THE SOBOLEV NORMS AND LOCALIZATION ON THE FOURIER SIDE FOR SOLUTIONS TO SOME EVOLUTION EQUATIONS

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ABSTRACT. In this paper, some evolution equations with rough time-dependent potential are studied in the case of one-dimensional torus. We show that the solution has higher regularity for the generic values of the coupling parameter. We also control the localization of these solutions on the Fourier side.

1. INTRODUCTION

Let $P(x)$ be an algebraic polynomial with real-valued time-dependent coefficients

$$P(x) = \sum_{j=1}^{d} p_j x^j, \quad p_j(t) \in L^1_{\rm loc}(\mathbb{R})$$

One can consider the following evolution equation

$$iu_t = (kP(i\partial_x) + V)u, \quad u(x,0,k) = 1, \quad x \in \mathbb{T}, \quad k \in \mathbb{R} \tag{1}$$

where the potential $V(x,t)$ satisfies

$$\|V(x,t)\|_{L^\infty(\mathbb{T})} \in L^1_{\rm loc}(\mathbb{R}^+) \tag{2}$$

Since $p_j \in \mathbb{R}$, the unperturbed evolution (i.e. when $V = 0$) defines a unitary group in $L^2(\mathbb{T})$. The assumption (2) allows one to iterate the Duhamel formula (see [10]) and show that the resulting series converges in $L^2(\mathbb{T})$. That implies the $L^2(\mathbb{T})$ norm of the solution is bounded for any t however it might grow as $t \to \infty$. One question we want to address in this paper is what happens to the Sobolev norms? Are they bounded for $t > 0$ and, if so, how fast can they grow as $t \to \infty$?

The case of real-valued V is a very special one and we will mostly focus on that situation. Indeed, if $V \in \mathbb{R}$, then $\|u(x,t,k)\|_{L^2(\mathbb{T})} = 1$ for all t. Let $\widehat{f}_n$ denote the Fourier transform of $f(x)$ in the variable $x \in \mathbb{T}$. For real-valued V, we will study the localization of solution on the Fourier side and its asymptotical behavior for large time. In particular, the following question is quite natural: if initially $\widehat{u}_n(0,k) = \delta_0$, then what can be said about the size of

$$\sum_{|n|>\mu(t)} |\widehat{u}_n(t,k)|^2$$

for various $\mu(t)$? If this sum is small, then a nontrivial ℓ^2 norm of $\widehat{u}$ should be supported on the first $\mu(t)$ frequencies because the total ℓ^2 norm is conserved and is equal to 1. This problem is related to the estimates on the Sobolev norms but is not equivalent to it. The results we obtain in this paper answer some of these questions. What makes our setting different from the earlier extensive work on the subject of large-time behavior of evolution equations (see, e.g., [11] and references there) is that we want to address these problems not for a particular k but for its "generic" value with respect to the Lebesgue measure. The current paper is a continuation of [3] where the analogous questions were studied mostly by the complex analysis technique. In the proofs that follow, we develop more robust perturbation theory. For example, we can handle equations in which the parameter k enters in a more complicated way, e.g. in (1), instead of k we can write $\lambda(k)$ where λ is smooth but not necessarily analytic. The main motivation to study these problems comes from the scattering theory of multidimensional Schrödinger operator with slowly decaying potential as explained in [4]. For the case of smooth V the nontrivial upper estimates for the growth of Sobolev norms were obtained in [2] where the Schrödinger evolution was handled and the arithmetic structure was used to gain extra regularity of solution. In the second section, the Schrödinger evolution with real V is considered. The third section also handles Schrödinger evolution but with general V. In the last section, an asymptotical result for the non-degenerate P, i.e. when $p_1 \neq 0$, is obtained. The Appendix contains two auxiliary lemmas.

For simplicity, we will study only the case of quadratic polynomials $P(x)$, i.e., $d = 2$, however the methods can be easily adjusted to other symbols. The symbol $\|f\|$ will refer to the $L^2(\mathbb{T})$ norm in case of a function f and $\|O\|$ will refer to the operator norm if O is an operator. We will denote the Hilbert-Schmidt norm of the operator O by $\|O\|_{\mathcal{S}_2}$. If $p \in [1, \infty]$, the symbol p' denotes the conjugate exponent, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1$$

For real α, $[\alpha]$ stands for the integer part, $f * g$ denotes the convolution of f and g, χ_A is the characteristic function of the set A. For the norms in Sobolev spaces we have

$$\|f\|^2_{H^\alpha(\mathbb{T})} = \sum_n (1+|n|)^{2\alpha} |\widehat{f}_n|^2, \quad \|f\|^2_{\dot{H}^\alpha(\mathbb{T})} = \sum_{n\neq 0} (1+|n|)^{2\alpha} |\widehat{f}_n|^2$$

For two operators A and B, the commutator $[A, B] = AB - BA$.

2. The Schrödinger evolution with real-valued V

The important special case of (1) is the Schrödinger evolution which corresponds to $P(x) = p_2 x^2$. We first assume that $p_2 = 1$ so (1) takes the form

$$iu_t = (-k\Delta + V)u, \quad u(x, 0) = 1, \quad x \in \mathbb{T}, \quad k \in \mathbb{R}, \quad \Delta = \partial^2_{xx} \tag{3}$$

Let

$$\upsilon_1(t) = \|V(x, t)\|_{L^\infty(\mathbb{T})}$$

and $w(t) \in C[0, T]$ be arbitrary positive function. Take

$$D_1(T) = \int_0^T \upsilon_1^2(\tau) \left(1 + \int_0^\tau \upsilon_1(\tau_1) d\tau_1\right) d\tau$$

and

$$D_2(T) = \left(\int_0^T \upsilon_1^2(\tau) w(\tau) d\tau\right) \left(\int_0^T \upsilon_1^2(\tau) \int_0^\tau w^{-1}(\tau_1) d\tau_1 d\tau\right)$$

Theorem 2.1. *Suppose V is real-valued and $\alpha < 1/2$. Then*

$$\int_{\mathbb{R}} \sup_{\tau \in [0,T]} \|u(x, \tau, k)\|^2_{\dot{H}^\alpha(\mathbb{T})} dk \lesssim D_1(T) + D_2(T), \quad \forall T > 0$$

In particular, if V is bounded on $\mathbb{T}\times\mathbb{R}^+$ then for Lebesgue a.e. k the solution is H^α regular for any t and the norm does not grow faster than $t^{1.5+\epsilon}$ with any fixed $\epsilon>0$. We expect much stronger result to hold and state it as an

Open problem. Prove that

$$\int_{\mathbb{R}}\sup_{\tau\in[0,T]}\|u(x,\tau,k)\|^2_{\dot H^1(\mathbb{T})}dk\lesssim 1+\int_0^T\int_{\mathbb{T}}V^2(x,\tau)dxd\tau,\quad \forall T>0$$

We will use the following notation

$$\widetilde{V}=e^{-ik\Delta t}Ve^{ik\Delta t}\tag{4}$$

Take any interval $S\subseteq[0,T]$ and define the operator $\widetilde{V}_S(k)$ by its matrix representation on the Fourier side

$$\widehat{\widetilde{V}_S(k)}(m,n)=\int_S e^{ik(m^2-n^2)\tau}\widehat{V}_{m-n}(\tau)d\tau,\quad m,n\in\mathbb{Z}$$

Let P_N be a projection to the first N Fourier modes

$$\widehat{P_Nf}(n)=\chi_{|n|\le N}\cdot\widehat{f}(n)\tag{5}$$

and $Q_N=I-P_N$.

Lemma 2.1. *We have*

$$\int_{\mathbb{R}}\sup_S\|P_N\widetilde{V}_S(k)Q_N\|^2_{\mathfrak{S}_2}dk\lesssim N^{-1}\log N\int_0^T\int_{\mathbb{T}}V^2(x,t)dxdt\tag{6}$$

Proof. For any S,

$$\sum_{|m|\le N}\sum_{|n|>N}\left|\int_S\widehat{V}_{m-n}(t)e^{ik(m^2-n^2)t}dt\right|^2\le\sum_{|n|<N}\sum_{|m|\ge N}|q_{m-n}(k(m^2-n^2))|^2$$

where

$$q_l(k)=\sup_S\left|\int_S\widehat{V}_l(t)e^{ikt}dt\right|$$

By Carleson's theorem on maximal functions [6], we have

$$v_l=\|q_l\|_2\lesssim\|\widehat{V}_l\|_2$$

The l.h.s. in (6) is bounded by T_1+T_2, where

$$T_1=\sum_{|m|\le N}\sum_{n>N}\frac{v^2_{m-n}}{n^2-m^2}$$

and

$$T_2=\sum_{|m|\le N}\sum_{n<-N}\frac{v^2_{m-n}}{n^2-m^2}$$

Since $T_1=T_2$, we only estimate T_1. If $\alpha=n-m,\beta=m+n$, then

$$T_1\lesssim\sum_{\alpha>2N}\frac{v_\alpha^2}{\alpha}\sum_{\beta=\alpha-2N}^{\alpha+2N}\frac{1}{\beta+1}+\sum_{\alpha=1}^{2N}\frac{v_\alpha^2}{\alpha}\sum_{\beta=2N-\alpha}^{2N+\alpha}\frac{1}{\beta+1}\lesssim N^{-1}\sum_{\alpha=1}^{\infty}v_\alpha^2+\sum_{\alpha=N}^{3N}\frac{v_\alpha^2}{\alpha}\log\frac{|2N+\alpha|}{|2N-\alpha|+1}\tag{7}$$

and that finishes the proof. □

Remark. The logarithmic factor in the estimate above is not present when the Laplacian Δ is restricted to the Hardy space $\mathcal{H}^2(\mathbb{T})$. It is also negligible when the average in N is taken. Indeed, we have

$$\sum_{N=1}^{\infty}\sum_{\alpha=N}^{3N}\frac{v_\alpha^2}{\alpha}\log\frac{|2N+\alpha|}{|2N-\alpha|+1}\lesssim\sum_{\alpha=1}^{\infty}\frac{v_\alpha^2}{\alpha}\int_{\alpha/3}^{\alpha}\log\frac{|2x+\alpha|}{|2x-\alpha|+1}dx\lesssim\sum_{\alpha=1}^{\infty}v_\alpha^2$$

so the second term in (7), rather than the first one, is in ℓ^1.

Now, we are ready to prove theorem 2.1. The technique will resemble the one used in [7] for the matrices 2×2.

Proof. (of theorem 2.1) Consider $\phi_n = P_n u, \psi_n = Q_n u$. Then,

$$i\partial_t\psi_n = (-kQ_n\Delta Q_n + Q_nVQ_n)\psi_n + Q_nV\phi_n, \quad \psi_n(x,0,k)=0$$

and so

$$\|\psi_n\|^2 = 2\,\mathrm{Im}\int_0^t\langle Q_nV\phi_n,\psi_n\rangle d\tau \tag{8}$$

If $\phi_n = e^{ik\Delta t}\widetilde{\phi}_n$ and $\psi_n = e^{ik\Delta t}\widetilde{\psi}_n$, then

$$i\partial_t\widetilde{\psi}_n = (Q_n\widetilde{V}Q_n)\widetilde{\psi}_n + (Q_n\widetilde{V}P_n)\widetilde{\phi}_n, \quad i\partial_t\widetilde{\phi}_n = (P_n\widetilde{V}P_n)\widetilde{\phi}_n + (P_n\widetilde{V}Q_n)\widetilde{\psi}_n \tag{9}$$

and integration by parts in (8) yields

$$\|\psi_n(t,k)\|^2 \lesssim I_1 + I_2$$

where

$$I_1 = \mathrm{Im}\int_0^t\langle\Big(Q_n\widetilde{V}_{[\tau,t]}P_n\Big)\widetilde{\phi}'_n,\widetilde{\psi}_n\rangle d\tau,\ I_2 = \mathrm{Im}\int_0^t\langle\Big(Q_n\widetilde{V}_{[\tau,t]}P_n\Big)\widetilde{\phi}_n,\widetilde{\psi}'_n\rangle d\tau$$

For I_1, we have

$$I_1 \lesssim \int_0^t\|Q_n\widetilde{V}_{[\tau,t]}P_n\|\cdot\|V\|\cdot\|\psi_n\|d\tau$$

due to (9) and $\|\phi_n\|\le 1, \|\psi_n\|\le 1$.

For I_2, we substitute (9) to get

$$I_2 \lesssim \int_0^t\|Q_n\widetilde{V}_{[\tau,t]}P_n\|\cdot\|V\|\cdot\|\psi_n\|d\tau +$$

$$\left|\mathrm{Re}\int_0^t\langle\Big(\int_\tau^t Q_n\widetilde{V}(\tau_1)P_nd\tau_1\Big)\widetilde{\phi}_n(\tau),(Q_n\widetilde{V}(\tau)P_n)\widetilde{\phi}_n(\tau)\rangle d\tau\right| \tag{10}$$

For the last term, we can use the following identity

$$2\,\mathrm{Re}\int_0^t\langle Z'(\tau)y(\tau),Z(\tau)y(\tau)\rangle d\tau = \|Z(\tau)y(\tau)\|^2\Big|_{\tau=0}^{\tau=t} - 2\,\mathrm{Re}\int_0^t\langle Z(\tau)y'(\tau),Z(\tau)y(\tau)\rangle d\tau$$

Thus, the second term in (10) is bounded by

$$a_1(t,k) = \|Q_n\widetilde{V}_{[0,t]}P_n\|^2 + \int_0^t\|Q_n\widetilde{V}_{[\tau,t]}P_n\|^2\|V(\tau)\|d\tau$$

If we denote

$$a_2(t,k) = \int_0^t\|Q_n\widetilde{V}_{[\tau,t]}P_n\|\cdot\|V\|d\tau$$

and $z(k)=\max_{\tau\in[0,T]}\|\psi_n\|=\|\psi_n(t(k),k)\|$, then the quadratic inequality

$$z^2\lesssim a_2z+a_1$$

gives

$$z\lesssim a_2+\sqrt{a_1} \tag{11}$$

In other words,

$$\sup_{\tau\in[0,T]}\sum_{|j|>n}|\widehat{u}_j(\tau,k)|^2\lesssim a_2^2(t(k),k)+a_1(t(k),k)\lesssim$$

$$\left(\int_0^T w(\tau)\|V(.,\tau)\|^2_{L^\infty(\mathbb{T})}d\tau\right)\left(\int_0^T w^{-1}(\tau)\sup_{S\subseteq[\tau,T]}\|Q_n\widetilde{V}_SP_n\|^2d\tau\right)+a_1(t(k),k)$$

where we applied Cauchy-Schwarz. By lemma 2.1, we have

$$\int_{\mathbb{R}}\sup_{\tau\in[0,T]}\sum_{|j|>n}|\widehat{u}_j(\tau,k)|^2dk\lesssim\frac{\log n}{n}(D_1+D_2)$$

where $D_{1(2)}$ were introduced above. Multiply the last estimate by $|n|^{-\epsilon}$ and sum in $n\neq 0$ to get theorem 2.1. □

Assume that $\alpha<1/2,\gamma>3/4$ is fixed and $v_1(t)\lesssim(1+t)^{-\gamma}$. Take $w(t)=(1+t)^{1/2}$. Then, we have the following striking estimate

$$\sup_{t>0}\|u\|_{H^\alpha(\mathbb{T})}<\infty$$

for a.e. k. This is a remarkable fact as we do not assume any smoothness of V at all.

In the rest of this section, we will consider the problem which is directly related to the multidimensional scattering [4]. We again take $P(x)=c_2x^2$ but now the coefficient decays in t

$$c_2(t)=\frac{1}{(1+t)^2}$$

The difficult problem in this area is to show that the solution has localization $\mu(t)\leq t$ for a.e. k as longs as potentials V satisfies some decay condition, e.g.

$$|V(x,t)|<C(1+t)^{-1/2-\epsilon},\quad \epsilon\in(0,1/2)$$

In fact, this is not known even for ϵ close to 1/2.

Below we will give a partial solution to this problem in the case when V oscillates. This will improve on the earlier result from [3]. The following proof is general enough to handle initial data of the form $u(x,0,k)=e^{ijx}$ for any j and thus it yields that the whole monodromy matrix for (1) is "almost diagonal".

Theorem 2.2. *Suppose that V can be written as $V=Q_x(x,t)/(t+1)$ where Q is real valued and*

$$\|Q\|_{L^\infty(\mathbb{T})}<\lambda(t+1)^{-\gamma},\ \|Q_x\|_{L^\infty(\mathbb{T})}<\lambda(t+1)^{1-\gamma},\quad \gamma>3/4 \tag{12}$$

Let $V_T=V\cdot\chi_{t>T}$ and u_T be corresponding solution. Then, we have

$$\int_{\mathbb{R}}\sup_{t>T}\|1-u_T(x,t,k)\|^4dk\lesssim\lambda^4(T+1)^{3-4\gamma}$$

Proof. We will suppress the dependence of u on T and will write u instead of u_T. Let $\widehat{u}_n(t,k) = e^{-in^2k/(t+1)}\widehat{\widetilde{u}}_n(t,k)$. For the zero Fourier mode of u, we have

$$\widehat{u}_0(t,k) = 1 + \sum_{|j|\geq 1}^{\infty} \int_T^t \frac{j\widehat{Q}_j(\tau)}{\tau+1} e^{ikj^2/(\tau+1)} \widehat{\widetilde{u}}_j(\tau,k)d\tau$$

Integration by parts gives

$$|\widehat{u}_0 - 1| \lesssim \left| \sum_{|j|\geq 1}^{\infty} \int_T^t \left(\int_\tau^t \frac{j\widehat{Q}_j(\tau_1)}{\tau_1+1} e^{ikj^2/(\tau_1+1)} d\tau_1 \right) \widehat{\widetilde{u}'}_j(\tau,k)d\tau \right|$$

From Cauchy-Schwarz and $\|\partial_t \widetilde{u}\| \lesssim \lambda(t+1)^{-\gamma}$, we have

$$\sup_{t>T} |\widehat{u}_0 - 1|^2 \lesssim$$

$$\lambda^2 \left(\int_T^\infty (\tau+1)^{-1-\epsilon} d\tau \right) \left(\int_T^\infty (\tau+1)^{-2\gamma+1+\epsilon} \sum_{|j|\geq 1}^{\infty} \left(\sup_{I\subseteq[\tau,\infty)} \left| \int_I \frac{j\widehat{Q}_j(\tau_1)}{\tau_1+1} e^{ikj^2/(\tau_1+1)} d\tau_1 \right| \right)^2 d\tau \right)$$

and the Carleson theorem implies (after the change of variables $\xi = (\tau_1+1)^{-1}$ for the integral in τ_1)

$$\int_{\mathbb{R}} \sup_{t>T} |\widehat{u}_0 - 1|^2 dk \lesssim \lambda^4 (T+1)^{3-4\gamma}$$

Then, notice that

$$\sum_j |\widehat{u}_j|^2 = 1, \quad 1 - |\widehat{u}_0|^2 \leq 2|1 - \widehat{u}_0| \tag{13}$$

and then

$$\|1-u\|^2 = |1-\widehat{u}_0|^2 + \sum_{j\neq 0} |\widehat{u}_j|^2 \lesssim |1-\widehat{u}_0|$$

This estimate finishes the proof. □

This theorem immediately implies for $T=0$ and small λ that for the positive measure set of k the solution has a zero mode bounded away from origin for all time. It also implies the localization on the Fourier side but only in the regime of small λ.

3. The Schrödinger flow with complex-valued V

In this section, we again study (3) but we do not assume that V is real-valued and thus the L^2 norm of the solution is not necessarily conserved. However, for generic k, not only the L^2 norm will be bounded but also the Sobolev norms $H^\alpha(\mathbb{T})$ where $\alpha < 1$. This is an improvement on the results from the second section however the bound will be exponential in time. The proof will be based on the concept of the variation norm as advocated in [9] with ideas going back to [8]. The key result from [9] we will use is the following estimate (see [9], (68), Appendix B): if $\mathcal{P} = \{\Delta_j\}, \Delta_j = [t_j, t_{j+1})$ is any partition of $\mathbb{R}$, then

$$\left\| \sup_{\mathcal{P}} \left(\sum_j \left| \int_{t_j}^{t_{j+1}} f(t)e^{ikt}dt \right|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R})} \lesssim \|f\|_p, \quad p\in[1,2) \tag{14}$$

We first restrict the problem (3) to the case of finite matrices. Instead of (3), consider

$$X_t = e^{-ik\Delta t} V^{(N)} e^{ik\Delta t} X, \quad X(0,k) = I_{(2N+1)\times(2N+1)} \tag{15}$$

where $V^{(N)} = P_N V P_N$ so we will be dealing with matrices of the finite size but the estimates we obtain must be independent of N. Assume first that the time $t \in [0,1]$, we will handle the intervals $[0,T]$ by scaling later.

Let us go to the Fourier side and then

$$\widehat{X}_t = \widetilde{V}^{(N)} \widehat{X}, \quad \widehat{X}(0,k) = I$$

where $\widetilde{V}$ is given by (4). Consider the scale $\ell^{2,\alpha}$ of the weighted ℓ^2 spaces with the norm

$$\|f\|_{2,\alpha} = \left(\sum_{|j|\le N} |f_j|^2 (1+|j|)^{2\alpha} \right)^{1/2}$$

If O is a linear operator in $\mathbb{C}^{2N+1}$, we will denote its operator norm in $\ell^{2,\alpha}$ by $\|O\|_\alpha$. On the group $\mathcal{G} = GL(2N+1,\mathbb{C})$, consider the following metric

$$d_{\mathcal{G}}(A,B) = \inf_\gamma \int_0^1 \|\gamma' \gamma^{-1}\|_\alpha dt$$

where $\gamma(t)$ is any continuously differentiable path in $\mathcal{G}$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Here we assume that both A and B lie in the same connected component of $GL(2N+1,\mathbb{C})$.

Remark. Let γ be any curve such that

$$\int_0^1 \|\gamma'\gamma^{-1}\|_\alpha dt < d_{\mathcal{G}}(A,B) + 1$$

Then

$$\gamma(t) = A + \int_0^t \gamma'(\tau)\gamma^{-1}(\tau)\gamma(\tau) d\tau$$

and therefore

$$\|\gamma(t)\|_\alpha \le \|A\|_\alpha + \int_0^t \|\gamma'(\tau)\gamma^{-1}(\tau)\|_\alpha \|\gamma(\tau)\|_\alpha d\tau$$

By Gronwall-Bellman, we have

$$\|B\|_\alpha \lesssim \|A\|_\alpha \exp\left(\int_0^1 \|\gamma'(\tau)\gamma^{-1}(\tau)\|_\alpha d\tau \right) \lesssim \|A\|_\alpha \exp\left(d_{\mathcal{G}}(A,B)\right) \tag{16}$$

Theorem 3.1. *Suppose $V(x,t) \in L^\infty(\mathbb{T}\times[0,1])$ and $X(t,k)$ is the solution to (15) on the interval $[0,1]$. Let $\alpha \in (0,1)$ and $p \in (4/3,2)$ so that $\alpha p < 2(p-1)$. Then, we have*

$$\sup_{0<t<1} \log(1 + \|\widetilde{X}(t,k)\|_\alpha) \lesssim 1 + \|V\|_\infty^{1+s_0} + \|V\|_\infty^{1-s_0^2} U^{s_0(1+s_0)}(k)$$

where

$$s_0 = \alpha p'/2 \quad \text{and} \quad \|U(k)\|_{p'} \lesssim \|V\|_\infty = \sup_{x\in\mathbb{T}, t\in[0,1]} |V(x,t)|$$

Proof. Recall [9] that for the continuous curve $\gamma(t)$ on $\mathcal{G}$ we can define

$$\|\gamma\|_{V^\beta} = \sup_{\mathcal{P}} \left(\sum_{j=0}^{n-1} d_{\mathcal{G}}^\beta(\gamma(t_j), \gamma(t_{j+1})) \right)^{1/\beta}, \quad \beta \in [1,\infty)$$

where $\mathcal{P}$ is any partition. Then, we have ([9], lemma C.3)

$$\|\gamma\|_{V^\beta} \le \|\gamma_r\|_{V^\beta} + C\min(\|\gamma_r\|^2_{V^\beta}, \|\gamma_r\|^\beta_{V^\beta}) \tag{17}$$

where

$$\gamma_r(t) = \int_0^t \gamma'(s)\gamma^{-1}(s)ds$$

and $\beta \in [1,2)$. Thus, (15), (16), (17) and the simple estimate (that follows from the definition of the variation norm)

$$d_{\mathcal{G}}(\gamma(0), \gamma(1)) \le \|\gamma\|_{V^\beta}$$

imply

$$\sup_{t\in[0,1]} \|\widehat{X}(t,k)\|_\alpha \lesssim \exp(Q + C\min\{Q^2, Q^\beta\}),\ Q = \left\| \int_0^t \widetilde{V}(\tau,k)d\tau \right\|_{V^\beta[0,1](\ell^{2,\alpha})} \tag{18}$$

and thus we only need to obtain a bound for Q. Let Λ^γ be a diagonal operator on the Fourier side with the diagonal elements equal to $(2+|n|)^\gamma$, $\gamma\in\mathbb{R}$.

To handle $\beta\in(1,2)$, we will use the standard complex interpolation between $\beta = 1$ and $\beta = 2$. Take $s_0\in(0,1)$ and $\mu\in(0,1)$. Suppose we fix a partition $\mathcal{P}$ of the interval $[0,1]$. Then, for any $s\in(0,1)$, take $\beta(s) = s+1, q(s) = (s+1)/s$. The ℓ^β norm can be written as

$$\left(\sum_j \left\| \int_{t_j}^{t_{j+1}} \Lambda^{s\mu}\widetilde{V}(\tau,k)\Lambda^{-s\mu}d\tau \right\|^{\beta(s)} \right)^{1/\beta(s)} =$$

$$\max_{\|\eta\|_{\ell^{q(s_0)}}=1} \sum_j \left\| \int_{t_j}^{t_{j+1}} \Lambda^{s\mu}\widetilde{V}(\tau,k)\Lambda^{-s\mu}d\tau \right\| |\eta_j|^{q(s_0)/q(s)}$$

since

$$\frac{1}{q(s)} + \frac{1}{\beta(s)} = 1$$

For the norm of the operator, we have another variational representation

$$\left\| \int_{t_j}^{t_{j+1}} \Lambda^{s\mu}\widetilde{V}(\tau,k)\Lambda^{-s\mu}d\tau \right\| =$$

$$\max_{\|f\|_{\ell^2}=\|g\|_{\ell^2}=1} \left| \int_{t_j}^{t_{j+1}} \sum_{|m|,|n|\le N} (2+|m|)^{s\mu}\widetilde{V}_{m,n}(\tau,k)(2+|n|)^{-s\mu} f_m g_n d\tau \right|$$

One can arrange the maximizers f', g' such that the last sum is equal to

$$\int_{t_j}^{t_{j+1}} \sum_{m,n} (2+|m|)^{s\mu}\widetilde{V}_{m,n}(\tau,k)(2+|n|)^{-s\mu} f'_m g'_n d\tau$$

i.e. the absolute value can be dropped. Thus, we only need to bound

$$F(s) = \sum_j |\eta_j|^{q(s_0)/q(s)} \int_{t_j}^{t_{j+1}} \sum_{m,n} (2+|m|)^{s\mu} \widetilde{V}_{m,n}(\tau,k)(2+|n|)^{-s\mu} f_m'^{(j)} g_n'^{(j)} d\tau$$

where $\|\eta\|_{\ell^{q(s_0)}} = 1$ and $\|f_j'\|_{\ell^2} = \|g_j'\|_{\ell^2} = 1$ for all j.

Notice that $F(s)$ is analytic in s in the strip $0 < \operatorname{Re} s < 1$ and we can apply the three lines lemma there [5].

$$M_0 = \sup_{\operatorname{Re} s=0} |F(s)| \le \sum_j (t_{j+1}-t_j) \sup_{t\in[t_j,t_{j+1}]} \|V(t)\| \lesssim \|V\|_{L^\infty(\mathbb{T}\times[0,1])}$$

and

$$M_1 = \sup_{\operatorname{Re} s=1} |F(s)| \le \left(\sum_j |\eta_j|^{q(s_0)}\right)^{1/2} \left(\sum_j \left\| \int_{t_j}^{t_{j+1}} \Lambda^\mu \widetilde{V}(\tau,k)\Lambda^{-\mu} d\tau \right\|^2\right)^{1/2}$$

$$\le \| \int_0^t \widetilde{V} \|_{V^2(\ell^{2,\mu})}$$

By the three line lemma, we have

$$|F(s_0)| \le M_0^{1-s_0} M_1^{s_0} = \|V\|_\infty^{1-s_0} \| \int_0^t \widetilde{V} \|^{s_0}_{V^2(\ell^{2,\mu})} \tag{19}$$

Taking the supremum over all partitions, we have the standard interpolation

$$\|\widetilde{V}\|_{V^{s_0+1}(\ell^{2,s_0\mu})} \le \|V\|_\infty^{1-s_0} \| \int_0^t \widetilde{V} \|^{s_0}_{V^2(\ell^{2,\mu})} \tag{20}$$

for any $s_0 \in [0,1]$. Next, we will focus on the bounds for the second variation norm because it enters into (20).

Lemma 3.1. *Suppose $p \in (4/3, 2)$ and $\mu = 2/p'$. Then, we have*

$$\left(\sup_{\mathcal{P}} \sum_j \|\Lambda^\mu \widetilde{V}_{\Delta_j}(k)\Lambda^{-\mu}\|^2\right)^{1/2} \lesssim \|V\|_\infty + U(k)$$

where

$$\|U(k)\|_{p'} \lesssim \|V\|_\infty, \quad \widetilde{V}_{\Delta_j}(k) = \int_{\Delta_j} \widetilde{V}(\tau,k) d\tau \tag{21}$$

Proof. We have

$$\Lambda^\mu \widetilde{V}_{\Delta_j}(k)\Lambda^{-\mu} = \widetilde{V}_{\Delta_j}(k) + [\Lambda^\mu, \widetilde{V}_{\Delta_j}(k)]\Lambda^{-\mu}$$

For the first term, we have an obvious estimate

$$\sum_j \|\widetilde{V}_{\Delta_j}(k)\|^2 \le \|V\|_\infty^2 \sum_j |\Delta_j|^2 \le \|V\|_\infty^2 \tag{22}$$

For the second one,

$$\sup_{\mathcal{P}} \sum_j \|[\Lambda^\mu, \widetilde{V}_{\Delta_j}(k)]\Lambda^{-\mu}\|^2_{\mathcal{S}_2} \lesssim \sum_{m,n} \left(\frac{|m|^\mu - |n|^\mu}{1+|n|^\mu}\right)^2 \sup_{\mathcal{P}} \sum_j \left| \int_{\Delta_j} \widehat{V}_{m-n}(t) e^{ik(m-n)(m+n)t} dt \right|^2$$

Let $\alpha = m - n$ and $\beta = m + n$. Notice that

$$||m|^\mu - |n|^\mu| \sim ||m| - |n|| \cdot ||m| + |n||^{\mu-1}$$

Therefore, we have two terms to bound

$$I_1 = \sum_{|\alpha|>1} \sum_{|\beta|\geq|\alpha|} \left(\frac{\alpha|\beta|^{\mu-1}}{1 + |\beta - \alpha|^\mu} \right)^2 \sup_{\mathcal{P}} \sum_j \left| \int_{\Delta_j} \widehat{V}_\alpha(t) e^{ik\alpha\beta t} dt \right|^2$$

and

$$I_2 = \sum_{|\alpha|>1} \sum_{|\beta|<|\alpha|} \left(\frac{|\alpha|^{\mu-1}|\beta|}{1 + |\beta - \alpha|^\mu} \right)^2 \sup_{\mathcal{P}} \sum_j \left| \int_{\Delta_j} \widehat{V}_\alpha(t) e^{ik\alpha\beta t} dt \right|^2$$

Now, (14) implies

$$\|I_2\|_{L^{p'/2}(\mathbb{R})} \lesssim \sum_{\alpha>1} \sum_{|\beta|<\alpha} \left(\frac{\alpha^{\mu-1}|\beta|}{1 + |\beta - \alpha|^\mu} \right)^2 \left\| \sup_{\mathcal{P}} \sum_j \left| \int_{\Delta_j} \widehat{V}_\alpha(t) e^{ik\alpha\beta t} dt \right|^2 \right\|_{L^{p'/2}(\mathbb{R})}$$

$$\lesssim \sum_{\alpha>1} \sum_{|\beta|<\alpha} \left(\frac{\alpha^{\mu-1}|\beta|}{|\alpha\beta|^{1/p'}(1 + |\beta - \alpha|^\mu)} \right)^2 \left(\int_0^1 |\widehat{V}_\alpha(t)|^p dt \right)^{2/p}$$

$$\leq \sum_{\alpha>1} \sum_{|\beta|<\alpha} \left(\frac{\alpha^{\mu-1}|\beta|}{|\alpha\beta|^{1/p'}(1 + |\beta - \alpha|^\mu)} \right)^2 \left(\int_0^1 |\widehat{V}_\alpha(t)|^2 dt \right) \lesssim$$

$$\sum_{\alpha>1} \|V_\alpha\|_2^2 (\alpha^{1-4/p'} + \alpha^{2\mu-4/p'}) \lesssim \|V\|_\infty^2$$

as long as

$$\frac{1}{2} < \mu \leq \frac{2}{p'}$$

For I_1, the estimate is similar

$$\|I_1\|_{L^{p'/2}(\mathbb{R})} \lesssim \sum_{\alpha>1} \sum_{|\beta|\geq\alpha} \left(\frac{\alpha|\beta|^{\mu-1}}{1 + |\beta - \alpha|^\mu} \right)^2 \left\| \sup_{\mathcal{P}} \sum_j \left| \int_{\Delta_j} \widehat{V}_\alpha(t) e^{ik\alpha\beta t} dt \right|^2 \right\|_{L^{p'/2}(\mathbb{R})}$$

$$\leq \sum_{\alpha>1} \sum_{|\beta|\geq\alpha} \left(\frac{\alpha|\beta|^{\mu-1}}{|\alpha\beta|^{1/p'}(1 + |\beta - \alpha|^\mu)} \right)^2 \left(\int_0^1 |\widehat{V}_\alpha(t)|^2 dt \right) \lesssim$$

$$\sum_{\alpha>1} \|V_\alpha\|_2^2 (\alpha^{2\mu-4/p'} + \alpha^{1-4/p'}) \lesssim \|V\|_\infty^2$$

Combining these bounds with (22), we have the statement of the lemma. □

The lemma gives a necessary bound for the variation norm so (18) and (20) then finish the proof of the theorem. □

The immediate corollary of this theorem is

Lemma 3.2. *Under the conditions of the theorem 3.1, assume that u is the solution to*

$$iu_t = (k\Delta + V)\,u, \quad u(x,0,k) = 1$$

Then, we have

$$\sup_{0<t<1} \log(1 + \|u(.,t,k)\|_{H^\alpha(\mathbb{T})}) \lesssim 1 + \|V\|_\infty^{1+s_0} + \|V\|_\infty^{1-s_0^2} U^{s_0(1+s_0)}(k)$$

where

$$\|U(k)\|_{p'} \lesssim \|V\|_\infty$$

Proof. Consider $u^{(N)}$ which solves

$$iu_t^{(N)} = \left(kP_N\Delta + V^{(N)}\right) u^{(N)}, \quad u^{(N)}(x,0,k) = 1$$

By approximating lemma (the lemma 4.1 in [3], which also works in our setting), we have

$$\sup_{t\in[0,1],\,k\in[-A,A]} \|u^{(N)}(x,t,k) - u(x,t,k)\| \to 0, \quad N \to \infty$$

for any fixed A. Therefore, given any fixed m, we have

$$\sup_{0<t<1} \log(1 + \|P_m u(x,t,k)\|_\alpha) \lesssim 1 + \|V\|_\infty^{1+s_0} + \|V\|_\infty^{1-s_0^2} U^{s_0(1+s_0)}(k)$$

with m-independent U. Taking $m \to \infty$, we have the statement of the lemma. □

We conclude this section with

Theorem 3.2. *If $V \in L^\infty(\mathbb{T} \times [0,T])$ and*

$$iu_t = (k\Delta + V)\,u, \quad u(x,0,k) = 1$$

then (under the conditions of the theorem 3.1)

$$\sup_{0<t<T} \log(1 + \|u(.,t,k)\|_{H^\alpha(\mathbb{T})}) \lesssim 1 + (T\|V\|_\infty)^{1+s_0} + (T\|V\|_\infty)^{1-s_0^2} U_1^{s_0(1+s_0)}(k)$$

and

$$\|U_1(k)\|_{p'} \lesssim T^{1-1/p'}\|V\|_\infty$$

Proof. It is sufficient to notice that $\psi(x,\tau,\kappa) = u(x,T\tau,\kappa/T)$ solves the problem

$$i\psi_\tau = \kappa\Delta\psi + TV(x,T\tau)\psi, \quad \psi(x,0,\kappa) = 1$$

for $\tau \in [0,1]$ and rescale using the lemma 3.2. □

The following corollary is immediate

Remark. Let $\alpha < 1$ and $V \in L^\infty(\mathbb{T} \times [0,\infty))$. We have

$$\sup_{t>0} \frac{\|u(x,t,k)\|_{H^\alpha(\mathbb{T})}}{\exp(t^2)} \leq C(k,\alpha) \tag{23}$$

for a.e. $k \in \mathbb{R}$. The simple example of $V(x,t) = i$ shows that the exponential growth is possible even for $L^2(\mathbb{T})$ norm. It is likely that $\exp(t^2)$ can be replaced by $\exp(t^\mu)$, $\mu > 1$.

4. The case of real V, small gaps, and the nondegenerate symbol

In this section, we will prove the localization result for a particular case of (1) when the symbol is nondegenerate, i.e.

$$P'(0) = 1 \neq 0$$

We will also assume that $c_j(t) = \alpha_j(t+1)^{-j}$ and α_j are constants. This is the hard case when the gaps between the eigenvalues of the differential operator are decreasing in t. That represents the real difficulty in the analysis of the multidimensional scattering [4].

In this paper, we will consider the quadratic polynomial only which leads to

$$iu_t = k\left(\frac{i\partial_x}{t+1} - \frac{\partial^2_{xx}}{(t+1)^2}\right)u + Vu, \quad u(x,0,k) = 1 \tag{24}$$

We will obtain the asymptotical result as $t \to \infty$ for the u with the standard WKB-type correction coming from the corresponding transport equation. This will done under the assumption that V is real and decays like $t^{-\gamma}$ with $\gamma < 1$ being very close to 1.

The following lemma is quite standard

Lemma 4.1. *Suppose $O_1(t)$ is an operator-valued function such that $\|O_1\| \in L^1[0,\infty)$. Consider two equations*

$$i\partial_t\psi_1 = O\psi_1, \quad \psi_1(0) = f_1$$

and

$$i\partial_t\psi_2 = (O + O_1)\psi_2 + j, \quad \psi_2(0) = f_2$$

where $O(t), O_1(t)$ are both self-adjoint and

$$\|j\|, \|O\|, \|O_1\| \in L^1_{\text{loc}}(\mathbb{R}^+)$$

Then,

$$\sup_{t\in[0,T]} \|\psi_1 - \psi_2\| \le \int_0^T (\|O_1(t)\| + \|j(t)\|)dt + \|f_2 - f_1\|$$

Proof. Let W_1 be the solution to

$$i\partial_t W_1(t_1,t) = O(t)W_1(t_1,t), \quad W_1(t_1,t_1) = I$$

Since O is self-adjoint, the operator W_1 is unitary. Then, by Duhamel's formula, we have

$$\psi_2(t) = W_1(0,t)f_2 - i\int_0^t W_1(\tau,t)(O_1(\tau)\psi_2(\tau) + j(\tau))d\tau$$

Since $W_1(0,t)f_1 = \psi_1$, $\|\psi_2\| = 1$, and W_1 is unitary,

$$\|\psi_2(t) - \psi_1(t)\| \le \|f_2 - f_1\| + \int_0^t (\|O_1(\tau)\| + \|j(\tau)\|)d\tau$$

□

This lemma will allow us to throw away any L^1 perturbations with small norm when the localization question is studied.

Restrict the problem (24) to the diadic intervals $I = [T, 2T]$ first. Then, on the Fourier side, we can write

$$i\widehat{u}_t = \left(k\left(\frac{n}{T}P_{[T^\alpha]} + Q_{[T^\alpha]}\left(\frac{n}{t+1} + \frac{n^2}{(t+1)^2}\right)\right) + \widehat{V} * + V_1\right)\widehat{u} \tag{25}$$

$$\widehat{u}(T,k) = \widehat{u}^{(0)}$$

where P_N and Q_N are defined in (5),

$$V_1 = kP_{[T^\alpha]}\frac{n^2}{(t+1)^2} + kP_{[T^\alpha]}\left(\frac{n}{t+1} - \frac{n}{T}\right)$$

and

$$\sup_{t\in I, k\in[-A,A]} \|V_1\| \lesssim T^{2\alpha-2} \tag{26}$$

for every fixed A.

We take

$$\alpha < 1/2 \tag{27}$$

to make sure that $\int_I \|V_1\|dt \sim T^{2\alpha-1}$ is small.

Consider the following problem now

$$i\phi_t = \left(k\left(\frac{n}{T}P_{[T^\alpha]} + Q_{[T^\alpha]}\left(\frac{n}{t+T+1} + \frac{n^2}{(t+T+1)^2}\right)\right) + \widehat{V}*\right)\phi, \tag{28}$$

$$\phi(x,0,k) = \phi^{(0)}(x,k)$$

It is obtained from (25) by dropping V_1 (the error made by doing that will be taken care of by lemma 4.1) and by shifting to the time interval $t \in [0, T]$.

Denote by $\nu = \exp(i\mu(x,t,k))$ the solution to the transport equation

$$i\nu_t = ik\nu_x/T + V\nu, \quad \nu(x,0,k) = 1$$

Theorem 4.1. *Let $\gamma \in (21/22, 1)$, $|V(x,t)| < C(t+1)^{-\gamma}$, and $\phi^{(0)}$ satisfies the following properties*

$$\sup_{k\in[-A,A]} \|\phi^{(0)}\|_{L^\infty(\mathbb{T})} \le 1, \qquad \sup_{k\in[-A,A]} \|\phi^{(0)}\|_{H^{1/2}(\mathbb{T})} \lesssim T^{1.5(1-\gamma)} \tag{29}$$

where $A > 1$ is fixed. Then, we have

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\phi(x,t,k) - \nu(x,t,k)\phi^{(0)}(x+kt/T,k)\|^4 dk \lesssim T^{-(1-\gamma)} \tag{30}$$

Proof. Write (28) in the block form

$$i\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{bmatrix} k\Lambda_1 + V_{11} & V_{12} \\ V_{21} & k\Lambda_2 + V_{22} \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{31}$$

where $k\Lambda_1 + V_{11} = P_{[T^\alpha]}\left(k\frac{n}{T} + \widehat{V}*\right)P_{[T^\alpha]}$. Notice that this operator is a restriction of the transport equation to the first T^α modes so we start with proving localization results for this operator.

Lemma 4.2. *Let $f(x,k)$ be such that*

$$\|f(x,k)\|_{L^\infty(\mathbb{T})} \le 1, \quad \|f(x,k)\|_{H^{1/2}(\mathbb{T})} \lesssim T^\beta$$

and y solve the problem

$$iy_t = (k\Lambda_1 + V_{11})y, \quad y(x,0,k) = P_{[T^\alpha]} f(x,k)$$

Then, we have the following representation

$$y(x,t,k) = \nu(x,t,k) f(x+kt/T,k) + \delta(x,t,k)$$

and

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\delta(x,k,t)\|^2 dk \lesssim T^{2-2\gamma+2\beta-\alpha} + T^{4-4\gamma-\alpha} \tag{32}$$

Proof. Notice that μ is real-valued. In ([3], lemma 2.1) we proved that

$$\int_{\mathbb{R}} \sup_{t\in[0,T]} \sum_j |j||\widehat{\mu}_j(t,k)|^2 dk \lesssim T \int_0^T \int_{\mathbb{T}} V^2(x,t)\,dxdt \lesssim T^{2-2\gamma}$$

That implies ([3], Appendix, lemma 6.2) that ν is unimodular and

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\nu(x,t,k)\|^2_{H^{1/2}(\mathbb{T})} dk \lesssim T^{2-2\gamma}$$

for any fixed A. Thus $\nu^{(1)}(x,t,k) = \nu(x,t,k) f(x+kt/T,k)$ solves the transport equation with initial data $f(x,k)$ and

$$\|\nu^{(1)}\|_{L^\infty(\mathbb{T})} \le 1$$

From the second lemma in Appendix, we have

$$\|\nu^{(1)}\|_{H^{1/2}(\mathbb{T})} \lesssim T^\beta + \|\nu\|_{H^{1/2}(\mathbb{T})}$$

Therefore

$$\int_{-A}^{A} \sup_{t\in[0,T]} \sum_{|n|>T^\delta} |\widehat{\nu}^{(1)}_n(t,k)|^2 dk \lesssim T^{-\delta+2-2\gamma} + T^{2\beta-\delta} \tag{33}$$

for every $\delta > 0$.

The function $\nu^{(2)} = P_{[T^\alpha]}\nu^{(1)}$ satisfies

$$i\nu^{(2)} = (k\Lambda_1 + V_{11})\nu^{(2)} + \delta^{(1)}, \quad \nu^{(2)}(x,0,k) = P_{[T^\alpha]} f(x,k)$$

where

$$\delta^{(1)} = P_{[T^\alpha]} V Q_{[T^\alpha]} \nu^{(1)}$$

and so (33) gives

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\delta^{(1)}\|^2 dk \lesssim T^{-2\gamma}(T^{-\alpha+2-2\gamma} + T^{2\beta-\alpha}) \tag{34}$$

The lemma 4.1 yields

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|y - \nu^{(2)}\|^2 dk \lesssim T^{2-2\gamma}(T^{-\alpha+2-2\gamma} + T^{2\beta-\alpha})$$

By (33),

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\nu^{(1)} - \nu^{(2)}\|^2 dk \lesssim T^{2\beta-\alpha} + T^{2-2\gamma-\alpha}$$

This gives (32). $\square$

Let us choose $\beta = 1.5(1-\gamma)$ as in (29) and consider the solution to (31). Notice first that

$$\|Q_{[T^\alpha]}\phi^{(0)}\| \lesssim T^{\beta-\alpha/2}$$

so, since evolution preserves the $L^2(\mathbb{T})$ norm, we have

$$\sup_{t\in[0,T]} \|\phi(x,t,k) - \zeta(x,t,k)\| \lesssim T^{\beta-\alpha/2} \tag{35}$$

where ζ solves (28) with initial condition $\zeta(x,0,k) = \zeta^{(0)} = P_{[T^\alpha]}\phi^{(0)}$, i.e.

$$i\partial_t \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{bmatrix} k\Lambda_1 + V_{11} & V_{12} \\ V_{21} & k\Lambda_2 + V_{22} \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad \zeta_1(0,k) = P_{[T^\alpha]}\phi^{(0)}, \zeta_2(0,k) = 0 \tag{36}$$

Let $W(t,k)$ be defined as

$$i\partial_t W = e^{ik\Lambda_1 t} V_{11} e^{-ik\Lambda_1 t} W, \quad W(0,k) = I$$

If

$$\zeta_1 = e^{-ik\Lambda_1 t} W \widetilde{\zeta}_1, \quad \zeta_2 = e^{-ik\Lambda_2 t}\widetilde{\zeta}_2$$

then

$$i\partial_t \begin{pmatrix} \widetilde{\zeta}_1 \\ \widetilde{\zeta}_2 \end{pmatrix} = \begin{bmatrix} 0 & W^{-1} e^{ik\Lambda_1 t} V_{12} e^{-ik\Lambda_2 t} \\ e^{ik\Lambda_2 t} V_{21} e^{-ik\Lambda_1 t} W & e^{ik\Lambda_2 t} V_{22} e^{-ik\Lambda_2 t} \end{bmatrix} \begin{pmatrix} \widetilde{\zeta}_1 \\ \widetilde{\zeta}_2 \end{pmatrix} \tag{37}$$

For $\widetilde{\zeta}_1$,

$$\widetilde{\zeta}_1(t,k) = \zeta^{(0)} - i\int_0^t W^{-1}(\tau,k) e^{ik\Lambda_1\tau} V_{12}(\tau) e^{-ik\Lambda_2\tau} \widetilde{\zeta}_2(\tau,k) d\tau$$

We take an inner product with $\zeta^{(0)}$ and integrate by parts

$$\langle \widetilde{\zeta}_1, \zeta^{(0)} \rangle = \|\zeta^{(0)}\|^2 - i\int_0^t \langle e^{ik\Lambda_1\tau} V_{12}(\tau) e^{-ik\Lambda_2\tau} \widetilde{\zeta}_2(\tau,k), W(\tau,k)\zeta^{(0)} \rangle d\tau \tag{38}$$

As $\|\widetilde{\zeta}_1\|^2 + \|\widetilde{\zeta}_2\|^2 = \|\zeta^{(0)}\|^2$, it is sufficient to show that the second term in (38) is small to guarantee that $\widetilde{\zeta}_1 = \zeta^{(0)} +$ "small" and $\widetilde{\zeta}_2 =$ "small" (see lemma 5.1 from Appendix).

Let

$$\widetilde{V}_{12}(t,k) = e^{ik\Lambda_1 t} V_{12}(t) e^{-ik\Lambda_2 t}$$

and write

$$\int_0^t \langle \widetilde{V}_{12}(\tau,k)\widetilde{\zeta}_2(\tau,k), \widetilde{y}(\tau,k) \rangle d\tau = I_1 + I_2$$

Here

$$\widetilde{y}(t,k) = W(t,k)\zeta^{(0)},\ I_1 = \int_0^t \langle \widetilde{\zeta}_2, \widetilde{V}_{12}^* P_{[T^\delta]}\widetilde{y} \rangle d\tau,\ I_2 = \int_0^t \langle \widetilde{\zeta}_2, \widetilde{V}_{12}^* Q_{[T^\delta]}\widetilde{y} \rangle d\tau$$

where δ will be chosen later. From the lemma 4.2, we have

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|Q_{[T^\delta]}\widetilde{y}(t,k)\|_2^2 dk \lesssim (T^{2-2\gamma} + T^{2\beta})T^{-\delta} + T^{2-2\gamma+2\beta-\alpha} + T^{4-4\gamma-\alpha}$$

since

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|Q_{[T^\delta]}\Big(\nu f\Big)\|^2 dk \lesssim T^{-\delta}(T^{2\beta}+T^{2-2\gamma})$$

Therefore, by Cauchy-Schwarz,

$$\left(\int_{-A}^{A} \sup_{t\in[0,T]} |I_2(t,k)|^2 dk\right)^{1/2} \lesssim T^{1-\gamma}\left((T^{1-\gamma}+T^{\beta})T^{-\delta/2}+T^{1-\gamma+\beta-\alpha/2}+T^{2-2\gamma-\alpha/2}\right)\sim T^{-\epsilon_1}$$

and

$$\epsilon_1=\min\{\delta/2-2(1-\gamma),\delta/2-\beta-(1-\gamma),\alpha/2-\beta-2(1-\gamma),\alpha/2-3(1-\gamma)\}$$

which yields the following conditions

$$2(1-\gamma)<\delta/2,\ 1-\gamma+\beta<\delta/2,\ 2(1-\gamma)<\alpha/2-\beta,\ 3(1-\gamma)<\alpha/2$$

For I_1, we integrate by parts and use $\widetilde{\zeta}_2(0,k)=0$ to get

$$I_1=\int_0^t \langle\widetilde{\zeta}_2', Q(\tau,t,k)\widetilde{y}\rangle d\tau+\int_0^t\langle\widetilde{\zeta}_2, Q(\tau,t,k)\widetilde{y}'\rangle d\tau$$

and

$$Q(\tau,t,k)=\int_\tau^t V_{12}^*(\tau_1,k)P_{[T^\delta]}d\tau_1$$

For the derivatives, we have

$$\|\widetilde{\zeta}_2'\|\lesssim T^{-\gamma},\ \|\widetilde{y}'\|\lesssim T^{-\gamma}$$

and for the Hilbert-Schmidt norm of Q,

$$\sup_{t,\tau\in[0,T]}\|Q\|_{\mathcal{S}_2}^2\lesssim\sum_{|l|>T^\alpha}\sum_{|j|<T^\delta}\sup_{\tau\in[0,T]}\left|\int_\tau^T \widehat{V}_{l-j}(\tau_1)\exp\left(ik(l/T+l^2/T^2-j/T)\tau_1\right)d\tau_1\right|^2$$

The Carleson's theorem on the maximal function again gives

$$\int\sup_{t,\tau\in[0,T]}\|Q\|_{\mathcal{S}_2}^2 dk\lesssim T^{1-2\gamma+1-\alpha}\cdot T^\delta$$

as long as

$$\delta<\alpha \tag{39}$$

That yields an estimate for I_1

$$\left(\int|\sup_{t\in[0,T]} I_1|^2 dk\right)^{1/2}\lesssim T^{2(1-\gamma)}\cdot T^{(\delta-\alpha)/2}$$

Combining the bounds obtained above, we have

$$\left(\int_{-A}^{A}\sup_{t\in[0,T]}|\langle\widetilde{\zeta}_1(t,k),\zeta^{(0)}\rangle-\|\zeta^{(0)}\|^2|^2dk\right)^{1/2}\lesssim T^{-\epsilon}$$

where

$$\epsilon=\min\{\epsilon_1,(\alpha-\delta)/2-2(1-\gamma)\}$$

so we add one more condition on the parameters

$$\alpha>\delta+4(1-\gamma) \tag{40}$$

Consequently, the lemma 5.1 from Appendix gives

$$\int \sup_{t\in[0,T]} \|\zeta(t,k) - e^{-ik\Lambda_1 t} W(0,t,k)\zeta^{(0)}\|^4 dk \lesssim T^{-2\epsilon}$$

since

$$\widetilde{\zeta}_1 = W^{-1} e^{ik\Lambda_1 t}\zeta_1$$

For $e^{-ik\Lambda_1 t}W(0,t,k)\zeta^{(0)}$, the lemma 4.2 is applicable and we have

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\phi - \nu\phi^{(0)}(x+kt/T,k)\|^4 dk \lesssim T^{4\beta-2\alpha} + T^{-2\epsilon} + T^{2-2\gamma+2\beta-\alpha} + T^{4-4\gamma-\alpha}$$

by using (35) and $\|\delta(x,k,t)\| \lesssim 1$. For simplicity, let us make the following choices of our parameters:

$$\delta = 6(1-\gamma),\ \alpha = 11(1-\gamma)$$

and so

$$\int_{-A}^{A} \sup_{t\in[0,T]} \|\phi - \nu\phi^{(0)}\|^4 dk \lesssim T^{-(1-\gamma)}$$

Since we will need $\alpha < 1/2$ (check (27)), the condition $\gamma > 21/22$ follows. □

Now, we are ready to prove the similar statement for the problem (25).

Theorem 4.2. *Let $\gamma \in [83/87, 1)$, $|V(x,t)| < C(t+1)^{-\gamma}$, and $u^{(0)}$ satisfies the following properties*

$$\sup_{k\in[-A,A]} \|u^{(0)}\|_{L^\infty(\mathbb{T})} \leq 1, \qquad \sup_{k\in[-A,A]} \|u^{(0)}\|_{H^{1/2}(\mathbb{T})} \lesssim T^{1.5(1-\gamma)}$$

where $A > 1$ is fixed. Then, for the solution of (25), we have

$$\int_{-A}^{A} \sup_{t\in[T,2T]} \|u(x,t,k) - \nu(x,t,k)u^{(0)}(x+kt/T,k)\|^4 dk \lesssim T^{-(1-\gamma)} \tag{41}$$

Proof. Apply the theorem 4.1, lemma 4.1, and (26) to get

$$\int_{-A}^{A} \sup_{t\in[T,2T]} \|u(x,t,k) - \nu(x,t,k)u^{(0)}(x+kt/T,k)\|^4 dk \lesssim T^{-(1-\gamma)} + T^{4(2\alpha-1)} \tag{42}$$

If $\gamma \geq 83/87$, the first term is larger and we have the statement of the theorem. □

Let us define the solution to the following transport equation

$$iG_t = ikG_x/(1+t) + VG, \quad G(x,0,k) = 1$$

Lemma 4.3. *For an arbitrary fixed $A > 0$, we have*

$$\int_{-A}^{A} \sup_{t<T} \|G(x,t,k)\|^2_{H^{1/2}(\mathbb{T})} dk \lesssim \int_0^T (1+t)\int_{\mathbb{T}} V^2(x,t)dxdt \tag{43}$$

Proof. Consider

$$H(x,t) = G(x, e^t - 1)$$

For H, we have

$$iH_t(x,t,k) = ikH_x(x,t,k) + V_s(x,t)H(x,t,k), \quad H(x,0,k) = 1$$

where

$$V_s(x,t) = e^t V(x, e^t - 1)$$

From [3], we have

$$\int \sup_{t\in[0,\tau]} \sum_j |j||\widehat{H}_j(t,k)|^2 dk \lesssim \int_0^\tau \int_{\mathbb{T}} V_s^2(x,t)dxdt$$

Taking $\tau = \log T$ with large T, we get the statement of the lemma. □

Now that we can control the behavior of u on every diadic interval, we can prove the main result of this section

Theorem 4.3. *Assume that $\gamma \in [83/87, 1)$ and V in (24) satisfies*

$$\|V(x,t)\|_{L^\infty(\mathbb{T})} \le \lambda(1+t)^{-\gamma}$$

Then

$$\int_{-A}^{A} \sup_{t>0} \|u(x,t,k) - G(x,t,k)\| dk \to 0 \tag{44}$$

as $\lambda \to 0$. Here A is any positive number.

Proof. Take $T_j = 2^j$ and consider the diadic intervals $[T_j, T_{j+1})$. On any fixed interval $[0, T_j]$ we have

$$\sup_{t\in[0,T_j]} \|u - 1\| \to 0, \ \sup_{t\in[0,T_j]} \|G - 1\| \to 0$$

when $\lambda \to 0$. This convergence is uniform in $k \in [-A, A]$. Therefore, it is sufficient to assume that we solve the problem on the interval $[T_N, \infty)$ instead where N is sufficiently large.

The estimate (43) implies that

$$\int_{-A}^{A} \|G(x,T_j,k)\|^2_{H^{1/2}(\mathbb{T})} dk \lesssim T_j^{2-2\gamma}$$

Therefore,

$$\int_{-A}^{A} \sum_j \frac{\|G(x,T_j,k)\|^2_{H^{1/2}(\mathbb{T})}}{T_j^{2\beta}} dk < \infty, \quad \beta \in (1-\gamma, 1.5(1-\gamma)) \tag{45}$$

If ν solves

$$i\nu_t = ikq(t)\nu_x + V\nu, \quad \nu(x,0,k) = 1$$

with q given by: $q(t) = T_j^{-1}$, $t \in [T_j, T_{j+1})$, then

$$\sup_{t>0} \|\nu - G\| \to 0$$

as $\lambda \to 0$ and similarly

$$\int_{-A}^{A} \sum_j \frac{\|\nu(x,T_j,k)\|^2_{H^{1/2}(\mathbb{T})}}{T_j^{2\beta}} dk < \infty, \quad \beta \in (1-\gamma, 1.5(1-\gamma)) \tag{46}$$

Then, we can consider Ω_λ: the set of those $k \in [-A, A]$ for which

$$\sum_j \frac{\|\nu(x, T_j, k)\|^2_{H^{1/2}(\mathbb{T})}}{T_j^{2\beta}} dk < 1$$

As $\lambda \to 0$, the measure of $\{[-A, A]\backslash\Omega_\lambda\}$ will converge to 0. Then, restricting k to the set Ω_λ, we can recursively apply theorem 4.2 to show that

$$\sup_{t>0} \|u - G\|$$

converges to zero in measure (as a function in $k \in \Omega_\lambda$) provided that $\lambda \to 0$. This is equivalent to (44) since $\|u\| = \|G\| = 1$. □

Remark. It is conceivable that the constant 83/87 can be decreased by more efficient choice of parameters. This theorem is important since it can handle a difficult case of $\gamma < 1$ for very general class of pseudodifferential operators. Indeed, we used the fact that V is multiplication operator but the polynomial $P(x)$ can be replaced by other nondegenerate symbols.

5. Appendix

We used the following elementary lemma in the main text.

Lemma 5.1. *Suppose $\mathcal{V}$ is a vector space with the inner product and v, a are two vectors such that $v = v_1 + v_2$, and*

$$v_1 \perp v_2, \quad a \perp v_2, \quad \|v_1\|^2 + \|v_2\|^2 = \|a\|^2$$

Then,

$$\|v - a\| \leq 2\sqrt{|\|a\|^2 - \langle v_1, a\rangle|} \tag{47}$$

Proof. Assume first that $\|a\| = 1$. We have an orthogonal decomposition

$$v = v_2 + \langle v_1, a\rangle a + \psi$$

Therefore

$$\|v_2\|^2 + \|\psi\|^2 + |\langle v_1, a\rangle|^2 = 1 \tag{48}$$

and

$$\|v - a\|^2 = \|v_2\|^2 + \|\psi\|^2 + |1 - \langle v_1, a\rangle|^2$$

However, (48) implies

$$\|v_2\|^2 + \|\psi\|^2 = 1 - |\langle v_1, a\rangle|^2$$

and so we have

$$\|v - a\|^2 \leq 4|1 - \langle v_1, a\rangle|$$

If $\|a\| \neq 1$, rescale the vectors by $\|a\|$ to get (47). □

The following lemma is quite standard in the description of Krein's algebra (e.g., [1], p. 123, formula (5.2) or [12], proposition 6.1.10)

Lemma 5.2. *Let $f, g \in H^{1/2}(\mathbb{T}) \cap L^\infty(\mathbb{T})$, then*

$$\|fg\|_{H^{1/2}(\mathbb{T})} \lesssim \|f\|_{L^\infty(\mathbb{T})}\|g\|_{H^{1/2}(\mathbb{T})} + \|g\|_{L^\infty(\mathbb{T})}\|f\|_{H^{1/2}(\mathbb{T})}$$

Proof. The proof immediately follows, e.g., from

$$\|f\|^2_{H^{1/2}(\mathbb{T})} \sim \|f\|^2 + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy$$

□

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