/2,k)\| = \|\psi_{n-1}(\infty,k)\|(1+O(T_n^{-\delta_3})) \tag{2.8}$$

where $k \in PC_{I,T_{n-1}}$ and δ_3 is a positive fixed number independent of n. To do that, we will use Lemma 5.2. Recall that $H_{(n)} = H_{(n-1)} + V \cdot \chi_{[T_{n-1},T_n]}$ and

$$R_{(n),k^2} F = R_{(n-1),k^2} F - R_{(n),k^2}(V \cdot \chi_{[T_{n-1},T_n]}) R_{(n-1),k^2} F\,.$$

Multiply the both sides with e^{-ikr} and recall the definition of ψ_n in (2.7). Since $\psi_{n-1}(r,k) = \psi_{n-1}(\infty,k)$ for $r \in [T_{n-1},\infty)$ and k is in the *zone of perfect control*, we can apply Lemma 5.2 to $R_{(n),k^2}$. This yields

$$\begin{aligned} \|\psi_n(T_n/2,k) - \psi_{n-1}(\infty,k)\| &\leqslant C_I \|\psi_{n-1}(\infty,k)\| \int_{T_{n-1}}^{T_n} e^{(\operatorname{Im} k)(T_{n-1} - c(\rho - T_{n-1}) - \rho)} T_n^{-\gamma} d\rho \\ &\leqslant C_I T_n^{-\gamma} (\operatorname{Im} k)^{-1} \|\psi_{n-1}(\infty,k)\| \leqslant C_I T_n^{\gamma-1-2\delta_1} \|\psi_{n-1}(\infty,k)\|, \end{aligned} \tag{2.9}$$

because $k \in PC_{I,T_{n-1}}$. Putting together (2.8) and (2.9) gives the desired result. □

3. Iteration and the proof of the main theorem

Recall that F is chosen to satisfy (2.1). First, we need an auxiliary lemma.

Lemma 3.1. *Suppose $\|V\| \in L^\infty(\mathbb{R}^+)$ and ψ_n is defined as in (2.7). Then,*

$$\sup_{0\leqslant y\leqslant 1} \int_I \|\psi_n(\infty, x+iy)\|^2 dx < \infty, \quad \inf_{0\leqslant y\leqslant 1} \int_I \log \|\psi_n(\infty, x+iy)\|^2 dx > -\infty$$

for every closed interval $I \subset \mathbb{R}^+$.

Proof. Since $V_{(n)}$ is compactly supported, $\psi_n(\infty, k)$ has continuous extension to any closed interval on the real line, and $\psi_n \not\equiv 0$. It is also analytic in k in every rectangle $R_{I,1}$ so the lemma follows from, e.g., the mean-value estimate for subharmonic function $\log \|\psi_n(\infty, k)\|$. □

To begin the iterative process which will be the key to the proof of our main result, we start with taking I, any closed interval in $\mathbb{R}^+$. Then, for this I, we choose $n_0 \in \mathbb{N}$, a fixed large parameter whose dependence on I will be specified later, and define two numbers A_{n_0} and B_{n_0} as follows

$$A_{n_0} := \sup_{0<y<\mathcal{L}_{T_{n_0}}} \int_I \|\psi_{n_0}(\infty, x+iy)\|^2 dx, \quad B_{n_0} := \sup_{0<y<\mathcal{L}_{T_{n_0}}} \int_I \log \|\psi_{n_0}(\infty, x+iy)\| dx \,. \tag{3.1}$$

From the last lemma, one knows that $A_{n_0} < \infty$ and $B_{n_0} > -\infty$ for every n_0. Next, we define the sequence of intervals $\{I_{(n)}\}, n \geqslant n_0$ by conditions

$$I_{(n_0)} := I,\ c_{I_{(n)}} = c_I,\ |I_{(n)}| = |I_{(n-1)}| - 2\tau_n \tag{3.2}$$

and $\tau_n = T_n^{-\upsilon}$, where $0 < \upsilon < 0.01(-\gamma + 1 + \delta_1)$ so $\mathcal{L}_n = T_n^{\gamma-1-\delta_1} \ll \tau_n = T_n^{-\upsilon}$, see Figure 1. Notice that

$$\sum_{n \geqslant n_0} \tau_n = \sum_{n \geqslant n_0} 2^{-\upsilon n} \sim C_\upsilon 2^{-\upsilon n_0}$$

and $\lim_{n_0 \to \infty} 2^{-\upsilon n_0} = 0$. Therefore, if I is given, we can always arrange for n_0 large enough that $\mathcal{L}_{T_{n_0}} < 1$ and that there is $\widetilde{I}_{(n_0)}$:

$$c_{\widetilde{I}_{(n_0)}} = c_I, \quad \widetilde{I}_{(n_0)} \subset \bigcap_{n \geqslant n_0} I_{(n)}, \quad \lim_{n_0 \to \infty} |I \backslash \widetilde{I}_{(n_0)}| \to 0 \,. \tag{3.3}$$

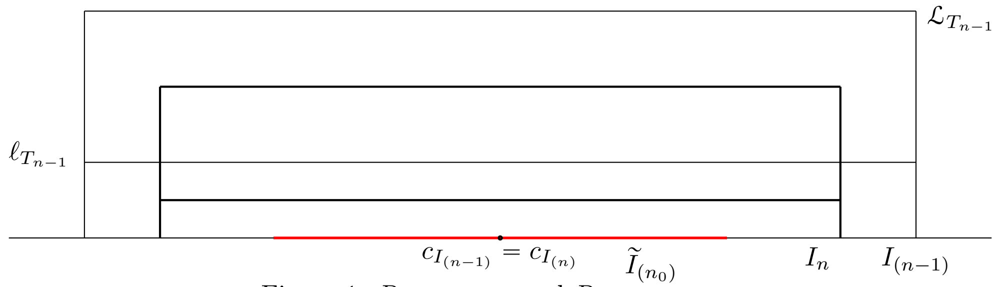

Figure 1: $R_{I_{(n-1)},T_{n-1}}$ and $R_{I_{(n)},T_n}$

Let us collect what we already know about the sequence $\{\psi_n\}$ below:

- **Rough upper bound**, Lemma 2.1:

$$\|\psi_n(\infty, k)\| \leqslant C(I', \alpha) \exp\left(2(\operatorname{Im} k)^{-(1-\alpha)/\alpha}\right), \quad k \in R_{I', \mathcal{L}_{T_n}} \,, \tag{3.4}$$

where I' can be chosen as any open interval in $\mathbb{R}^+$ that contains $I_{(n_0)} = I$. The parameter α is related to γ by $\alpha = 1 + \delta_1 - \gamma$.

- **The first step**: by construction, A_{n_0} and B_{n_0} are defined for every n_0.

- **Estimate in the** *zone of perfect control*, Lemma 2.5: if $k \in PC(I, T_{n-1})$, then

$$\|\psi_n(\infty, k)\| = \|\psi_{n-1}(\infty, k)\|(1 + \epsilon_n'), \quad |\epsilon_n'| \leqslant C_I T_n^{-\delta_2} \,. \tag{3.5}$$

- **Uniform bounds on the real line**, formula (1.4): for every $I' \subset \mathbb{R}^+$, we get

$$\sup_{n \geqslant n_0} \int_{I'} \|\psi_n(\infty, k)\|^2 dk < C_{I'} \,. \tag{3.6}$$

To control $\psi_n(\infty,k)$ in $R_{I_{(n)},\mathcal{L}_{T_n}}$, one can use apriori estimates (3.4), (3.6) along with (3.5). To interpolate the bounds on $\psi_n(\infty,k)$ from the *zone of perfect control* all the way to $R_{I_{(n)},\mathcal{L}_{T_n}}$, we will use a few estimates on the subharmonic functions that are collected and proved in the Appendix for reader's convenience. Our immediate goal is to prove the following lemma.

Lemma 3.2. *For every closed interval $J\subset\mathbb{R}^+$, we have the estimates*

$$\limsup_{n\to\infty}\sup_{0<y<\mathcal{L}_{T_n}}\int_J\|\psi_n(\infty,x+iy)\|^2dx<\infty \tag{3.7}$$

and

$$\|\psi_n(\infty,x+iy)\|^2\leqslant C_J\left(1+y^{-1}+(\mathcal{L}_{T_n}-y)^{-1}\right),\quad x\in J,\ 0<y<\mathcal{L}_{T_n}\,. \tag{3.8}$$

Proof. We start with any interval I and define the sequence $\{I_{(n)}\}$ as before in (3.2). For each $n\geqslant n_0$, one lets

$$A_n:=\sup_{0<y<\mathcal{L}_{T_n}}\int_{I_{(n)}}\|\psi_n(\infty,x+iy)\|^2dx\,.$$

We will control how A_n changes when n is increased by one. Given $n-1$ and A_{n-1}, the goal is to estimate A_n. To do that, we apply (3.5) and write

$$\sup_{\ell_{T_{n-1}}<y<\mathcal{L}_{T_{n-1}}}\int_{I_{(n-1)}}\|\psi_n(\infty,x+iy)\|^2dx\leqslant(1+\epsilon_n')^2\sup_{\ell_{T_{n-1}}<y<\mathcal{L}_{T_{n-1}}}\int_{I_{(n-1)}}\|\psi_{n-1}(\infty,x+iy)\|^2dx$$
$$\leqslant A_{n-1}(1+\epsilon_n')^2\,.$$

Next, we apply (3.4), (3.6), and Lemma 4.3 with $\kappa=(1-\alpha)/\alpha,\delta\sim\tau_n,\epsilon_1\sim\mathcal{L}_{T_{n-1}},\epsilon_2=\ell_{T_{n-1}}$ to get

$$\sup_{0<y<\ell_{T_{n-1}}}\int_{I_{(n)}}\|\psi_n(\infty,x+iy)\|^2dx\leqslant C_{I'}+O(T_n^{-\delta_4}(1+C_{I'}+A_{n-1})),\quad\delta_4>0\,.$$

In the end, we have

$$A_n\leqslant\max\Big\{C_{I'}+O(T_n^{-\delta_4}(1+C_{I'}+A_{n-1})),A_{n-1}(1+O(T_n^{\delta_5}))\Big\}$$

with positive δ_4 and δ_5. That is supplemented by fixing A_{n_0}. The previous bound yields

$$A_n\leqslant A_{n-1}(1+O(T_n^{\delta_5}))+O(T_n^{-\delta_4})$$

and $A_n\leqslant C_IA_{n_0}$. Consequently,

$$\limsup_{n\to\infty}\sup_{0<y<\mathcal{L}_{T_n}}\int_{\widetilde{I}_{(n_0)}}\|\psi_n(\infty,x+iy)\|^2dx<\infty\,. \tag{3.9}$$

Due to (3.3), we can start with any J, choose I that contains it and then n_0 so large that $\widetilde{I}_{(n_0)}$ contains J too. That will give us the first statement of the lemma. Now, the bound (3.8) follows from (4.8). ☐

Lemma 3.3. *For every closed interval $J\subset\mathbb{R}^+$, we have an estimate*

$$\liminf_{n\to\infty}\int_J\log\|\psi_n(\infty,k)\|dk>-\infty\,. \tag{3.10}$$

Proof. As in the previous proof, we define

$$B_n:=\inf_{0<y<\mathcal{L}_{T_n}}\int_{I_{(n)}}\log\|\psi_n(\infty,x+iy)\|dx\,.$$

We will control how B_n changes when n is increased by one. Given B_{n-1} and the previous lemma, we want to estimate B_n. To control $\log\|\psi_n(\infty,x+iy)\|$ in the upper part of $R_{I_{(n)},T_n}$, we use estimates in $PC(I_{n-1},T_{n-1})$. Applying (3.5), one has

$$\inf_{\ell_{T_{n-1}}\leqslant y\leqslant\mathcal{L}_{T_n}}\int_{I_{(n-1)}}\log\|\psi_n(\infty,x+iy)\|dx= \tag{3.11}$$
$$O(\epsilon_n')+\inf_{\ell_{T_{n-1}}\leqslant y\leqslant\mathcal{L}_{T_n}}\int_{I_{(n-1)}}\log\|\psi_{n-1}(\infty,x+iy)\|dx\geqslant O(\epsilon_n')+B_{n-1}.$$

and

$$\inf_{\ell_{T_{n-1}}\leqslant y\leqslant\mathcal{L}_{T_n}}\int_{I_{(n)}}\log\|\psi_n(\infty,x+iy)\|dx=O(\epsilon_n')+\inf_{\ell_{T_{n-1}}\leqslant y\leqslant\mathcal{L}_{T_n}}\int_{I_{(n)}}\log\|\psi_{n-1}(\infty,x+iy)\|dx\,.$$

Notice that for the chosen range of y we have

$$\int_{I_{(n)}} \log \|\psi_{n-1}(\infty, x+iy)\| dx = \int_{I_{(n-1)}} \log \|\psi_{n-1}(\infty, x+iy)\| dx - \int_{I_{(n-1)}\backslash I_{(n)}} \log \|\psi_{n-1}(\infty, x+iy)\| dx$$

and

$$-\int_{I_{(n-1)}\backslash I_{(n)}} \log \|\psi_{n-1}(\infty, x+iy)\| dx \geqslant -\int_{I_{(n-1)}\backslash I_{(n)}} \log_+ \|\psi_{n-1}(\infty, x+iy)\| dx .$$

Then,

$$\int_{I_{(n-1)}\backslash I_{(n)}} \log_+ \|\psi_{n-1}(\infty, x+iy)\| dx \lesssim_I \tau_n^{\frac{1}{2}}$$

as follows from the estimate $\log_+ t \leqslant |t|$, Cauchy-Schwarz inequality, (3.7), and the bound $|I_{(n)}\backslash I_{(n-1)}| \lesssim \tau_n$. In the end, we get

$$\inf_{\ell_{T_{n-1}} \leqslant y \leqslant \mathcal{L}_{T_n}} \int_{I_{(n)}} \log \|\psi_n(\infty, x+iy)\| dx \geqslant B_{n-1} + O(\tau_n^{\frac{1}{2}}) + O(\epsilon_n') \,.$$

To control the integral for the smaller values of y, i.e., when $y < \ell_{T_{n-1}}$, we apply Lemma 4.4 with $\epsilon_1 = \mathcal{L}_{T_n}, \epsilon_2 = 2\ell_{T_{n-1}}$ and $\delta \sim \tau_n$. The base of smaller rectangle is $I_{(n)}$ and the base of the larger one is $I_{(n-1)}$. Given Lemma 3.2, we can write

$$\inf_{0<y<\ell_{T_{n-1}}} \int_{I_{(n)}} \log \|\psi_n(\infty, x+iy)\| dx \geqslant (1 + O(T_n^{-\delta_6})) \int_{I_{(n-1)}} \log \|\psi_n(\infty, x+2i\ell_{T_{n-1}})\| dx - O(T_n^{-\delta_7}) \,.$$

with positive δ_6 and δ_7. For the integral on the right-hand side, apply (3.11). In the end, one has

$$B_n \geqslant (1 + O(T_n^{-\delta_8}))B_{n-1} + O(T_n^{-\delta_9}) \,, \quad \delta_8 > 0, \delta_9 > 0.$$

Consequently $\liminf_{n\to\infty} B_n > -\infty$ and thus $\liminf_{n\to\infty} \int_{I_{(n)}} \log \|\psi_n(\infty, x)\| dx > -\infty$. Since

$$\int_{I_{(n)}} \log \|\psi_n(\infty, x)\| dx = \int_{I_{(n)}} \log_- \|\psi_n(\infty, x)\| dx + \int_{I_{(n)}} \log_+ \|\psi_n(\infty, x)\| dx$$

and (3.6) guarantees that $\limsup_{n\to\infty} \int_{I_{(n)}} \log_+ \|\psi_n(\infty, x)\| dx < \infty$, we have

$$\liminf_{n\to\infty} \int_{\tilde{I}_{(n_0)}} \log_- \|\psi_n(\infty, x)\| dx \geqslant \liminf_{n\to\infty} \int_{I_{(n)}} \log_- \|\psi_n(\infty, k)\| dk > -\infty.$$

The reasoning given at the end of the proof of the previous lemma can be used again to deduce (3.10). □

The last two results provide the crucial estimates for $\|\psi_n(\infty, k)\|$ when $\operatorname{Im} k \in (0, \mathcal{L}_{T_n})$. They control the behavior of $\|(R_{(n),k^2}F)(r)\|$ for large r without giving precise asymptotics for $(R_{(n),k^2}F)(r)$. That, however, is enough to prove Theorem 1.1.

Proof of Theorem 1.1. Take any closed interval $J \subset \mathbb{R}^+$ and recall that $V_{(n)} = V \cdot \chi_{r<T_n}$. Define $\sigma_{(n),F}$, the spectral measure of F relative to $H_{(n)} = H^{(0)} + V_{(n)}$. The spectral measure of F relative to H is σ_F. Then, the previous lemma yields

$$\liminf_{n\to\infty} \int_{\Delta^2} \log \sigma'_{(n),F}(E) dE > -\infty \,.$$

Since $\lim_{n\to\infty} \|R_{(n),z}F - R_zF\|_{\mathcal{H}} = 0$, $z \in \mathbb{C}^+$, we get $\sigma_{(n),F} \to \sigma_F$ in the weak–$(*)$ sense. Hence, (see [12], section 5),

$$\int_{\Delta^2} \log \sigma'_F dE > -\infty$$

which implies that Δ^2 supports a.c. spectrum of the original H. Since Δ was arbitrary, we get the statement of the theorem. □

4. Appendix 1: some estimates on subharmonic functions

For the reader's convenience, we collect some elementary estimates on subharmonic functions in this appendix. Start with the estimates for the subharmonic function of a thin isosceles trapezoid. We denote this trapezoid by $\mathcal{T}_{I,\epsilon,\beta}$ where the height is ϵ, the side angles at the lower base are both equal to π/β, and the projection of the upper base to the real line is a given interval $I \subset \mathbb{R}$. First, we will need some estimates on the harmonic measure of that trapezoid. It is instructive to start with giving the exact formula for harmonic measure of the infinite tube which is "infinitely long" rectangle. If $Cyl_\epsilon := \{k : 0 < \operatorname{Im} k < \epsilon\}$, then the density of harmonic measure on its lower side is

$$\omega_k'(t) = \frac{1}{2\epsilon} \frac{\sin(\pi\epsilon^{-1}y)}{\cosh(\pi\epsilon^{-1}(x-t)) - \cos(\pi\epsilon^{-1}y)}\,,\ t \in \mathbb{R},\ k = x+iy \in Cyl_\epsilon\,. \tag{4.1}$$

That formula can be verified directly. Let $\Gamma := \partial\mathcal{T}_{J,\epsilon,\beta} = \Gamma_1 \cup \ldots \cup \Gamma_4$, where Γ_1 is an upper base, Γ_2 the lower base, Γ_3 the left leg, and Γ_4 the right leg of the trapezoid. Denote the harmonic measure at point k by ω_k.

Lemma 4.1. *Suppose the $\Gamma_2 = [0,2]$ and the positive parameters β, ϵ, δ are chosen such that $\beta > 2, \beta \sim 1$, $\epsilon < \delta^2 \ll 1$, $k = x+iy \in R_{(\delta,2-\delta),0.5\epsilon}$, and $\xi \in \Gamma$. Then, the derivative of harmonic measure in the corresponding trapezoid with respect to its arclength satisfies*

$$\xi = s + \epsilon i \in \Gamma_1, \qquad \omega_k'(\xi) \lesssim \frac{\epsilon^{-2}y}{\cosh(\pi\epsilon^{-1}(x-s))}\,, \tag{4.2}$$

$$\xi = s \in \Gamma_2, \qquad \omega_k'(\xi) \leqslant \frac{y}{\pi((s-x)^2+y^2)}\,, \tag{4.3}$$

$$\xi = te^{i\pi/\beta} \in \Gamma_3, \qquad \omega_k'(\xi) \leqslant C_\beta \frac{(xt)^{\beta-1}y}{(t^2+x^2)^\beta}\,, \tag{4.4}$$

$$\xi = 2 + te^{i(\pi-\pi/\beta)} \in \Gamma_4,\ x<1 \qquad \omega_k'(\xi) \leqslant C_\beta y t^{\beta-1}\,. \tag{4.5}$$

Proof. See Figure 2.

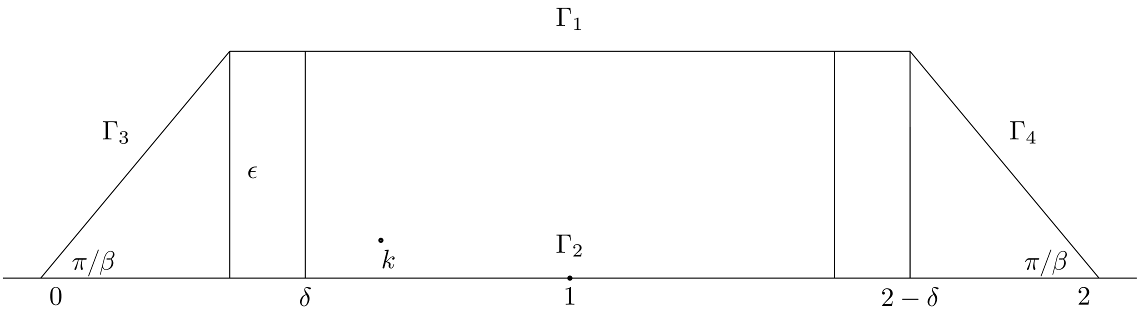

Figure 2, $\epsilon < \delta^2 \ll 1$

Recall the following monotonicity property of harmonic measure. If $\Omega_1 \subset \Omega_2$ and $E \subset \partial\Omega_1 \cap \partial\Omega_2$, then $\omega_{k,\Omega_1}(E) \leqslant \omega_{k,\Omega_2}(E)$ for $k \in \Omega_1$ ([10], p. 36) where $\omega_{k,\Omega}$ denotes harmonic measure at point k relative to the domain Ω. This monotonicity helps us get the required upper bounds by comparing to harmonic measure of an angle, an infinite cylinder, or a half-plane. We obtain (4.2) by comparing with infinite cylinder and (4.3) by comparing with the upper half-plane. The other two formulas are deduced by making a comparison with an infinite angle. □

Remark. The estimates in the upper part of rectangle can be obtained in a similar way.

We will need the following result later. Recall that I_r denotes the interval on the real line with radius r centered at the origin.

Lemma 4.2. *Suppose the positive parameters $\epsilon_2, \epsilon_1, \delta$ satisfy $2\epsilon_2 < \epsilon_1 < \delta^2 \ll 1$ and let ω_k be a harmonic measure for $R_{I_{1+\delta},\epsilon_1}$. Then, for $k = x + i\epsilon_2$, we have*

$$\sup_{|\xi|<1-\delta} \left| \int_{I_1} \omega_{x+i\epsilon_2}'(\xi)dx - 1 \right| \lesssim \epsilon_2\epsilon_1^{-1}\,. \tag{4.6}$$

Proof. The required density of harmonic measure can be written via harmonic measure of infinite cylinder through proper extension from $I_{1+\delta}$ to $\mathbb{R}$. The resulting formula shows that the contribution from the left and righ sides of rectangle are exponentially small and the desired density can be well approximated by the density of harmonic measure of the infinite cylinder. Then, we use formula (4.1) to obtain required bound. □

Lemma 4.3. *Suppose the positive parameters ϵ_1, ϵ_2 and δ satisfy $2\epsilon_2 < \epsilon_1 < \delta^2 \ll 1$. Assume that h is $\ell^2(\mathbb{N})$-valued function holomorphic in $R_{I_2,1}$, continuous in $\overline{R_{I_2,1}}$, and*

$$\|h(k)\| \leqslant C_1 \exp(C_2(\operatorname{Im} k)^{-\kappa}),\ \ k \in R_{I_2,1}\,,\ 1 < \kappa,\ \kappa \sim 1. \tag{4.7}$$

Then, we have

$$\|h(x+iy)\|^2 \leqslant C_\kappa \left(1 + y^{-1}A + (\epsilon_1 - y)^{-1}B\right), \tag{4.8}$$

$$A := \int_{I_2} \|h(t)\|^2 dt, \quad B := \int_{-1-\delta}^{1+\delta} \|h(t+i\epsilon_1)\|^2 dt$$

provided that $k = x + iy \in R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. Moreover,

$$\sup_{0<y<\epsilon_2} \int_{-1}^{1} \|h(x+iy)\|^2 dx \leqslant A + C_\kappa \epsilon_2 \epsilon_1^{-1}(A + B + \epsilon_1)\,. \tag{4.9}$$

Proof. See Figure 3.

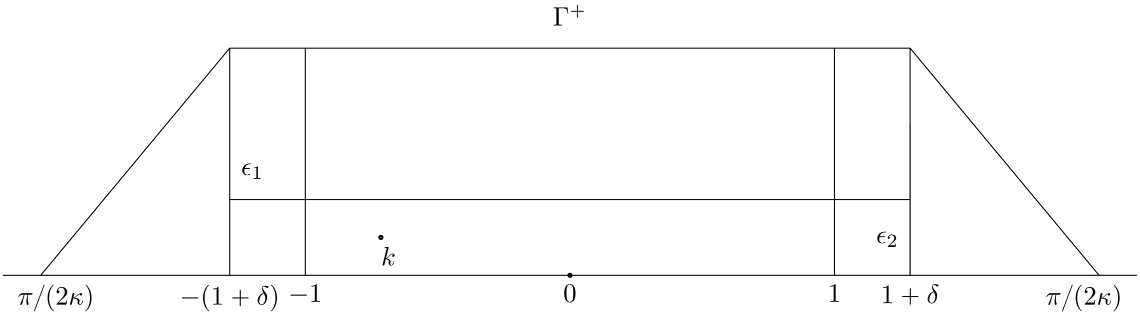

Figure 3

We can assume $h \not\equiv 0$. Let $k = x + iy \in R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. Consider the isosceles trapezoid $\mathcal{T}_{I_{1+\delta},\epsilon_1,\pi/(2\kappa)}$. Denote its upper base by Γ^+ and its lower base by Γ^-. We write the mean-value inequality for subharmonic function $2\log_+ \|h\|$ and use the estimate (4.4) on the density of harmonic measure on the legs to get

$$2\log_+ \|h(k)\| \leqslant 2\int_{\partial \mathcal{T}_{I_{1+\delta},\epsilon_1,\pi/(2\kappa)}} \log_+ \|h\| d\omega_k \leqslant C_\kappa y \delta^{-1-2\kappa}\epsilon_1^\kappa +$$

$$2\int_{\Gamma^+\cup\Gamma^-} \log_+ \|h\|\omega_k'(\xi)d\xi \leqslant C_\kappa + 2\int_{\Gamma^+\cup\Gamma^-} \log_+ \|h\|\omega_k'(\xi)d\xi$$

where we applied the given estimates on $\|h\|$ along with $\epsilon_1 < \delta^2$. Define $Q(k) = \max\{1, \|h\|\}$ and notice that $\log Q = \log_+ Q \geqslant 0$ so

$$\log Q^2 \leqslant C_\kappa + \int_{\Gamma^+\cup\Gamma^-} (\log Q^2)\omega_k'(\xi)d\xi \leqslant C_\kappa + \int_{\Gamma^+\cup\Gamma^-} (\log Q^2) d\mu, \quad \mu := \frac{\omega_k|_{\Gamma^-\cup\Gamma^+}}{\|\omega_k|_{\Gamma^-\cup\Gamma^+}\|} \geqslant \omega_k|_{\Gamma^-\cup\Gamma^+}\,.$$

Taking the exponential of both sides and using Jensen's inequality

$$\exp\left(\int \log f d\mu\right) \leqslant \int f d\mu, \quad \|\mu\| = 1$$

we get

$$Q^2 \leqslant C_\kappa \frac{\displaystyle\int_{\Gamma^-\cup\Gamma^+} Q^2 \omega_k'(\xi)d\xi}{\|\omega_k|_{\Gamma^-\cup\Gamma^+}\|}\,.$$

For considered k, we have $\|\omega_k|_{\Gamma^-\cup\Gamma^+}\| \sim 1$. Thus,

$$Q^2 \lesssim_\kappa 1 + \int_{-2}^{2} \frac{\pi^{-1}y}{(\xi - x)^2 + y^2}\|h(\xi)\|^2 d\xi + \int_{-1-\delta}^{1+\delta} \frac{\pi^{-1}(\epsilon_1 - y)}{(\xi - x)^2 + (\epsilon_1 - y)^2}\|h(\xi + i\epsilon_1)\|^2 d\xi$$

$$\lesssim_\kappa 1 + C\left(y^{-1}\int_{I_2} \|h\|^2 d\xi + (\epsilon_1 - y)^{-1}\int_{-1-\delta}^{1+\delta} \|h(\xi + i\epsilon_1)\|^2 d\xi\right).$$

To obtain (4.9), we take $k \in R_{I_1,\epsilon_2}$ and apply the mean-value inequality to subharmonic function $\|h(k)\|^2$ inside the domain $R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. The symbol $\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}$ will stand for an upper base of this rectangle. Then,

$$\|h(k)\|^2 \leqslant \int_{\partial R_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 d\omega_k \leqslant I + \int_{I_{1+\frac{\delta}{2}}} \|h\|^2 \omega_k'(\xi) d\xi + \int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 \omega_k'(\xi) d\xi \,. \tag{4.10}$$

To estimate the first term, we use (4.8). That gives

$$I \lesssim_\kappa \int_0^{0.5\epsilon_1} \left(1+t^{-1}A+(\epsilon_1-t)^{-1}B\right)\left(\frac{xyt}{(x^2+t^2)^2}\right) dt +$$
$$\int_{0.5\epsilon_1}^{\epsilon_1} \left(1+t^{-1}A+(\epsilon_1-t)^{-1}B\right)\left(\frac{xy(\epsilon_1-t)}{(x^2+(\epsilon_1-t)^2)^2}\right) dt \lesssim_\kappa (A+B+\epsilon_1)y\epsilon_1\delta^{-3}$$

as follows from (4.8) and the estimates for the harmonic measure of rectangle. For the last term in the right hand side of (4.10), one employs the bound on harmonic measure to write

$$\int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 \omega_k'(\xi) d\xi \lesssim \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \|h(\xi+i\epsilon_1)\|^2 \frac{\epsilon_1^{-2}y}{\cosh(\pi\epsilon_1^{-1}(x-\xi))} d\xi \,. \tag{4.11}$$

Next, we integrate (4.10) in $x \in I_1$. Integration of (4.11) yields

$$\int_{I_1}\left(\int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 \omega_k'(\xi) d\xi\right) dx \lesssim \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \|h(\xi+i\epsilon_1)\|^2 \left(\int_{I_1} \frac{\epsilon_1^{-2}y}{\cosh(\pi\epsilon_1^{-1}(x-\xi))} dx\right) d\xi \lesssim By\epsilon_1^{-1} \,.$$

The second term on the right-hand side of (4.10) contributes

$$\int_{I_1}\left(\int_{I_{1+\frac{\delta}{2}}} \|h\|^2 \omega_k'(\xi) d\xi\right) dx \leqslant \int_{I_{1+\frac{\delta}{2}}} \|h\|^2 \left(\int_{I_1} \omega_k'(\xi) d\xi\right) dx \leqslant \int_{I_2} \|h\|^2 dx$$

where the estimate

$$\omega_k'(\xi) \leqslant \frac{\pi^{-1}y}{(\xi-x)^2+y^2}$$

was used. Combining the bounds, we get (4.9) after our assumption $\epsilon_1 < \delta^2$ is taken into account. $\square$

Lemma 4.4. *Suppose the positive parameters ϵ_1, ϵ_2 and δ are chosen such that $\epsilon_2 \leqslant \epsilon_1 |\log \epsilon_1|$, $\epsilon_1 < \delta^2$ and $\delta \ll 1$. Assume that $\ell^2(\mathbb{N})$-valued function h is holomorphic in $R_{I_{1+\delta},\epsilon_1}$, $h \in C(\overline{R_{I_{1+\delta},\epsilon_1}})$, $h \not\equiv 0$,*

$$W := \sup_{0<y<\epsilon_1} \int_{I_{1+\delta}} \|h(x+iy)\|^2 dx, \quad \|h(k)\|^2 \leqslant L(y^{-1}+(\epsilon_1-y)^{-1}), \quad k = x+iy \in R_{I_{1+\delta},\epsilon_1}\,, \quad L > 2\,.$$

Then, we have

$$\inf_{0<y<\epsilon_2/2} \int_{I_{1-\delta}} \log\|h(x+iy)\| dx \geqslant (1+O(\epsilon_2\epsilon_1^{-1}))\left(\int_{I_1} \log\|h(x+i\epsilon_2)\| dx - \eta\right),$$
$$|\eta| < C\Big(\epsilon_2\epsilon_1^{-1}\Big(W^{0.5}+|\log L|+|\log\epsilon_1|\Big)+(\delta W)^{0.5}\Big)\,.$$

Proof. It is enough to prove

$$\int_{I_{1-\delta}} \log\|h(x)\| dx \geqslant (1+O(\epsilon_2\epsilon_1^{-1}))\left(\int_{I_1} \log\|h(x+i\epsilon_2)\| dx - \eta\right). \tag{4.12}$$

Take $k = x+i\epsilon_2, x \in I_1$ and apply the mean-value inequality to the subharmonic function $\log\|h\|$ within $R_{I_{1+\delta},\epsilon_1}$. We define $\Gamma_1 = \{k : \mathrm{Re}\, k \in I_{1+\delta}, \mathrm{Im}\, k = \epsilon_1\}$, $\Gamma_2 = \{k : \mathrm{Im}\, k \in (0,\epsilon_1), k \in \partial R_{I_{1+\delta},\epsilon_1}\}$, $\Gamma_3 = \{k : \mathrm{Re}\, k \in I_{1+\delta}, \mathrm{Im}\, k = 0\}$. Check Figure 4.

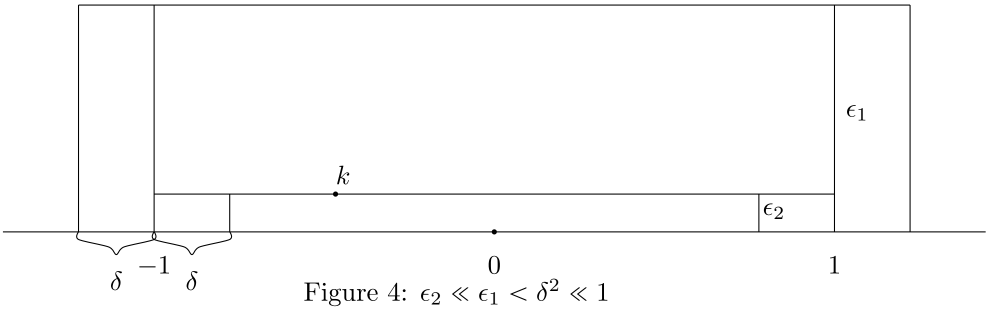

Figure 4: $\epsilon_2 \ll \epsilon_1 < \delta^2 \ll 1$

We get

$$\int_{\Gamma_3} \log \|h\| d\omega_k \geqslant \log \|h(x+i\epsilon_2)\| - E_1 - E_2 \,, \tag{4.13}$$

where

$$E_1 = \int_{\Gamma_1} \log_+ \|h\| d\omega_k, \quad E_2 = \int_{\Gamma_2} \log_+ \|h\| d\omega_k \,.$$

One applies the given estimates on h and the estimates on a harmonic measure to bound $E_{1(2)}$:

$$E_2 \lesssim (|\log L| + |\log \epsilon_1|)\delta^{-3}\epsilon_1^2\epsilon_2, \quad E_1 \lesssim \int_{I_{1+\delta}} \log_+ \|h(\xi+i\epsilon_1)\| \frac{\epsilon_1^{-2} y}{\cosh(\pi\epsilon_1^{-1}(x-\xi))} d\xi \,.$$

Now, we integrate (4.13) in x over I_1 and recall that $\Gamma_3 = I_{1+\delta}$. That gives

$$\int_{I_1} E_1 dx \lesssim \int_{I_{1+\delta}} \log_+ \|h(\xi+i\epsilon_1)\| \left(\int_{I_1} \frac{\epsilon_1^{-2} y}{\cosh(\pi\epsilon_1^{-1}(x-\xi))} dx \right) d\xi \leqslant W^{\frac{1}{2}} y \epsilon_1^{-1} \,.$$

Then,

$$\int_{I_1} \left(\int_{I_{1+\delta}} \log \|h\| d\omega_k \right) dx = \int_{I_{1+\delta}} \log \|h\| \left(\int_{I_1} \omega_k' dx \right) d\xi \leqslant$$

$$(1+O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log \|h\| dx + O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log_+ \|h\| dx + \int_{I_{1+\delta}\backslash I_{1-\delta}} \log_+ \|h\| \left(\int_{I_1} \omega_k' dx \right) d\xi \leqslant$$

$$(1+O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log \|h\| dx + C\epsilon_2\epsilon_1^{-1} W^{\frac{1}{2}} + C \int_{I_{1+\delta}\backslash I_{1-\delta}} \log_+ \|h\| d\xi$$

after we use the bound (4.6) from Lemma 4.2. Finally,

$$\int_{I_{1+\delta}\backslash I_{1-\delta}} \log_+ \|h\| d\xi \leqslant C(\epsilon) W^{0.5} \delta^{0.5}$$

by Cauchy-Schwarz inequality. Combining obtained estimates, we get the statement of the lemma. □

5. Appendix 2: rough bounds on Green's function

We need the following standard bounds "a la Combes-Thomas" (see, e.g., [9]) for Green's function $G(r,\rho,k^2)$ of $H = H^{(0)} + V$. In this section, we assume that I is a fixed closed interval in $\mathbb{R}^+$ and $k \in R_{I,1}$.

Lemma 5.1. *Suppose $\|V\|_{L^\infty(\mathbb{R}^+)} < \infty$. Then, we have*

$$\|G(r,\rho,k^2)\| \leqslant C_I' e^{-0.5(\operatorname{Im} k)|r-\rho|} \tag{5.1}$$

for all $k \in R_{I,1}, \operatorname{Im} k > C_I \|V\|_{L^\infty(\mathbb{R}^+)}$ with some $C_I > 0$ and $C_I' > 0$.

Proof. This is immediate from the analysis of perturbation identity for the Green's kernel G:

$$G(r,\rho,k^2) = G^{(0)}(r,\rho,k^2) - \int_0^\infty G^{(0)}(r,\xi,k^2) V(\xi) G(\xi,\rho,k^2) d\xi \,.$$

Multiply the both sides by $e^{0.5(\operatorname{Im} k)|r-\rho|}$ and apply the contraction mapping principle in $L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$. We use (1.2) to get

$$e^{0.5(\operatorname{Im} k)|r-\rho|}\int_0^\infty e^{-(\operatorname{Im} k)|r-\xi|}\|V(\xi)\|e^{-0.5(\operatorname{Im} k)|\xi-\rho|}d\xi \leqslant 4\|V\|_{L^\infty(\mathbb{R}^+)}(\operatorname{Im} k)^{-1}$$

and (5.1) follows provided $\operatorname{Im} k > C_I\|V\|_{L^\infty(\mathbb{R}^+)}$ with suitable C_I. □

Finally, we can focus on the lemma we need in the main text.

Lemma 5.2. *Let $\|V\|\leqslant \lambda(1+r)^{-\gamma}$, $H=H^{(0)}+V$, $k\in R_{I,1}$, where I is a closed interval in $\mathbb{R}^+$, $\gamma\in(0,1)$, and $T>1$. Then, there are positive T-independent constants C, C_1 and c such that*

$$\|G(r,\rho,k)\| < Ce^{-c(\operatorname{Im} k)|r-\rho|}$$

for $\operatorname{Im} k > C_1T^{-\gamma}$, $0.5T<r<T$, and $0.5T<\rho<T$.

Proof. Define $H' = -\partial^2_{rr} + V\cdot\chi_{r>\frac14 T}$. By the previous lemma, the corresponding Green's kernel G' satisfies the bound

$$\|G'(r,\rho,k)\|\leqslant Ce^{-0.5(\operatorname{Im} k)|r-\rho|} \tag{5.2}$$

if $\operatorname{Im} k > C_1T^{-\gamma}$. Next, we again write the second resolvent identity

$$G(r,\rho,k^2)=G'(r,\rho,k^2)-\int_0^{\frac14 T} G(r,\xi,k^2)V(\xi)G'(\xi,\rho,k^2)d\xi\,.$$

For the first term, we use (5.2). To estimate the second one, we apply a general bound: for every $h\in\mathcal{H}$, one has $\|R_{k^2}h\|_{L^\infty(\mathbb{R}^+)}\leqslant C(\|R_{k^2}h\|_{L^2(\mathbb{R}^+)}+\|(R_{k^2}h)''\|_{L^2(\mathbb{R}^+)})\leqslant C_{I,\lambda}(\operatorname{Im} k)^{-1}\|h\|_{L^2(\mathbb{R}^+)}$ which follows from Sobolev's embedding, the equation for $R_{k^2}h$, and the Spectral Theorem. Then, since $r,\rho\in[0.5T,T]$, one deduces

$$\left\|\int_0^{\frac14 T} G(r,\xi,k^2)V(\xi)G'(\xi,\rho,k^2)d\xi\right\| \leqslant C_{I,\lambda}(\operatorname{Im} k)^{-1}\left(\int_0^{\frac14 T} e^{-(\operatorname{Im} k)|\xi-\rho|}d\xi\right)^{\frac12}\leqslant C_{I,\lambda}(\operatorname{Im} k)^{-2}e^{-0.1(\operatorname{Im} k)T}\,.$$

Since $\operatorname{Im} k > C_1T^{-\gamma}$ and $\gamma\in(0,1)$, we have $(\operatorname{Im} k)^{-2}e^{-0.1(\operatorname{Im} k)T} < Ce^{-c_1(\operatorname{Im} k)T}$ with positive c_1. The result now follows because $e^{-c_1(\operatorname{Im} k)T}\leqslant e^{-c(\operatorname{Im} k)|\rho-r|}$ with positive c provided that $0.5T<r,\rho<T$. □

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