

ON THE GROWTH OF THE POLYNOMIAL ENTROPY INTEGRALS FOR MEASURES IN THE SZEGŐ CLASS

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ABSTRACT. Let σ be a probability Borel measure on the unit circle \mathbb{T} and $\{\phi_n\}$ be the orthonormal polynomials with respect to σ . We say that σ is a Szegő measure, if it has an arbitrary singular part σ_s , and $\int_{\mathbb{T}} \log \sigma' dm > -\infty$, where σ' is the density of the absolutely continuous part of σ , m being the normalized Lebesgue measure on \mathbb{T} . The entropy integrals for ϕ_n are defined as

$$\epsilon_n = \int_{\mathbb{T}} |\phi_n|^2 \log |\phi_n| d\sigma$$

It is not difficult to show that $\epsilon_n = \bar{o}(\sqrt{n})$. In this paper, we construct a measure from the Szegő class for which this estimate is sharp (over a subsequence of n 's).

1. INTRODUCTION

Let σ be a probability Borel measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. The moments $c_k = c_k(\sigma)$, the Schur parameters $\gamma_k = \gamma_k(\sigma)$, the orthonormal polynomials $\phi_n = \phi_n(\sigma)$ with respect to the measure as well as their monic versions $\Phi_n = \Phi_n(\sigma)$ are defined in the standard way, see Simon [7, Ch. 1] for definitions and terminology. We often indicate the dependence on the measure explicitly to avoid the misunderstanding.

It is quite reasonable to ask the following question: does some additional condition on the measure provide nontrivial bounds on the size of the polynomials ϕ_n beyond the normalization

$$\int_{\mathbb{T}} |\phi_n(z)|^2 d\sigma = 1 ?$$

The size can be controlled by $L^p(d\sigma)$ norm ($p > 2$) or by other quantities. This problem is classical and was addressed, for instance, in the framework of Steklov's conjecture [6] by Rakhmanov (see also [1]) where the $L^\infty(\mathbb{T})$ norms were studied.

In this paper, we measure the size of the orthonormal polynomials by taking the entropy integrals

$$(1.1) \quad \epsilon_n(\sigma) = \int_{\mathbb{T}} |\phi_n|^2 \log |\phi_n| d\sigma$$

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Set $\log x = \log^+ x - \log^- x$. Since $0 \leq x^2 \log^- x \leq 1/(2e)$, $x \geq 0$, one has

$$(1.2) \quad \int_{\mathbb{T}} |\phi_n|^2 \log^- |\phi_n| d\sigma < 1$$

Now, if one considers

$$\epsilon_n^+ = \int_{\mathbb{T}} |\phi_n|^2 \log^+ |\phi_n| d\sigma$$

it becomes clear that only ϵ_n^+ can contribute to the growth of ϵ_n .

We say that σ is a Szegő measure (notation: $\sigma \in (S)$) if its singular part σ_s is arbitrary and

$$\int_{\mathbb{T}} \log \sigma' dm > -\infty,$$

where σ' is the density of the absolutely continuous (a.c., for shorthand) part of σ and $dm = dm(t) = d\theta/(2\pi)$, $t = e^{i\theta} \in \mathbb{T}$ is the normalized Lebesgue measure on \mathbb{T} . One might think that the Szegő condition is relevant to the entropy integrals for the following reason. Assume first that σ is purely absolutely continuous with the smooth positive density: $d\sigma = p(\theta)dm$ and $p(\theta) = |\pi(\theta)|^{-2}$ where $\pi(z)$ is an outer function on \mathbb{D} such that $\pi^{-1}(z)$ is in the Hardy space $H^2(\mathbb{D})$. Then, one can easily show that $\phi_n^*(z)$ goes to $\pi(z)$ uniformly on $\overline{\mathbb{D}}$. What happens to the entropy integrals? Obviously,

$$(1.3) \quad \epsilon_n = \int_{\mathbb{T}} |\phi_n|^2 \log |\phi_n| \frac{dm}{|\pi(z)|^2} \rightarrow \int_{\mathbb{T}} \log |\pi(z)| dm$$

Now, if one considers $\sigma \in (S)$ instead, then the convergence of the polynomials is not uniform but the right-hand side in (1.3) does exist. So, one can conjecture that ϵ_n has a limit without any smoothness assumptions and that the only condition needed is $\sigma \in (S)$. This conjecture is well-known in the orthogonal polynomials community and attracted some attention recently (see Beckermann et al. [4] and Aptekarev et al. [2, 3]). In [4], for example, the entropies were studied for the polynomials on the real line and under additional assumption that the measure is absolutely continuous.

In this paper, we do not make this additional assumption. We conjecture that the construction from theorem 1.1 can be adjusted to produce an a.c. measure σ (see remark 2.4 below).

In the following theorem, we construct a Szegő measure with unbounded ϵ_n 's thus proving that the above reasoning (1.3) is not true for general Szegő measures.

Theorem 1.1. *Let $h : \mathbb{N} \rightarrow \mathbb{R}_+$ be an arbitrary positive decreasing function such that $\lim_{n \rightarrow \infty} h(n) = 0$. Then there is $\sigma \in (S)$ and a sequence $\{M_k\}$ such that*

$$\epsilon_{M_k}(\sigma) = \int_{\mathbb{T}} |\phi_{M_k}(\sigma)|^2 \log |\phi_{M_k}(\sigma)| d\sigma \geq h(M_k) \cdot \sqrt{M_k}$$

as $k \rightarrow \infty$.

It follows from the discussion in section 2 that this result is sharp and, in particular, bound (2.5) cannot be improved.

The by-product of the proof of this theorem is a result on the growth of other integrals that measure the size of the polynomials, see corollary 3.1.

The simple counterpart of theorem 1.1 also holds for orthogonal polynomials with respect to a Szegő measure on an interval of the real line.

Let $F, G > 0$ be two functions. We write $F \gtrsim G$ ($F \lesssim G$) if $F \geq cG$ ($F \leq cG$, respectively) with an absolute constant $c > 0$ for all values of the arguments. Naturally, writing $F \simeq G$ stands for $F \lesssim G$ and $F \gtrsim G$ simultaneously; equivalently, this means that $0 < c_1 \leq F/G \leq c_2 < \infty$ with absolute constants $c_1 > 0, c_2 > 0$ for all arguments. The symbol $\xrightarrow{*}$ stands for the weak- $(*)$ convergence of measures. Non-essential constants are denoted by C and they can change from one formula to another.

2. PRELIMINARIES AND THE MAIN LEMMA

We begin with several simple observations:

- If $\|\gamma\|_\infty \leq 1/2$, one has

$$(2.1) \quad \int_{\mathbb{T}} \log \sigma' dm = \sum_k \log(1 - |\gamma_k|^2) \simeq - \sum_k |\gamma_k|^2$$

where the both sides could be equal to $-\infty$. They are finite iff $\sigma \in (S)$ (see [8], [7, p. 136, formula (2.3.1)]).

- Let κ_n be the leading coefficient of ϕ_n . It is well-known that

$$(2.2) \quad \kappa_n^2 = \frac{1}{\prod_{k=0}^{n-1} (1 - |\gamma_k|^2)},$$

so $\sup_n \kappa_n < \infty$ iff $\sigma \in (S)$. Hence, we can study the entropy of monic polynomials $\Phi_n = \phi_n/\kappa_n$ instead, *i.e.*

$$(2.3) \quad \widehat{\epsilon}_n = \int_{\mathbb{T}} |\Phi_n|^2 \log |\Phi_n| d\sigma, \quad \widehat{\epsilon}_n^+ = \int_{\mathbb{T}} |\Phi_n|^2 \log^+ |\Phi_n| d\sigma$$

We will do just that, the estimates obtained will imply theorem 1.1.

- An upper bound for $\widehat{\epsilon}_n$ is easy to obtain. Recalling the Szegő recurrence formulas [7, theorem 1.5.2] (notice that our γ_n are $-\alpha_n$ from the book)

$$(2.4) \quad \begin{cases} \Phi_{n+1} &= z\Phi_n + \bar{\gamma}_n \Phi_n^*, & \Phi_0 &= 1, \\ \Phi_{n+1}^* &= \Phi_n^* + \gamma_n z \Phi_n, & \Phi_0^* &= 1 \end{cases}$$

and $|\Phi_n(z)| = |\Phi_n^*(z)|$, $z \in \mathbb{T}$ we see that

$$|\Phi_n(z)| \leq \prod_{k=0}^{n-1} (1 + |\gamma_k|)$$

for $z \in \mathbb{T}$. Since $\sigma \in (S)$, one has $\{\gamma_k\} \in \ell^2(\mathbb{Z}_+)$, and

$$\begin{aligned} \log |\Phi_n(z)| &\leq \sum_{k=0}^{n-1} \log(1 + |\gamma_k|) \leq \sum_{k=0}^{n-1} |\gamma_k| \\ &= \sum_{k=0}^{[\sqrt{n}]} |\gamma_k| + \sum_{k=[\sqrt{n}]+1}^{n-1} |\gamma_k| \leq n^{1/4} \|\{\gamma_k\}\|_2 + n^{1/2} v_n \end{aligned}$$

by Cauchy-Schwarz inequality. Above, $v_n = \left(\sum_{k=[\sqrt{n}]+1}^{n-1} |\gamma_k|^2 \right)^{1/2} \rightarrow 0$ as n goes to infinity, and $[x]$ is the integer part of a real number x .

Consequently, the right-hand side of the above inequality is $\bar{o}(\sqrt{n})$, and

$$(2.5) \quad \widehat{\epsilon}_n = \int_{\mathbb{T}} |\Phi_n|^2 \log |\Phi_n| d\sigma \leq \left(\sum_{k=0}^{n-1} |\gamma_k| \right) \int_{\mathbb{T}} |\Phi_n|^2 d\sigma = \bar{o}(\sqrt{n})$$

Now we need to introduce some definitions to be used later in the text.

Let μ be a probability measure on \mathbb{T} with Schur parameters $\{\gamma_k(\mu)\}$ and corresponding orthogonal polynomials $\{\phi_n(\mu)\}$. Given integers $N' < N$ and arbitrary $\kappa > 0$, we introduce the so-called $(N', N; \kappa)$ -transformation of the measure (or, equivalently, of its Schur parameters). Strictly speaking, the $(N', N; \kappa)$ -transformation depends also on $\{\gamma'_k\}_{k=N'+1, \dots, N}$, a “new interval” of Schur parameters we want to “incorporate” into $\{\gamma_k\}$. However, we will suppress this dependence to keep the notation reasonably simple.

Definition of $(N', N; \kappa)$ -transformation.

First, consider

$$d\mu_0[\mu] = \frac{dm}{|\phi_{N'+1}(\mu)|^2}$$

This measure is the so-called Bernstein-Szegő approximation to $d\mu$. Its Schur coefficients $\gamma_k(\mu_0)$ satisfy ([8, 7]): $\gamma_k(\mu_0) = \gamma_k(\mu)$, $k = 0, \dots, N'$, $\gamma_k(\mu_0) = 0$, $k > N'$. Secondly, define the new sequence of Schur parameters by

$$(2.6) \quad \gamma_k(\mu_1) = \begin{cases} \gamma_k(\mu_0) & , \quad k = 0, \dots, N' \\ \gamma'_k & , \quad k = N' + 1, \dots, N \\ 0 & , \quad k > N \end{cases}$$

This corresponds of course to writing $d\mu_1[\mu] = 1/|\phi_{N+1}(\mu)|^2 dm$ and the polynomial $\phi_{N+1}(d\mu_1)$ is defined by Schur parameters through (2.4). Next, we let

$$(2.7) \quad d\sigma[\mu] = \frac{1}{1 + \kappa} (d\mu_1 + \kappa d\delta_1)$$

where δ_1 is the Dirac's delta measure at $z = 1$ on the unit circle. The measure σ and its Schur coefficients $\{\gamma_k(\sigma)\}$ are called the $(N', N; \kappa)$ -transformation of μ and its Schur coefficients $\{\gamma_k(\mu)\}$, respectively. Notice that the normalization in (2.7) guarantees that σ is a probability measure.

We now define the functions $\Gamma_n, \Psi_n : \mathbb{R}^n \rightarrow \mathbb{R}_+$ depending on n -tuples $\{x_k\}_{k=1, \dots, n}$ as

$$(2.8) \quad \begin{aligned} \Gamma_n &= \Gamma_n(\{x_k\}) = \frac{\left(\sum_{j=1}^n x_j \right) \exp \left(\sum_{j=1}^n x_j \right)}{\sum_{j=1}^n \exp \left(\sum_{k=1}^j x_k \right)} \\ \Psi_n &= \Psi_n(\{x_k\}) = \frac{\exp \left(\sum_{j=1}^n x_j \right)}{\sum_{j=1}^n \exp \left(\sum_{k=1}^j x_k \right)} \end{aligned}$$

Lemma 2.1. *Let $0 < L < 10^{-6}$ be arbitrary small. Then, there exists an increasing sequence $\{N_k\} \subset \mathbb{N}$ and the corresponding N_k -tuples*

$\{x_{k,j}\}_{j=1,\dots,N_k}$, $0 \leq x_{k,j} < 1$, such that

$$(2.9) \quad \sum_{j=1}^{N_k} x_{k,j}^2 \simeq L^2$$

and

$$(2.10) \quad \Gamma_{N_k}(\{x_{k,j}\}) \gtrsim L^4 \sqrt{N_k}, \quad \Psi_{N_k}(\{x_{k,j}\}) \gtrsim L^3$$

Proof. Rewrite Γ_n as

$$\Gamma_n = \Gamma_n(\{x_t\}) = \frac{\sum_{t=1}^n x_t}{1 + \sum_{m=1}^{n-1} \exp(-\sum_{t=m+1}^n x_t)}$$

and make the change of summation index $t \rightarrow n - t + 1$, $\hat{x}_t = x_{n-t+1}$. We have

$$(2.11) \quad \Gamma_n = \Gamma_n(\{x_t\}) = \frac{\sum_{t=1}^n \hat{x}_t}{1 + \sum_{m=1}^{n-1} \exp(-\sum_{t=1}^m \hat{x}_t)}$$

Define the sequence $\{N_k\}$ by recursion

$$(2.12) \quad N_{k+1} = N_k + \left[\frac{1}{k^2} \exp \left(L \sum_{j=1}^k \frac{\sqrt{N_j - N_{j-1}}}{j^2} \right) \right], \quad k = 1, 2, \dots$$

where $N_0 = 0$ and $[y]$ is the integer part of a real y . Take $N_1 = 10L^{-3}$. Then, one obtains by induction that

$$(2.13) \quad N_{k+1} - N_k \geq L^{-3(k+1)}$$

and

$$(2.14) \quad N_{k+1} - N_k \simeq N_{k+1}$$

for any $k = 0, 1, \dots$ and for $0 < L < 10^{-6}$.

Indeed, let us consider (2.13). For $k = 0$, this follows from the choice of N_0 and N_1 . For $k > 0$, we have

$$(2.15) \quad N_{k+1} - N_k \geq \frac{1}{k^2} \exp \left(\frac{L \sqrt{N_k - N_{k-1}}}{k^2} \right) - 1$$

by (2.12). Assuming that (2.13) holds for $k - 1$, we continue as

$$N_{k+1} - N_k \geq \frac{1}{k^2} \exp \left(\frac{L^{1-3k/2}}{k^2} \right) - 1$$

It is left to notice that

$$\frac{1}{k^2} \exp \left(\frac{L^{1-3k/2}}{k^2} \right) - 1 > L^{-3(k+1)}$$

holds for any $k = 1, 2, \dots$ and for $0 < L < 10^{-6}$. Estimate (2.13) is hence proved and it implies

$$(2.16) \quad N_k \geq L^{-3k} \frac{1 - L^{3k}}{1 - L^3}$$

To show (2.14), it is sufficient to prove

$$N_{k+1} - N_k \geq \frac{N_{k+1}}{2}$$

which is the same as

$$(2.17) \quad N_{k+1} - N_k \geq N_k$$

We proceed by induction. Again, for $k = 0$, this follows from the choice of N_0 and N_1 . For $k > 0$, assuming (2.17) for $k - 1$ and using (2.15), we get

$$N_{k+1} - N_k \geq \frac{1}{k^2} \exp\left(\frac{L\sqrt{N_k}}{k^2}\right) - 1$$

Now, it is sufficient to notice that

$$\frac{1}{k^2} \exp\left(\frac{L\sqrt{N_k}}{k^2}\right) - 1 > N_k$$

follows from (2.16). Bound (2.17) is proved and so we have (2.14).

For each k , we choose the following $\{\hat{x}_{k,t}\}_{t=1,\dots,N_k}$

$$(2.18) \quad \hat{x}_{k,t} = \frac{L\beta_j}{\sqrt{N_j - N_{j-1}}}, \quad t \in (N_{j-1}, N_j],$$

$j = 1, \dots, k$. Furthermore, $\beta_j = j^{-2}$, $j = 1, \dots, k - 1$, and $\beta_k = 1$. We have

$$(2.19) \quad \sum_{t=1}^{N_k} \hat{x}_{k,t}^2 = \sum_{j=1}^k \sum_{t=N_{j-1}+1}^{N_j} \hat{x}_{k,t}^2 \simeq L^2 \left(\sum_{j=1}^{k-1} \frac{1}{j^4} + 1 \right) \simeq L^2$$

and hence condition (2.9) on the N_k -tuple $\{\gamma_j\}$ is satisfied. Let us now compute (2.11). For the numerator, one has

$$\sum_{t=1}^{N_k} \hat{x}_{k,t} = L \left(\sum_{j=1}^{k-1} \frac{\sqrt{N_j - N_{j-1}}}{j^2} + \sqrt{N_k - N_{k-1}} \right) \geq L\sqrt{N_k - N_{k-1}} \stackrel{(2.14)}{\gtrsim} L\sqrt{N_k}$$

Bound (2.19) and the Cauchy-Schwarz inequality imply

$$\sum_{t=1}^{N_k} \hat{x}_{k,t} \lesssim L\sqrt{N_k}$$

Therefore

$$(2.20) \quad \sum_{t=1}^{N_k} \hat{x}_{k,t} \simeq L\sqrt{N_k}$$

Next, estimating the denominator in (2.11), we have

$$\sum_{t=1}^m \hat{x}_{k,t} > \sum_{l=1}^{j-1} (N_l - N_{l-1}) \hat{x}_{k,N_l}, \quad m \in [N_{j-1}, N_j]$$

so

$$\sum_{m=1}^{N_k-1} \exp\left(-\sum_{t=1}^m \hat{x}_{k,t}\right) \lesssim N_1 + \sum_{j=2}^k (N_j - N_{j-1}) \exp\left(-\sum_{l=1}^{j-1} (N_l - N_{l-1}) \hat{x}_{k,N_l}\right)$$

$$\begin{aligned}
&= N_1 + \sum_{j=2}^k (N_j - N_{j-1}) \exp \left(-L \sum_{l=1}^{j-1} \frac{\sqrt{N_l - N_{l-1}}}{l^2} \right) \\
&\stackrel{(2.12)}{\lesssim} L^{-3} + \sum_{j=2}^k \frac{1}{(j-1)^2} \lesssim L^{-3}
\end{aligned}$$

Combining this bound and (2.20), we get the first inequality in (2.10). The second one follows similarly. \square

The estimates obtained are not sharp in L when L is small. However, they are sharp in n and this is all we need.

Remark 2.1. *Estimate (2.20) yields*

$$(2.21) \quad \sum_{j=1}^{N_k} \exp \left(\sum_{l=1}^j x_l \right) \geq \exp \left(\sum_{l=1}^{N_k} x_l \right) \geq \exp(CL\sqrt{N_k})$$

Remark 2.2. *Let $\Sigma_L = \{\{x_j\}_{j=1, \dots, N_k} : \sum_{j=1}^{N_k} x_j^2 = L^2\}$. Then, for any $L, 0 < L < 10^{-6}$, we have*

$$\max_{\{x_j\} \subset \Sigma_L} \Gamma_{N_k}(\{x_j\}) \gtrsim L^4 \sqrt{N_k}$$

The bound

$$\max_{\{x_j\} \subset \Sigma_L} \Gamma_{N_k}(\{x_j\}) \lesssim L\sqrt{N_k}$$

trivially follows from definition (2.8) and the Cauchy-Schwarz inequality.

Remark 2.3. *The reasoning of the above lemma can be adapted to handle any sufficiently large n , not necessarily constructed as a sequence $\{N_k\}$.*

Recalling the definition of the $(N', N; \kappa)$ -transformation, we have the following key lemma.

Lemma 2.2. *Let $\mu \in (S)$ be a probability measure on \mathbb{T} with real Schur parameters. For any natural N' and arbitrary small positive δ and L , there is σ , a $(N', N; \kappa)$ -transformation of μ such that:*

- (1) $N \geq 2N'$,
- (2) *the coefficients $\{\gamma_k\}_{k=N'+1, \dots, N}$ are positive and*

$$\sum_{k=N'+1}^N |\gamma_k|^2 \lesssim L^2,$$

- (3) $0 < \kappa < \delta$,
- (4) *Finally,*

$$\widehat{\epsilon}_N(\sigma) = \int_{\mathbb{T}} |\Phi_N(\sigma)|^2 \log |\Phi_N(\sigma)| d\sigma \gtrsim L^4 \sqrt{N}$$

Proof. We start the proof with some simple observations. Let, as above, $\mu_0 = \mu_0[\mu]$, $\mu_1 = \mu_1[\mu]$ and $\sigma = \sigma[\mu]$. First, assuming that such a transformation exists and using (2.1), we see that

$$\int_{\mathbb{T}} \log \mu'_0 dm = \sum_{k=0}^{N'} \log(1 - |\gamma_k(\mu)|^2)$$

and

$$(2.22) \quad \sum_{k=0}^{\infty} \log(1 - |\gamma_k(\sigma)|^2) = \int_{\mathbb{T}} \log \sigma' dm \stackrel{(2.7)}{=} \int_{\mathbb{T}} \log \mu'_1 dm - \log(\kappa + 1)$$

$$\stackrel{(2.6)}{=} \sum_{k=0}^{N'} \log(1 - |\gamma_k(\mu)|^2) + \sum_{k=N'+1}^N \log(1 - |\gamma_k|^2) - \log(1 + \kappa)$$

This estimate controls the growth of the ℓ^2 -norm of Schur coefficients under our transformation.

We assign $N - N' = N_k$ where N_k is an element of the sequence $\{N_k\}$ constructed in lemma 2.1; the precise choice of k will be made below. From now on, we already assume that these N', N satisfy (1). We define the Schur coefficients $\{\gamma_k\}_{k=N'+1, \dots, N}$ as

$$\gamma_{N+1-t} = \frac{\widehat{x}_{k,t}}{2}, \quad t = 1, \dots, N_k$$

where $\{\widehat{x}_{k,t}\}_{t=1, \dots, N-N'}$ comes from (2.18). Notice that $\gamma_t > 0$ for these t and the N_k -tuple $\{\gamma_t\}_{t=N'+1, \dots, N}$ satisfies (2).

Introduce the Christoffel-Darboux kernel

$$(2.23) \quad K_n(\mu_1)(z, w) = \sum_{k=0}^n \phi_k(\mu_1)(z) \overline{\phi_k(\mu_1)(w)}$$

We define κ as

$$(2.24) \quad \kappa = \frac{1}{K_{N-1}(\mu_1)(1, 1)}$$

and now we need a bound from below for $K_{N-1}(\mu_1)(1, 1)$. Notice that all Schur coefficients are real so $\Phi_j(\mu_1)(1)$ are real and $\Phi_j(\mu_1)(1) = \Phi_j^*(\mu_1)(1)$. Let $B = |\Phi_{N'+1}(\mu_1)(1)|$. All zeroes of $\Phi_j(\mu_1)(z)$ are inside \mathbb{D} so $B > 0$. Then, by Szegő recurrence relations (2.4), one has

$$(2.25) \quad |\Phi_m(\mu_1)(1)| = B \cdot \left| \prod_{t=N'+1}^{m-1} (1 + \gamma_t) \right|, \quad m > N' + 1$$

Hence,

$$(2.26) \quad K_{N-1}(\mu_1)(1, 1) = \sum_{t=0}^{N-1} |\phi_t(\mu_1)(1)|^2 \gtrsim \sum_{t=N'+2}^{N-1} |\Phi_t(\mu_1)(1)|^2$$

$$= B^2 \sum_{t=N'+2}^{N-1} \prod_{j=N'+1}^{t-1} (1 + \gamma_j)^2 \simeq B^2 \sum_{t=N'+2}^{N-1} \exp\left(2 \sum_{j=N'+1}^{t-1} \gamma_j\right)$$

where we used inequality

$$(2.27) \quad 1 + x \leq e^x \leq 1 + x + (e/2)x^2, \quad 0 \leq x < 1$$

along with Szegő condition and (2.22). By remark 2.1, the latter quantity goes to infinity through a constructed subsequence in $N = N' + N_k$. So, (2.24) implies (3) for k large enough.

To start with (4), recall the following formula usually attributed to Geronimus (see, e.g., [6, p. 253] or [5, p. 38, (3.30)]; this very formula was used by Rakhmanov in his paper on the Steklov's conjecture [6])

$$\Phi_N(\sigma)(z) = \Phi_N(\mu_1)(z) - \frac{\kappa \Phi_N(\mu_1)(1)}{1 + \kappa K_{N-1}(\mu_1)(1, 1)} K_{N-1}(\mu_1)(z, 1)$$

Our choice of κ (see (2.24)) yields

$$\Phi_N(\sigma)(1) = \frac{\Phi_N(\mu_1)(1)}{2}$$

and

$$\begin{aligned} \widehat{\epsilon}_N(\sigma) &= \int_{\mathbb{T}} |\Phi_N(\sigma)|^2 \log |\Phi_N(\sigma)| d\sigma \\ &= \int_{\mathbb{T}} |\Phi_N(\sigma)|^2 \log^+ |\Phi_N(\sigma)| d\sigma - \int_{\mathbb{T}} |\Phi_N(\sigma)|^2 \log^- |\Phi_N(\sigma)| d\sigma \\ &\stackrel{(1.2)}{\geq} \int_{\mathbb{T}} |\Phi_N(\sigma)|^2 \log^+ |\Phi_N(\sigma)| d\sigma - C \end{aligned}$$

In the first term of the right-hand side we leave only the integration against $\kappa(1 + \kappa)^{-1} \delta_1$, see (2.7). This leads us to

$$\widehat{\epsilon}_N(\sigma) \gtrsim \kappa |\Phi_N(\sigma)(1)|^2 \log^+ |\Phi_N(\sigma)(1)| - C$$

For the first term

$$\begin{aligned} &\kappa |\Phi_N(\sigma)(1)|^2 \log^+ |\Phi_N(\sigma)(1)| \\ &\stackrel{(2.25), (2.26)}{\gtrsim} \frac{B^2 \prod_{t=N'+1}^{N-1} (1 + \gamma_t)^2 \cdot \log^+ \left(B \prod_{t=N'+1}^{N-1} (1 + \gamma_t) / 2 \right)}{\sum_{t=0}^{N'} |\Phi_t(\mu_1)(1)|^2 + \sum_{t=N'+1}^{N-1} B^2 \prod_{j=N'+1}^t (1 + \gamma_j)^2} \\ &\gtrsim \frac{B^2 \exp(\sum_{t=N'+1}^{N-1} 2\gamma_t) \left(2 \log B - 2 \log 2 + \sum_{t=N'+1}^{N-1} 2\gamma_t \right)}{\sum_{t=0}^{N'} |\Phi_t(\mu_1)(1)|^2 + B^2 \sum_{t=N'+1}^{N-1} \exp(\sum_{j=N'+1}^t 2\gamma_j)} \end{aligned}$$

where we again used (2.27) along with the Szegő condition and (2.22). The last expression can be written as $I_1 + I_2$ where

$$I_1 = \frac{B^2 \exp(\sum_{t=N'+1}^{N-1} 2\gamma_t) (2 \log B - 2 \log 2)}{\sum_{t=0}^{N'} |\Phi_t(\mu_1)(1)|^2 + B^2 \sum_{t=N'+1}^{N-1} \exp(\sum_{j=N'+1}^t 2\gamma_j)}$$

and

$$I_2 = \frac{1}{Q} \frac{\exp(\sum_{t=N'+1}^{N-1} 2\gamma_t) \left(\sum_{t=N'+1}^{N-1} 2\gamma_t \right)}{1 + \sum_{t=N'+1}^{N-1} \exp(\sum_{j=N'+1}^t 2\gamma_j)}$$

where

$$Q = 1 + \frac{-1 + B^{-2} \sum_{t=0}^{N'} |\Phi_t(\mu_1)(1)|^2}{1 + \sum_{t=N'+1}^{N-1} \exp(\sum_{j=N'+1}^t 2\gamma_j)}$$

For I_1 , we have $|I_1| < |2 \log B - 2 \log 2|$. In the expression for Q , the denominator goes to infinity as $N \rightarrow \infty$ due to (2.21) and so $Q \rightarrow 1$ as $N \rightarrow \infty$. Applying (2.10) to the second factor in the formula for I_2 , we get

$$I_2 \gtrsim L^4 \sqrt{N - N'} \gtrsim L^4 \sqrt{N}$$

whenever $N = N' + N_k$ is large enough and N_k belongs to the subsequence from lemma 2.1. Thus, bound (4) for $\widehat{\epsilon}_N(\sigma)$ is proven, and so is the lemma. \square

Remark 2.4. *The lemma above along with remark 2.3 imply the sharp bound*

$$\sup_{\sigma: \{\|\gamma\|_2 < 1/2\}} \epsilon_n(\sigma) \simeq \sqrt{n}$$

In our construction, the measure σ yielding the lower bound contained a mass point. However, taking the Bernstein-Szegő approximations σ_j to σ , we obtain for $j > n$

$$\begin{aligned} \epsilon_n(\sigma_j) &= \int_{\mathbb{T}} |\phi_n(\sigma_j)|^2 \log |\phi_n(\sigma_j)| d\sigma_j = \int_{\mathbb{T}} |\phi_n(\sigma)|^2 \log |\phi_n(\sigma)| d\sigma_j \rightarrow \\ &\int_{\mathbb{T}} |\phi_n(\sigma)|^2 \log |\phi_n(\sigma)| d\sigma, \quad j \rightarrow \infty \end{aligned}$$

since $\sigma_j \xrightarrow{*} \sigma$ as $j \rightarrow \infty$. Thus, we have

$$\sup_{\sigma_{ac}: \{\|\gamma\|_2 < 1/2\}} \epsilon_n(\sigma_{ac}) \simeq \sqrt{n}$$

where σ_{ac} stands for a purely absolutely continuous measure.

3. PROOF OF THEOREM 1.1 AND SOME COROLLARIES

Proof of theorem 1.1. Let h be the given function from the formulation of the theorem, $\delta_k > 0$, $0 < L_k < 10^{-6}$, and

$$\sum_{k=1}^{\infty} \delta_k < 10^{-3}, \quad \sum_{k=1}^{\infty} L_k^2 < 10^{-3}$$

The construction will recursively use lemma 2.2 from the previous section. We will construct the sequence of probability measures σ_j by applying the $(M', M; \kappa)$ -transformation consecutively (properly choosing parameters M', M, κ at every step) and then will take the weak limit of $\{\sigma_j\}$. The measure σ obtained in this way will have the necessary properties.

First step: $k = 1$. Let $d\mu^0 = d\sigma^0 = dm$, the Lebesgue measure on \mathbb{T} . Take $M'_1 = 0$; then, by lemma 2.2, there is a $(M'_1, M_1; \kappa_1)$ -transformation of σ^0 which is denoted by σ^1 ; it depends on the sequence of Schur parameters $\{\gamma_{1k}\}_{k=M'_1+1, \dots, M_1}$. We can arrange M_1 to be large enough, *i.e.*, pick $M_1 \geq 2^1$ so that $h(M_1) \leq L_1^4/2$ and $\kappa_1 < \delta_1$, see (3) in lemma 2.2. Also,

$$\sum_{k=M'_1+1}^{M_1} |\gamma_{1k}|^2 \lesssim L_1^2$$

and (again by the same lemma)

(3.1)

$$\widehat{\epsilon}_{M_1}(\sigma^1) = \int_{\mathbb{T}} |\Phi_{M_1}(\sigma^1)|^2 \log |\Phi_{M_1}(\sigma^1)| d\sigma^1 \gtrsim L_1^4 \sqrt{M_1} \geq 2h(M_1) \sqrt{M_1}$$

Notice that for the Schur parameters of σ^1 we have

$$(3.2) \quad \sum_{l=0}^{\infty} \log(1 - |\gamma_l(\sigma^1)|^2) = \sum_{l=M'_1+1}^{M_1} \log(1 - |\gamma'_{1l}|^2) - \log(1 + \kappa_1)$$

due to (2.22).

For each measure σ^k we construct later, let us introduce

$$T_{M_j}(\sigma^k) = |\Phi_{M_j}(\sigma^k)|^2 \log |\Phi_{M_j}(\sigma^k)|, \quad j \leq k$$

This function is continuous on \mathbb{T} since $\Phi_l(\sigma^k)$ has all its zeroes inside \mathbb{D} .

Second step: $k = 2$. Consider $T_{M_1}(\sigma^1)$; this is a continuous function and hence there is a trigonometric polynomial f_1 such that

$$(3.3) \quad \|T_{M_1}(\sigma^1) - f_1\|_{\infty} < \varepsilon'$$

for any fixed $\varepsilon' > 0$. Let $M'_2 = \max\{\deg f_1, M_1\}$. Define σ^2 as $(M'_2, M_2; \kappa_2)$ -transformation of σ^1 . Once again, we choose $M_2 \geq 2^2$ large enough to guarantee $h(M_2) \leq L_2^4/2$, $\kappa_2 \leq \delta_2$ by (3) of lemma 2.2, and

$$\sum_{k=M'_2+1}^{M_2} |\gamma_{2k}|^2 \lesssim L_2^2$$

Notice that the inequality $\kappa_2 \leq \delta_2$ allows us to have $\|\gamma(\sigma^2)\|_2$ under control

$$(3.4) \quad \sum_{l=0}^{\infty} \log(1 - |\gamma_l(\sigma^2)|^2) = \sum_{l=0}^{M'_2} \log(1 - |\gamma_l(\sigma^1)|^2) + \sum_{l=M'_2+1}^{M_2} \log(1 - |\gamma'_{2l}|^2) - \log(1 + \kappa_2)$$

Once again, by lemma 2.2

$$\widehat{\varepsilon}_{M_2}(\sigma^2) = \int_{\mathbb{T}} |\Phi_{M_2}(\sigma^2)|^2 \log |\Phi_{M_2}(\sigma^2)| d\sigma^2 \gtrsim L_2^4 \sqrt{M_2} \geq 2h(M_2) \sqrt{M_2}$$

Now, we could continue to apply the same procedure to generate measures σ_k with

$$(3.5) \quad \widehat{\varepsilon}_{M_k}(\sigma^k) = \int_{\mathbb{T}} |\Phi_{M_k}(\sigma^k)|^2 \log |\Phi_{M_k}(\sigma^k)| d\sigma^k \gtrsim L_k^4 \sqrt{M_k} = 2h(M_k) \sqrt{M_k}$$

However, we want more than (3.5): we want every measure σ^k to have all of the entropies $\{\widehat{\varepsilon}_{M_1}(\sigma^k), \dots, \widehat{\varepsilon}_{M_{k-1}}(\sigma^k)\}$ large. That, as we will see next, can also be achieved by the choice of large M_k .

Let us handle the case $k = 2$ first. We need to make $\widehat{\varepsilon}_{M_1}(\sigma^2)$ large. Recall the definition of $(N', N; \kappa)$ -transform. Set $\mu^2 = \mu_0[\sigma^1]$ and $\sigma^2 = \sigma[\sigma^1]$. We have $\gamma_l(\sigma^1) = \gamma_l(\mu^2)$, $c_l(\sigma^1) = c_l(\mu^2)$ for $l \leq M'_2$ where $c_l(\cdot)$ are the moments of the measures. Therefore

$$(3.6) \quad \Phi_{M_1}(\sigma^1) = \Phi_{M_1}(\mu^2), \quad T_{M_1}(\sigma^1) = T_{M_1}(\mu^2)$$

Since f_1 is a trigonometric polynomial of degree smaller than M'_2 , we have

$$\int_{\mathbb{T}} T_{M_1}(\sigma^1) d\sigma^1 - \varepsilon' \leq \int_{\mathbb{T}} f_1 d\sigma^1 = \int_{\mathbb{T}} f_1 d\mu^2 \leq \int_{\mathbb{T}} T_{M_1}(\mu^2) d\mu^2 + \varepsilon'$$

by (3.3). Consequently

$$(3.7) \quad \int_{\mathbb{T}} T_{M_1}(\sigma^1) d\sigma^1 - 2\varepsilon' \leq \int_{\mathbb{T}} T_{M_1}(\mu^2) d\mu^2$$

Now, notice that

$$c_j(\sigma^2) \rightarrow c_j(\mu^2), \quad j \leq M'_2$$

as $\kappa_2 \rightarrow 0$ and this convergence is uniform in the choice of M_2 and $\{\gamma_j(\sigma^2), j > M'_2\}$. Indeed, it follows from (2.7)

$$c_j(\sigma^2) = \frac{c_j(\mu^2) + \kappa_2}{1 + \kappa_2}, \quad j \leq M'_2$$

Therefore

$$\Phi_{M_1}(\sigma^2) \rightarrow \Phi_{M_1}(\mu^2)$$

as $\kappa_2 \rightarrow 0$ and, recalling that $\Phi_{M_1}(\mu^2)$ has no zeroes on \mathbb{T} ,

$$T_{M_1}(\sigma^2) = |\Phi_{M_1}(\sigma^2)|^2 \log |\Phi_{M_1}(\sigma^2)| \rightarrow |\Phi_{M_1}(\mu^2)|^2 \log |\Phi_{M_1}(\mu^2)| = T_{M_1}(\mu^2)$$

uniformly on \mathbb{T} . Thus,

$$(3.8) \quad \int_{\mathbb{T}} T_{M_1}(\sigma^2) d\sigma^2 \rightarrow \int_{\mathbb{T}} T_{M_1}(\mu^2) d\sigma^2$$

and

$$(3.9) \quad \int_{\mathbb{T}} f_1 d\sigma^2 \rightarrow \int_{\mathbb{T}} f_1 d\mu^2, \quad \text{as } \kappa_2 \rightarrow 0$$

again, uniformly in the choice of M_2 and $\{\gamma_j(\sigma^2), j > M'_2\}$. On the other hand, (3.3) and (3.6) guarantee

$$(3.10) \quad \left| \int_{\mathbb{T}} f_1 d\mu^2 - \int_{\mathbb{T}} T_{M_1}(\mu^2) d\mu^2 \right| \leq \varepsilon', \quad \left| \int_{\mathbb{T}} f_1 d\sigma^2 - \int_{\mathbb{T}} T_{M_1}(\mu^2) d\sigma^2 \right| \leq \varepsilon'$$

Thus, recalling (3.8), (3.9) and (3.10), we only need to make sure that κ_2 is small enough to guarantee

$$\int_{\mathbb{T}} T_{M_1}(\sigma^2) d\sigma^2 > \int_{\mathbb{T}} T_{M_1}(\mu^2) d\mu^2 - 3\varepsilon'$$

and then

$$\int_{\mathbb{T}} T_{M_1}(\sigma^2) d\sigma^2 \stackrel{(3.7)}{>} \int_{\mathbb{T}} T_{M_1}(\sigma^1) d\sigma^1 - 5\varepsilon' \stackrel{(3.1)}{\gtrsim} L_1^4 \sqrt{M_1} - 5\varepsilon' \geq h(M_1) \sqrt{M_1}$$

whenever ε' is small enough.

k -th step. Similarly, we construct the measure σ^k such that (3.5) holds along with

$$(3.11) \quad \int_{\mathbb{T}} T_{M_j}(\sigma^k) d\sigma^k > \int_{\mathbb{T}} T_{M_j}(\sigma^j) d\sigma^j - 5\varepsilon' > L_j^4 \sqrt{M_j} - 5\varepsilon', \quad j < k$$

Moreover, we have

$$(3.12) \quad \sum_{l=0}^{\infty} \log(1 - |\gamma_l(\sigma^k)|^2) \gtrsim - \sum_{j=1}^k L_j^2 - \sum_{j=1}^k \delta_j$$

by induction (check, *e.g.*, (3.2) and (3.4)). It is now clear from the construction that $\{\sigma^k\}$ converges $(*)$ -weakly to some σ . Indeed, recall that the

p -th moment of a measure σ is denoted by $c_p(\sigma)$. At each step we have a recursion

$$c_p(\sigma^{k+1}) = \frac{c_p(\sigma^k) + \kappa_{k+1}}{1 + \kappa_{k+1}}$$

where p is fixed. Since $\kappa_k < \delta_k$, we have convergence of $c_p(\sigma^k)$ by Cauchy criterion. That is equivalent to $\sigma^k \xrightarrow{*} \sigma$.

Repeating the arguments given above and using (3.5) and (3.11), we obtain

$$\begin{aligned} \int_{\mathbb{T}} |\Phi_{M_j}(\sigma)|^2 \log |\Phi_{M_j}(\sigma)| d\sigma &= \lim_{k \rightarrow \infty} \int_{\mathbb{T}} |\Phi_{M_j}(\sigma^k)|^2 \log |\Phi_{M_j}(\sigma^k)| d\sigma^k \\ &\geq \epsilon_{M_j}(\sigma^j) - 5\epsilon' \gtrsim L_j^4 \sqrt{M_j} - 5\epsilon' \geq h(M_j) \sqrt{M_j} \end{aligned}$$

for ϵ' small enough and any fixed j . For the ℓ^2 -norm of the Schur parameters, we have

$$(3.13) \quad \sum_{l=0}^{\infty} |\gamma_l(\sigma^k)|^2 \lesssim \sum_{k=1}^{\infty} L_k^2 + \sum_{k=1}^{\infty} \delta_k$$

The theorem is proved. \square

One can obtain the following striking generalization. Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing continuous function such that $\lim_{x \rightarrow \infty} F(x)/x = \infty$. The proof of the next statement repeats the arguments of the previous proof word for word.

Corollary 3.1. *For any gauge F , there is $\sigma \in (S)$ such that*

$$\epsilon_{M_k, F}(\sigma) = \int_{\mathbb{T}} F(|\Phi_{M_k}(\sigma)|^2) d\sigma \rightarrow \infty$$

for some $M_k \rightarrow \infty$.

As one can expect, theorem 1.1 can be transferred to the polynomials on an interval of the real line. We say that $\tilde{\sigma}$ is a Szegő measure on $[-1, 1]$ if it has an arbitrary singular part and

$$\int_{-1}^1 \frac{\log \tilde{\sigma}'}{\sqrt{1-x^2}} dx > -\infty$$

The orthonormal polynomials with respect to the measure $\tilde{\sigma}$ are denoted $p_n = p_n(\tilde{\sigma})$.

Corollary 3.2. *Let h be a function as in theorem 1.1. Then, there is a Szegő measure $\tilde{\sigma}$ on $[-1, 1]$ and a subsequence $\{M_k\}$ such that*

$$\epsilon_{M_k}(\tilde{\sigma}) = \int_{-1}^1 |p_{M_k}(\tilde{\sigma})|^2 \log |p_{M_k}(\tilde{\sigma})| d\tilde{\sigma} \geq h(M_k) \cdot \sqrt{M_k}$$

as $k \rightarrow \infty$.

Sketch of the proof. Notice that the measure σ from theorem 1.1 is symmetric on \mathbb{T} . Consequently, its Schur coefficients as well as coefficients of corresponding orthonormal polynomials are real. Let $\tilde{\sigma}$ denote the image of the measure σ constructed in theorem 1.1 from \mathbb{T} to $[-1, 1]$ by the (standard) transformation $x = \cos \theta = (z + z^{-1})/2$, $z = e^{i\theta} \in \mathbb{T}$.

The classical formula [8, theorem 11.5] implies that

$$p_n(x) \simeq \bar{z}^n \phi_{2n}(z) + z^n \phi_{2n}(\bar{z})$$

if $\sigma \in (S)$. We adjust the construction of theorem 1.1 such that M_k are all even. Then $|p_{M_k/2}(1)| \simeq |\phi_{M_k}(1)|$.

Consequently,

$$\begin{aligned} \varepsilon_{M_k/2}^+(\tilde{\sigma}) &= \int_{-1}^1 |p_{M_k/2}(\tilde{\sigma})|^2 \log^+ |p_{M_k/2}(\tilde{\sigma})| d\tilde{\sigma} \\ &\gtrsim \epsilon_{M_k}^+(\sigma^k) \geq h(M_k) \sqrt{M_k} \end{aligned}$$

by theorem 1.1 as the value at $z = 1$ alone provides the necessary growth of the entropy. \square

It is an interesting question to find a natural class of measures for which the polynomial entropy integrals are bounded. It might be that by improving the technique of [6, 1] one can show that the Steklov's condition on the measure is not good enough for the entropies to be uniformly bounded. In the meantime, it is quite possible that fairly mild conditions are sufficient for the averages of ϵ_n to be under control.

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