ON THE GROWTH OF THE SUPPORT OF POSITIVE VORTICITY FOR 2D EULER EQUATION IN AN INFINITE CYLINDER

K. CHOI, S. DENISOV

ABSTRACT. We consider the incompressible 2D Euler equation in an infinite cylinder $\mathbb{R}\times\mathbb{T}$ in the case when the initial vorticity is non-negative, bounded, and compactly supported. We study $d(t)$, the diameter of the support of vorticity, and prove that it allows the following bound: $d(t) \leqslant Ct^{1/3}\log^2 t$ when $t\to\infty$.

1. Introduction

Consider the incompressible 2D Euler equation in vorticity form on an infinite cylinder $S := \mathbb{R}\times\mathbb{T}$, where $\mathbb{T} = [0, 2\pi)$ is a unit circle:

$$\partial_t\theta + u\cdot\nabla\theta = 0, \quad \theta|_{t=0} = \theta_0. \tag{1}$$

The velocity $u(x, y, t)$ is related to the scalar vorticity θ via a cylindrical Biot-Savart law, which will be introduced in the next section (see formula (4)). This problem is identical to the Euler equation in $\mathbb{R}^2$ with θ_0 being 2π-periodic in y in the sense that we can obtain the two-dimensional cylinder S from the infinite strip $\mathbb{R}\times[0, 2\pi]$ by identifying its sides. In the paper, we use notation $z = (x, y)$, $\xi = (\xi_1, \xi_2)$, and $dz = dxdy, d\xi = d\xi_1 d\xi_2$ for shorthand.

We assume that θ_0 has a compact support and $\theta_0(x, y) \in L^\infty(S)$. For the 2D Euler equation on a cylinder, the existence and uniqueness of compactly supported solution in the sense of distributions from the class $L^\infty(S)$ can be proved in a similar manner as in the case of the whole space [14]. We refer the reader to [9] or Appendix in [3]. If the initial data assumes further $C^{m,\gamma}$-regularity, one can obtain $C^{m,\gamma}$-regular solution for all time by adapting the method in Chapter 4 of [10]. In this paper, however, we do not need smoothness that high and from now on a solution means a solution of (1) in the sense of distributions with u given by (4).

For any function f compactly supported on S, we define

$$d_f := \sup_{z,\xi\in\text{supp}(f)} |x-\xi_1|,$$

where $\text{supp}(f)$ denotes the essential support of f.

In this paper, we are interested in controlling the support of nonnegative vorticity for large time. The main result is the following upper estimate on $d_{\theta(t)}$:

Theorem 1.1. *Suppose that an initial data θ_0 is non-negative, compactly supported, and belongs to $L^\infty(S)$. Then, the corresponding solution θ satisfies*

$$d(t) := d_{\theta(t)} \leqslant C(t+1)^{\frac{1}{3}}\log^2(2+t) \quad \textit{for any } t > 0\,, \tag{2}$$

where the constant C depends only on d_{θ_0} and $\|\theta_0\|_{L^\infty}$.

An important example of θ_0 is the characteristic function χ_{Ω_0} of a compact subset Ω_0 of S, a patch. Then, $\theta(z, t) = \chi_{\Omega(t)}$ and one can study dynamics of $\Omega(t)$ in time. Note that the periodic

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extension of θ_0 into the whole space $\mathbb{R}^2$ is not compactly supported, in general.

For the problem when the data is compactly supported in $\mathbb{R}^2$ (so it's not periodic in y), the upper bound $d(t) \leqslant C(t+1)^{1/3}$ was obtained in [11]. Later, it was improved to $((t+1)\log(t+2))^{1/4}$ in [8] (see also [13]). The key idea of the proof in [8] was to use the following conserved quantities for Euler equation in $\mathbb{R}^2$:

$$\text{the total mass} \int_{\mathbb{R}^2} \theta(z)dz,$$

$$\text{the center of mass} \int_{\mathbb{R}^2} z\theta(z)dz, \quad \text{and}$$

$$\text{the moment of inertia} \int_{\mathbb{R}^2} |z|^2\theta(z)dz\,.$$

In particular, the moment of inertia plays an important role because its conservation in time shows that, when the initial vorticity is non-negative and compactly supported near the origin, only a small portion of θ can concentrate far away from zero at any given time. For exterior domains, we refer to [7, 12].

In order to have an analogous confinement for the 2D Euler on a cylinder $\mathbb{R} \times \mathbb{T}$, one needs to establish conserved quantities first. It does not seem to be the case that the Euler evolution on a cylinder preserves the second moment

$$\int_S x^2\theta(x,y,t)dz.$$

However, the following quantity:

$$e_0 := \int_S \theta(z,t)\Psi(z,t)dz$$

is conserved, where the stream function Ψ will be introduced in the next section. This allows us to show that

$$\int_S |x|\theta(x,y,t)dz \tag{3}$$

is uniformly bounded in time if the initial vorticity is non-negative (see Proposition 3.1). This quantity e_0 can be regarded as a regularized energy. In fact, the standard kinetic energy given by

$$\int_S |u(z,t)|^2dz$$

is not finite for non-negative vorticity, in general.

The second ingredient of the proof is related to the cylindrical Biot-Savart law. It shows that, for the horizontal component of velocity $u_1 := k_1 * \theta$, the kernel k_1 takes the form

$$k_1 = \frac{-\sin(y)}{2(\cosh(x) - \cos(y))}\,.$$

Thus, it is smaller than $\frac{C}{|z|}$ near 0 and decays exponentially for large $|x|$. The decay so strong makes interaction between the distant parts of vorticity essentially negligible. Note that the exponential bound for the kernel k_1 has been used in [6] to study 2D Navier-Stokes equation on a cylinder.

Then, our proof proceeds by controlling the integrals

$$\int_{x>r} \theta(z,t)dz$$

for different values of $r \in \mathbb{R}^+$. Using the strong decay of k_1, we establish the following inequality for $r \gtrsim 1$ (see (11)):

$$\int_{|x|>4r} \theta(z,t)dz \lesssim r^{-2} \int_0^t \left(\left(\int_{|x|>r} \theta(z,\tau)dz \right)^2 + \text{ small error} \right) d\tau.$$

Then, we analyze the sequence of these estimates taking $r \sim 4^n, n \in \mathbb{N}$ to obtain a bound for u_1. It shows that u_1 is very small outside the region $|x| \gtrsim t^{1/3} \log^2 t$ for $t \gtrsim 1$ (see (15)) and this will imply the main estimate (2).

For stability questions, there were several publications in which stability of steady states on S was studied. In the paper [2], the Couette flow was considered. In [4], the case of increasing steady vorticity was studied and, more recently, the stability of a rectangular patch was investigated in [3].

The structure of the paper is as follows. In the next section, the cylindrical Biot-Savart law and some conserved quantities of the Euler equation will be introduced. In Section 3, we will prove the bound for (3) and discuss some of its easy consequences. Section 4 contains the proof of Theorem 1.1. We collect auxiliary results in the last section.

In this paper, we use the following standard notation. If two non-negative functions f_1 and f_2 satisfy $f_1 \leqslant Cf_2$ with some absolute constant C, we write $f_1 \lesssim f_2$. If $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$, we use $f_1 \sim f_2$. If $f_{1(2)}(t)$ satisfy $f_1(t) \leqslant Cf_2(t)$ for all $t > 1$, we write $f_1 = \mathcal{O}(f_2)$ as $t \to \infty$, here C might depend on some fixed parameters but not on t. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}^+ := \mathbb{N} \cup \{0\}$.

2. Preliminaries

In this paper, we use the cylindrical Biot-Savart law (see [1] or [5, 3] for the detail):

$$u(x,y) = (u_1, u_2) = k * \theta = \int_S k(x-\xi_1, y-\xi_2)\theta(\xi_1,\xi_2)d\xi, \tag{4}$$

where the kernel k is given by

$$k(x,y) = (k_1,k_2) = \frac{(-\sin(y), \sinh(x))}{2(\cosh(x)-\cos(y))}.$$

Note that if we define the stream function Ψ of θ by $\Psi(x,y) = \Gamma * \theta$, where

$$\Gamma(x,y) = \frac{1}{2}\log(\cosh(x)-\cos(y)),$$

then the function Ψ solves the elliptic problem

$$(2\pi)^{-1}\Delta\Psi = \theta, \quad \lim_{x\to+\infty} \partial_x\Psi(x,y) = -\lim_{x\to-\infty} \partial_x\Psi(x,y), \quad |\Psi(x,y)| \leqslant C(|x|+1)$$

and the velocity can be recovered by $u = \nabla^\perp\Psi = (-\partial_y\Psi, \partial_x\Psi)$. We observe that $|\Gamma(z)| \sim |\log|z||$ for small $|z|$ and $\Gamma(z) \sim |x|$ for large $|x|$.

For any bounded and compactly supported θ_0, we denote

$$\text{the total mass } m_0 := \int_S \theta_0(z)dz,$$

$$\text{the horizontal center of mass } h_0 := \int_S x\theta_0(z)dz, \quad \text{and}$$

$$\text{the regularized energy } e_0 := \int_S \theta_0(z)\Psi_0(z)dz = \int_S\int_S \theta_0(z)\theta_0(\xi)\Gamma(z-\xi)d\xi dz.$$

Remark. Both m_0 and e_0 are controlled by diameter d_{θ_0} and $\|\theta_0\|_{L^\infty(S)}$. Indeed, we have

$$|m_0| \leqslant \|\theta_0\|_{L^1(S)} \lesssim d_{\theta_0} \cdot \|\theta_0\|_{L^\infty(S)}. \tag{5}$$

For the regularized energy e_0, we notice that there is $l \in \mathbb{R}$, such that θ_0 is supported in the rectangle $\{z \in S \,|\, |x-l| \le d_{\theta_0}\}$. For any such z, the stream function Ψ_0 of θ_0 satisfies

$$|\Psi_0(z)| \leqslant \|\theta_0\|_{L^\infty(S)} \cdot \int_{\{\xi\in S\,|\,|\xi_1-l|\le d_{\theta_0}\}} |\Gamma(z-\xi)|d\xi \leqslant \|\theta_0\|_{L^\infty(S)} \cdot C_{d_{\theta_0}}$$

since $|z-\xi| \lesssim 1 + d_{\theta_0}$ and $\Gamma(\cdot)$ is locally integrable thanks to

$$|\Gamma(z)| \sim |\log|z||$$

which holds for small $|z|$. So we have

$$|e_0| \leqslant \|\theta_0\|_{L^1(S)} \cdot \|\theta_0\|_{L^\infty(S)} \cdot C_{d_{\theta_0}} \lesssim d_{\theta_0} \cdot \|\theta_0\|^2_{L^\infty(S)} \cdot C_{d_{\theta_0}}. \tag{6}$$

Remark. The kinetic energy $\int_S |u|^2 dz$ is not finite, in general. Indeed, assume that the data θ_0 is non-negative and non-trivial. Since $k_2 \to \pm\frac{1}{2}$ as $x \to \pm\infty$, we get $|u_2| = |k_2 * \theta| \to \frac{1}{2}m_0 \neq 0$ as $x \to \pm\infty$. This implies divergence of the integral $\int_S |u|^2 dz$.

Since we consider an incompressible flow and its vorticity is transported by the flow, the L^1-norm and $L^\infty-$norm of $\theta(z,t)$ are preserved in time. In addition to these norms, we have the following conserved quantities.

Lemma 2.1. *For any bounded and compactly supported θ_0, the Euler evolution on S preserves the total mass, the horizontal center of mass, and the regularized energy:*

$$m_0 = \int_S \theta(z,t)dz, \quad h_0 = \int_S x\theta(z,t)dz, \quad e_0 = \int_{S\times S} \theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz \quad \textit{for all } t \geqslant 0.$$

Its proof can be found in Proposition 2.1 of [3].

Remark. If θ is a smooth solution, this lemma easily follows from the following arguments. The quantity

$$\int_S \theta(z,t)dz$$

is time-independent because the velocity u is incompressible. To handle the center of mass, we multiply equation (1) by x and integrate over S to get

$$\frac{d}{dt}\int_S x\theta(x,y,t)dz = -\int_S x(-\Psi_y\theta_x + \Psi_x\theta_y)dz\,,$$

where θ is smooth and compactly supported. Integration by parts gives

$$-\int_S x(-\Psi_y\theta_x + \Psi_x\theta_y)dz = -\int_S \Psi_y\theta dz = \int_{S\times S} \frac{-\sin(y-\xi_2)}{2(\cosh(x-\xi_1)-\cos(y-\xi_2))}\theta(z,t)\theta(\xi,t)dzd\xi = 0\,,$$

because the kernel in this quadratic form is antisymmetric. Consider the regularized energy. Differentiation in time gives

$$\frac{d}{dt}\left(\int_{S\times S}\theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz\right) = 2\int_{S\times S}\theta_t(z,t)\theta(\xi,t)\Gamma(z-\xi)dzd\xi$$

$$= 2\int_S(\Psi_y\theta_x - \Psi_x\theta_y)\Psi dz = \int_S (\Psi^2)_y\theta_x - (\Psi^2)_x\theta_y dz = 0$$

after integration by parts. For solutions in the sense of distributions, a mollification argument has been used in [3].

Remark. For any bounded and compactly supported initial vorticity, we have a trivial bound

$$d(t) = \mathcal{O}(t) \qquad \text{as } t \to \infty.$$

Indeed, since the vorticity in the Euler equation is transported by the corresponding velocity and our domain is a horizontal cylinder, we only need to estimate the horizontal velocity $u_1 = k_1 * \theta$, where

$$k_1(z) = \frac{-\sin(y)}{2(\cosh(x) - \cos(y))}.$$

We can use an estimate $|k_1(z)| \lesssim |z|^{-1}e^{-|x|/2}$ to get the bound $|u_1| \lesssim \|\theta\|_{L^1(S)} + \|\theta\|_{L^\infty(S)}$. Since L^p-norms of θ are preserved by the Euler evolution, we get

$$\sup_{z\in S, t\geqslant 0} |u_1(z,t)| \lesssim \|\theta_0\|_{L^1(S)} + \|\theta_0\|_{L^\infty(S)} \lesssim d_{\theta_0} \cdot \|\theta_0\|_{L^\infty(S)} + \|\theta_0\|_{L^\infty(S)}, \tag{7}$$

which implies $d(t) = \mathcal{O}(t)$.

3. One proposition and another rough bound on $d(t)$

In the following proposition, the non-negativity of θ will be crucial.

Proposition 3.1. *Suppose that θ_0 is non-negative, bounded, and compactly supported on S. If the horizontal center of mass is at 0, then*

$$\sup_{t\geqslant 0} \int_S |x|\theta(z,t)dz \leqslant C(d_{\theta_0}, \|\theta_0\|_{L^\infty(S)}) \,.$$

Proof. Bounds (5) and (6) show that it is enough to estimate $\int_S |x|\theta(z,t)dz$ by m_0, e_0, and $\|\theta_0\|_{L^\infty(S)}$. Let $t \geqslant 0$. Then, by Lemma 2.1, we write

$$e_0 = \int_S\int_S \theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_{n<x<n+1}\int_{m<\xi_1<m+1} \theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz.$$

For each term in the sum which satisfies $|n-m| < 10$, we can write

$$\left|\int_{n<x<n+1}\int_{m<\xi_1<m+1} \theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz\right| \lesssim \|\theta(t)\|_{L^\infty(S)} \cdot \int_{n<x<n+1} \theta(z,t)dz \,.$$

Indeed, it follows from the logarithmic estimate for the kernel

$$|\Gamma(\eta)| \sim |\log|\eta||$$

which holds for small $|\eta|$. Thus, the sum of all terms for which $|n-m| < 10$ is bounded by $C\|\theta(t)\|_{L^\infty(S)} \cdot \int_S \theta(z,t)dz = C\|\theta_0\|_{L^\infty(S)} \cdot m_0$.

On the other hand, since $\Gamma(\eta) \sim |\eta_1|$ for $|\eta_1| \geqslant 2$, all terms for which $|n-m| \geqslant 10$ satisfy

$$\int_{n<x<n+1}\int_{m<\xi_1<m+1} \theta(z,t)\theta(\xi,t)\Gamma(z-\xi)d\xi dz \sim \int_{n<x<n+1}\int_{m<\xi_1<m+1} \theta(z,t)\theta(\xi,t)|x-\xi_1|d\xi dz$$

and, in particular, they are all positive. Suppose that θ_0 is non-trivial so that $\int_S \theta(z,t)dz = m_0 > 0$. Then, we have $\int_{x>0} \theta(z,t)dz \geqslant m_0/2$, or $\int_{x<0}\theta(z,t)dz \geqslant m_0/2$, or the both estimates are true. Suppose, e.g.,

$$\int_{x>0} \theta(z,t)dz \geqslant m_0/2.$$

Then, we can write

$$\frac{m_0}{2}\int_{x<-10}|x|\theta(z,t)dz \leqslant \int_{x<-10}\int_{\xi_1>0}\theta(z,t)\theta(\xi,t)|x|d\xi dz$$
$$\leqslant \int_{x<-10}\int_{\xi_1>0}\theta(z,t)\theta(\xi,t)|x-\xi_1|d\xi dz \lesssim |e_0| + m_0\cdot\|\theta_0\|_{L^\infty(S)}.$$

Thus, we get

$$\int_{x<0}|x|\theta(z,t)dz \lesssim \frac{1}{m_0}\cdot(|e_0| + m_0\cdot\|\theta_0\|_{L^\infty(S)}) + \|\theta_0\|_{L^\infty(S)}\,.$$

Since the horizontal center of mass is at zero for all time by Lemma 2.1, we have

$$\int_{x>0} x\theta(z,t)dz = \int_{x<0}|x|\theta(z,t)dz$$

and, therefore,

$$\int_S |x|\theta(z,t)dz \leqslant C(e_0, m_0, \|\theta_0\|_{L^\infty(S)})\,,$$

where the constant is independent of $t \geqslant 0$.

□

Before proving Theorem 1.1, we will show how Proposition 3.1 can be used to obtain a rough upper estimate on the diameter. Under its assumptions, we get

$$\int_{r\leqslant|x|}\theta(z,t)dz \leqslant \frac{1}{r}\int_{r\leqslant|x|}|x|\theta(z,t)dz \leqslant \frac{C(d_{\theta_0}, \|\theta_0\|_{L^\infty(S)})}{r} \quad \text{for any } r>0 \text{ for any } t\geqslant 0. \tag{8}$$

Then, we have an estimate for the first component of the velocity:

$$|u_1(z,t)| \lesssim \int_S \theta(\xi,t)\left|\frac{\sin(y-\xi_2)}{\cosh(x-\xi_1)-\cos(y-\xi_2)}\right|d\xi \lesssim \int_{|x-\xi_1|\geqslant 1}\theta(\xi,t)e^{-|x-\xi_1|}d\xi + \int_{|x-\xi_1|<1}\frac{\theta(\xi,t)}{|z-\xi|}d\xi. \tag{9}$$

The first integral can be estimated as

$$\int_{|x-\xi_1|\geqslant 1}\theta(\xi,t)e^{-|x-\xi_1|}d\xi = \sum_{n\in\mathbb{Z}}\int_{|x-\xi_1|>1, n<\xi_1<n+1}\theta(\xi,t)e^{-|x-\xi_1|}d\xi$$

$$\lesssim \sum_{n\in\mathbb{Z}} e^{-|x-n|}\int_{n<\xi_1<n+1}\theta(\xi,t)d\xi \leqslant C(d_{\theta_0}, \|\theta_0\|_{L^\infty(S)})\sum_{n\in\mathbb{Z}} e^{-|x-n|}(|n|+1)^{-1} \leqslant \frac{C(d_{\theta_0}, \|\theta_0\|_{L^\infty(S)})}{|x|+1}\,.$$

Here we used (8) to bound $\int_{n<\xi_1<n+1}\theta d\xi$ for $n\neq\{0,-1\}$. For $n=\{0,-1\}$, we wrote $\int_{n<\xi_1<n+1}\theta d\xi \lesssim \|\theta\|_{L^\infty(S)}$.

The second integral in (9) can be bounded by Hölder inequality as follows

$$\int_{|x-\xi_1|<1}\frac{\theta(\xi,t)}{|z-\xi|}d\xi \leqslant \||z-\xi|^{-1}\|_{L^p(|x-\xi_1|<1)}\|\theta\|_{L^{p'}(|x-\xi_1|<1)} \lesssim C(\epsilon)\left(\int_{|x-\xi_1|<1}\theta^{p'}d\xi\right)^{\frac{1}{p'}}\,,$$

where $p = 2-\epsilon, \epsilon\in(0,1)$, and p' is defined by $p^{-1}+p'^{-1}=1$. Finally, we have

$$\int_{|x-\xi_1|<1}\theta^{p'}d\xi \lesssim \|\theta\|^{p'-1}_{L^\infty(S)}\int_{|x-\xi_1|<1}\theta d\xi \lesssim \frac{C(\epsilon, d_{\theta_0}, \|\theta_0\|_{L^\infty(S)})}{|x|+1}\,.$$

Thus, for the first component in the Lagrangian dynamics of a point $(x(t), y(t))$ in the support of $\theta(z,t)$, we have

$$|\dot{x}| \leqslant C(\delta, d_{\theta_0}, \|\theta_0\|_{L^\infty(S)})(|x|+1)^{-\frac{1}{2}+\delta} \quad \text{for } t>0$$

with arbitrary $\delta > 0$. This gives the bound

$$d(t) \leqslant C(\delta_1, d_{\theta_0}, \|\theta_0\|_{L^\infty})(1+t)^{\frac{2}{3}+\delta_1} \quad \text{for} \quad \text{every } \delta_1 > 0. \tag{10}$$

4. Proof of Theorem 1.1

In this section, we show that the bound (10) can be improved to $d(t) = \mathcal{O}(t^{1/3}\log^2 t)$. To do that we will exploit the decay of u_1 both in z and t.

Proof of Theorem 1.1. Without loss of generality, we suppose that $0 \leqslant \theta_0 \leqslant 1$, $d_{\theta_0} \leqslant 1$, $\int_S \theta_0(z)dz > 0$, and $\int_S x\theta_0(z)dz = 0$. Then, θ_0 is supported in $[-1,1]\times\mathbb{T}$ and $\int_S \theta_0(z)dz \leqslant 2\pi$. Indeed, notice first that if $\theta(z,t)$ is the solution to (1) and if $\widehat{x} \in \mathbb{R}$, then $\theta(x-\widehat{x}, y, t)$ is the solution to (1) with the initial data $\theta_0(x-\widehat{x}, y)$. Thus, choosing $\widehat{x}$ suitably, we can always assume that $h_0 = 0$. Then, if θ_0 is an arbitrary non-trivial, non-negative, bounded, and compactly supported function satisfying $\int_S x\theta_0(z)dz = 0$ and giving rise to the solution θ, we can rescale θ as follows. Put $M := \|\theta_0\|_{L^\infty} > 0$ and choose $N \in \mathbb{N}$ large enough so that $N \geqslant d(0)$. Then, define $\widetilde{\theta}$ by

$$\widetilde{\theta}(x,y,t) := \frac{1}{M}\theta(N\cdot x, N\cdot y, \frac{t}{M}).$$

Notice now that $\widetilde{\theta}$ is $2\pi/N$-periodic in y, solves the Euler equation (1), and satisfies

$$0 \leqslant \widetilde{\theta}_0 \leqslant 1,\ d_{\widetilde{\theta}_0} \leqslant 1,\ \int_S \widetilde{\theta}_0(z)dz > 0,\ \int_S x\widetilde{\theta}_0(z)dz = 0.$$

We will need the following lemma.

Lemma 4.1. *Take $a \in 2\mathbb{N}$. We have*

$$\int_{|x|>2a} \theta(z,t)dz \lesssim a^{-2}\int_0^t \left(\left(\int_{|x|>a/2}\theta(z,\tau)dz\right)^2 + e^{-a/4}\left(\int_{|x|>a/2}\theta(z,\tau)dz\right)\right)d\tau \tag{11}$$

for any $t \geqslant 0$.

Proof. For $a \in 2\mathbb{N}$, consider

$$k_a(t) := \int_{x>a}(x-a)^2\theta(z,t)dz.$$

If θ is smooth, taking the time derivative of k_a gives

$$\begin{aligned} k_a'(t) &= \int_{x>a}(x-a)^2(\partial_t\theta)dz = -\int_{x>a}(x-a)^2(u\cdot\nabla\theta)dz \\ &= \int_{x>a}(x-a)^2(\Psi_y\theta_x - \Psi_x\theta_y)dz = 2\int_{x>a}(x-a)\cdot\theta\cdot(-\Psi_y)dz. \end{aligned}$$

Recall $-\Psi_y = u_1 = k_1 * \theta$. We estimate the time derivative:

$$\begin{aligned} |k_a'(t)| &= \left|2\int_{x>a}(x-a)\cdot\theta\cdot u_1 dz\right| \lesssim \left|\int_{x>a}\int_S \frac{(x-a)\sin(y-\xi_2)}{\cosh(x-\xi_1)-\cos(y-\xi_2)}\theta(z,t)\theta(\xi,t)d\xi dz\right| \\ &\leqslant \left|\int_{x>a}\int_{\xi_1<a}\dots d\xi dz\right| + \left|\int_{x>a}\int_{\xi_1>a}\dots d\xi dz\right|. \end{aligned}$$

Using the bound $|x-a| \leqslant |x-\xi_1|$ in the first term and symmetrizing in the second one, we get

$$|k_a'(t)| \lesssim \int_{x>a}\int_S \left|\frac{(x-\xi_1)\sin(y-\xi_2)}{\cosh(x-\xi_1)-\cos(y-\xi_2)}\right|\theta(z,t)\theta(\xi,t)d\xi dz.$$

We observe that

$$\left|\frac{(x-\xi_1)\sin(y-\xi_2)}{\cosh(x-\xi_1)-\cos(y-\xi_2)}\right| \lesssim (1+|x-\xi_1|)e^{-|x-\xi_1|} \lesssim e^{-\frac{1}{2}|x-\xi_1|}\,.$$

Let us denote $\int_{a\leqslant\xi<b}\theta(z,t)dz$ by $\int_a^b\theta$ for shorthand. The above estimate implies

$$|k_a'| \lesssim \sum_{j=-\infty}^{\infty}\sum_{l=a}^{\infty} e^{-\frac{1}{2}|j-l|}\left(\int_j^{j+1}\theta\right)\left(\int_l^{l+1}\theta\right).$$

We get

$$\sum_{j=a}^{\infty}\sum_{l=a}^{\infty} e^{-\frac{1}{2}|j-l|}\left(\int_j^{j+1}\theta\right)\left(\int_l^{l+1}\theta\right) \leqslant \left(\sum_{j=a}^{\infty}\int_j^{j+1}\theta\right)^2 = \left(\int_a^{\infty}\theta\right)^2$$

and

$$\begin{aligned}
&\sum_{j=-\infty}^{a}\sum_{l=a}^{\infty} e^{-\frac{1}{2}|j-l|}\left(\int_j^{j+1}\theta\right)\left(\int_l^{l+1}\theta\right)\\
&\leqslant \sum_{j=a/2}^{a}\sum_{l=a}^{\infty} e^{-\frac{1}{2}|j-l|}\left(\int_j^{j+1}\theta\right)\left(\int_l^{l+1}\theta\right) + \sum_{j=-\infty}^{a/2}\sum_{l=a}^{\infty} e^{-\frac{1}{2}|j-l|}\left(\int_j^{j+1}\theta\right)\left(\int_l^{l+1}\theta\right)\\
&\lesssim \left(\int_{a/2}^{\infty}\theta\right)^2 + e^{-a/4}\left(\int_{a/2}^{\infty}\theta\right).
\end{aligned}$$

Since $\int_a^{\infty}(x-a)^2\theta_0 = 0$, we can write

$$\int_a^{\infty}(x-a)^2\theta \lesssim \int_0^t\left(\left(\int_{a/2}^{\infty}\theta\right)^2 + e^{-a/4}\left(\int_{a/2}^{\infty}\theta\right)\right)d\tau$$

and

$$\int_{2a}^{\infty}\theta \lesssim a^{-2}\int_0^t\left(\left(\int_{a/2}^{\infty}\theta\right)^2 + e^{-a/4}\left(\int_{a/2}^{\infty}\theta\right)\right)d\tau.$$

The estimate for $\int_{-\infty}^{-2a}\theta$ can be proved similarly. Thus, we get (11) for smooth solutions. For solutions in the sense of distributions, one can use the mollification argument following, e.g., [3], Proposition 2.1.

□

We denote

$$f_0(t) := \int_S \theta(z,t)dz \quad \text{and} \quad f_n(t) := \int_{|x|>4^n}\theta(z,t)dz, \quad n\in\mathbb{N}.$$

So, $f_0(t) = f_0(0) = \int_S \theta_0(z)dz \lesssim 1$ for $t\geqslant 0$ and $f_n(0) = 0$ for $n\in\mathbb{N}$. Taking $a = 2\cdot 4^n$ in Lemma 4.1, we get the bounds

$$f_{n+1}(t) \lesssim 4^{-2n}\int_0^t (f_n^2(\tau) + e^{-\frac{1}{2}4^n}f_n(\tau))d\tau \quad \text{for any } n\geqslant 0, t>0$$

and Proposition 3.1 yields $f_n(t)\lesssim 4^{-n}$ for any $n\geqslant 0, t\geqslant 0$. We combine these two estimates into

$$f_0(t) = c_1, f_{n+1}(t) \leqslant c_2\min\left(4^{-2n}\int_0^t (f_n^2(\tau) + e^{-\frac{1}{2}4^n}f_n(\tau))d\tau, 4^{-(n+1)}\right) \quad \text{for any } n\geqslant 0, t\geqslant 0\,,$$

where time-independent parameters c_1 and c_2 satisfy $0 < c_1 \lesssim 1, 0 < c_2 \lesssim 1$.

For any bounded and non-negative function h and for $n \geqslant 0$, we define the operator M_n by

$$(M_n h)(t) = c_2 \min \left(4^{-2n} \int_0^t h(\tau) \cdot (h(\tau) + e^{-\frac{1}{2}4^n}) d\tau, 4^{-(n+1)} \right) \quad \text{for } t \geqslant 0.$$

Without loss of generality, we can assume $c_2 \geqslant 2$. Define $\{g_n(t)\}$ recursively by

$$g_0(t) = c_1, g_{n+1} := M_n(g_n)$$

for all $t \geqslant 0$. We can use induction argument to show that

$$f_j(t) \leqslant g_j(t) \quad \text{for any } j \geqslant 0,\, t \geqslant 0\,. \tag{12}$$

From Lemma 5.1, proved in the next section, we know that there exist $n_0 \in \mathbb{N}$ and positive constants c_3, c_4, c_5 such that

$$g_{n+j}(t) \leqslant c_3 4^{-n-c_4 2^j}\,, \tag{13}$$

for all $n \geqslant n_0$, $j \in \mathbb{Z}^+$, and $t \in [0, c_5 4^{3n}]$.

We now can estimate the first component of velocity. By (12) and (13), we have

$$\int_{|x| \geqslant 4^{n+j}} \theta(z,t) dz \leqslant c_3 4^{-n-c_4 2^j} \tag{14}$$

for any $n \geqslant n_0$, $j \geqslant 0$, and $0 \leqslant t \leqslant c_5 4^{3n}$. If necessary, redefine n_0 to satisfy $e \leqslant c_5 4^{3(n_0-1)}$ and let

$$T := c_5 4^{3(n_0-1)} \geqslant e.$$

Now, given any $t \geqslant T$, we choose n to depend on t in such a way that $c_5 4^{3(n-1)} \leqslant t < c_5 4^{3n}$. Substituting this bound into (14), we get

$$\int_{|x| \geqslant 4\left(\frac{t}{c_5}\right)^{1/3} 4^j} \theta(z,t) dz \leqslant c_3 \left(\frac{c_5}{t}\right)^{\frac{1}{3}} 4^{-c_4 2^j}.$$

for all $t \geqslant T$ and $j \geqslant 0$.

For $A > 1/2$ and for $t \geqslant T$, we can take the integer $j = j(A,t) \geqslant 0$ such that $2^{j-1} < A \log t \leqslant 2^j$. Then,

$$\int_{|x| \geqslant 16 A^2 \left(\frac{t}{c_5}\right)^{1/3} \log^2 t} \theta(z,t) dz \leqslant (c_3 \cdot c_5^{\frac{1}{3}}) t^{-(\frac{1}{3} + (c_4 \log 4) A)}.$$

Introducing

$$\phi(L) := 16 \left(\frac{L - \frac{1}{3}}{c_4 \log 4} \right)^2 / c_5^{\frac{1}{3}}, \quad c_7 := c_3 \cdot c_5^{\frac{1}{3}}, \quad L := \frac{1}{3} + (c_4 \log 4) A\,,$$

and assuming that $L > L_0 := (1/3) + (c_4 \log 4)/2$, we can rewrite the last inequality in more convenient form

$$\int_{|x| \geqslant \phi(L) t^{1/3} \log^2 t} \theta(z,t) dz \leqslant c_7 t^{-L} \tag{15}$$

for all $t \geqslant T$. Note that $\phi(L) \sim L^2$ for $L > L_0$.

For $L > L_0$, we define

$$R_L(t) := 2(\phi(L) t^{1/3} \log^2 t + 1)$$

for $0 \leqslant t < \infty$. We need the following lemma.

Lemma 4.2. *There exists $L_1 > 0$ such that*

$$|u_1(z,t)| \leqslant \frac{d}{dt} R_L(t)$$

holds whenever $L \geqslant L_1, |x| = R_L(t)$, and $t \geqslant T$.

Proof. Let $L > L_0$. We have a bound

$$|u_1(z,t)| \lesssim \int_{|x-\xi_1|\geqslant 1} \theta(\xi,t) e^{-|x-\xi_1|} d\xi + \int_{|x-\xi_1|<1} \frac{\theta(\xi,t)}{|z-\xi|} d\xi$$

for any z and for any t. Notice that $\phi(L) t^{1/3} \log^2 t \leqslant \min(R_L(t)/2, R_L(t) - 1)$. Suppose $x = R_L(t)$, the case $x = -R_L(t)$ can be handled similarly. Thus, for $t \geqslant T$, we get

$$\begin{aligned}
&|u_1(R(t), y, t)| \lesssim \\
&\int_{\xi_1 < R_L(t)/2} \theta(\xi,t) e^{-|R_L(t)-\xi_1|} d\xi + \int_{\xi_1 \geqslant R_L(t)/2} \theta(\xi,t) e^{-|R_L(t)-\xi_1|} d\xi + \int_{|R_L(t)-\xi_1|<1} \frac{\theta(\xi,t)}{|z-\xi|} d\xi \\
&\lesssim e^{-R_L(t)/2} \cdot \int_{\xi_1 < R_L(t)/2} \theta(\xi,t) d\xi + \int_{\xi_1 \geqslant R_L(t)/2} \theta(\xi,t) d\xi + \Big(\int_{|R_L(t)-\xi_1|<1} |\theta(\xi,t)|^3 d\xi \Big)^{\frac{1}{3}},
\end{aligned}$$

where we used Hölder inequality to get the last term. Recall that $\|\theta\|_{L^\infty(S)} \leqslant 1, \|\theta\|_{L^1(S)} \lesssim 1$, so

$$\begin{aligned}
&|u_1(R_L(t), y, t)| \lesssim \\
&e^{-R_L(t)/2} \cdot \int_S \theta(\xi,t) d\xi + \int_{\xi_1 \geqslant R_L(t)/2} \theta(\xi,t) d\xi + \|\theta\|_{L^\infty}^{\frac{2}{3}} \Big(\int_{\xi_1 > R_L(t)-1} |\theta(\xi,t)| d\xi \Big)^{1/3} \\
&\lesssim e^{-(\phi(L) t^{1/3} \log^2 t + 1)} + \int_{\xi_1 \geqslant \phi(L) t^{1/3} \log^2 t} \theta(\xi,t) d\xi + \Big(\int_{\xi_1 \geqslant \phi(L) t^{1/3} \log^2 t} |\theta(\xi,t)| d\xi \Big)^{1/3}.
\end{aligned}$$

We now use (15) to get

$$|u_1(R_L(t), y, t)| \leqslant c_8 \Big(e^{-\phi(L) t^{1/3} \log^2 t} + t^{-L} + t^{-\frac{L}{3}} \Big)$$

with some constant c_8.

The derivative of $R_L(t)$ can be computed explicitly:

$$\frac{d}{dt} R_L(t) = 2\phi(L) t^{-2/3} \log t \left(\frac{1}{3} \log t + 2 \right).$$

Recalling that $\phi(L) \sim L^2$ when $L \to \infty$, we can take L_1 large enough to have

$$c_8 \Big(e^{-\phi(L) t^{1/3} \log^2 t} + t^{-L} + t^{-L/3} \Big) \leqslant \frac{d}{dt} R_L(t)$$

uniformly in $t \geqslant T$ and $L \geqslant L_1$.

□

We are ready to finish the proof of Theorem 1.1. We claim that there is an absolute constant L such that

$$d_{\theta(t)} \leqslant 2R_L(t) \tag{16}$$

for all $t \geqslant 0$.

Indeed, notice first that $u_{max} := \sup_{t \geqslant 0} \|u_1(t)\|_{L^\infty} < \infty$ by (7). Since $\phi(L) \sim L^2$, there is $L_2 > 0$ so that for each $L \geqslant L_2$ we get

$$R_L(t) \geqslant 1 + u_{max} \cdot t$$

uniformly in $t \in [0,T]$. Since θ_0 is supported in $[-1,1]\times\mathbb{T}$, the Euler solution $\theta(z,t)$ is supported in $[-R_L(t), R_L(t)]\times\mathbb{T}$ for all $t\in[0,T]$ and for all $L \geqslant L_2$.

Take $L \geqslant \max\{L_0, L_1, L_2\}$. From Lemma 4.2, we conclude that for any particle trajectory $Z_{(x,y)}(t) = (X_{(x,y)}(t), Y_{(x,y)}(t))$ in Lagrangian dynamics, satisfying

$$\begin{cases} \frac{d}{dt} Z_{(x,y)}(t) & = u(Z_{(x,y)}(t), t) \quad \text{ for } t > T \\ Z_{(x,y)}(T) & = (x,y) \in [-R_L(T), R_L(T)] \times \mathbb{T} \end{cases},$$

we have $Z_{(x,y)}(t) \in [-R_L(t), R_L(t)]\times\mathbb{T}$ for any $t \geqslant T$. Indeed, we argue by contradiction: if there is a particle trajectory escaping from the region, then there should be a moment $T_0 \geq T$ and a point $z_0 = (x_0, y_0) \in S$ such that $|x_0| = R_L(T_0)$ and $|u_1(z_0, T_0)| > \frac{d}{dt} R_L(t_0)$, which contradicts the lemma.

Estimate (16) finishes the proof of Theorem.

□

5. Some auxiliary results

Recall that the operator M_n has been defined as

$$(M_n h)(t) = c_2 \min\left(4^{-2n}\int_0^t h(\tau)\cdot(h(\tau) + e^{-\frac{1}{2}4^n})d\tau, 4^{-(n+1)}\right) \quad \text{ for } t \geqslant 0$$

and $c_2 \geqslant 2$. Take $c_1 > 0$ and define $\{g_n(t)\}$ recursively by

$$g_0(t) = c_1, g_{n+1} := M_n(g_n) \quad \text{ for all } t \geqslant 0. \tag{17}$$

Lemma 5.1. *There exists an integer $n_0 \in \mathbb{N}$ and positive constants c_3, c_4, c_5 such that for any $n \geqslant n_0$ and for $0 < t \leqslant c_5 4^{3n}$, we have*

$$g_{n+j}(t) \leqslant c_3 4^{-n-c_4 2^j} \quad \textit{for any } j \geqslant 0. \tag{18}$$

Proof. Without loss of generality, we assume $c_2 \geqslant 2$. For any bounded and non-negative function h and for $n \geqslant 0$, $M_n h(\cdot)$ is non-decreasing in t. We denote by $T_n(h)$ the first time when $M_n h(t) = c_2 4^{-n-1}$. If h is non-decreasing and if h is not identically zero, we have $0 < T_n(h) < \infty$. Moreover,

$$T_n(h_1) \leqslant T_n(h_2)$$

if $h_1 \geqslant h_2 \geqslant 0$ for all t. Function g_n defined in (17) satisfies the following properties.

- For all $n \in \mathbb{Z}^+$, g_n is non-decreasing, bounded, non-negative, and $g_n(t) > 0$ for any $t > 0$.
- Denote $t_n := T_{n-1}(g_{n-1}) < \infty$ for $n \geqslant 1$ and $t_0 := 0$. In other words, $t_n = \min\{\tau : g_n(\tau) = c_2 4^{-n}\}$ for $n \geqslant 1$. We will need some estimates on t_n later on so we start with getting a lower bound.

Since $e^{-\alpha} < 1/\alpha$ for $\alpha > 0$ and $c_2 \geqslant 2$, we get $e^{-(\frac{1}{2})4^n} \leqslant 2\cdot 4^{-n} \leqslant c_2 4^{-n}$. Then, the estimate $g_n \leqslant c_2 4^{-n}$ yields

$$g_{n+1}(t) \leqslant c_2 4^{-2n}\int_0^t c_2 4^{-n}(c_2 4^{-n} + e^{-\frac{1}{2}4^n})d\tau \leqslant c_2 4^{-2n}\int_0^t 2(c_2)^2 4^{-2n} d\tau \leqslant 2c_2^3 t 4^{-4n}.$$

Thus,

$$t_{n+1} \geqslant 4^{3n-1}/(2c_2^2), \quad \text{for}\, n \in \mathbb{N}. \tag{19}$$

An upper bound on t_n can be obtained as follows. Since $g_n \geqslant 0$ on $t \in [0, t_n]$ and $g_{n+1} = c_2 4^{-(n+1)}$ for $t \geqslant t_{n+1}$, we have

$$t_{n+1} \leqslant t_n + 4^{3n-1}/(c_2^2),\, n \in \mathbb{N}\,. \tag{20}$$

Indeed, this inequality holds trivially if $t_{n+1} \leqslant t_n$. For the case $t_{n+1} \geqslant t_n$, we have

$$c_2 4^{-(n+1)} = g_{n+1}(t_{n+1}) = M_n(g_n)(t_{n+1}) \geqslant c_2 4^{-2n} \int_0^{t_{n+1}} g_n^2(\tau) dt \geqslant c_2 4^{-2n} \int_{t_n}^{t_{n+1}} c_2^2 4^{-2n} dt.$$

Summing up (20) in n, we get

$$t_n \leqslant t_1 + \sum_{k=1}^{n} 4^{3k-4}/(c_2^2) \leqslant t_1 + 4^{3n-3}/(c_2^2)\,.$$

Since $t_1 = (4c_1(c_1 + e^{-\frac{1}{2}}))^{-1}$, the last estimate and (19) imply that there are positive constants c_5 and c_6 so that

$$c_5 4^{3n} \leqslant t_n \leqslant c_6 4^{3n}, \quad n \in \mathbb{N}\,. \tag{21}$$

- Let $n \geqslant 1$. Since g_k is non-decreasing in t, we can write the following bound for every $j \in \mathbb{Z}^+$:

$$\begin{aligned} g_{(n+j)+1}(t_n) &\leqslant c_2 4^{-2(n+j)} \int_0^{t_n} g_{(n+j)}(\tau) \cdot (g_{(n+j)}(\tau) + e^{-\frac{1}{2}4^{(n+j)}}) d\tau \\ &\leqslant c_2 4^{-2(n+j)} g_{(n+j)}(t_n) \cdot (g_{(n+j)}(t_n) + e^{-\frac{1}{2}4^{(n+j)}}) \cdot t_n \\ &\leqslant c_2 4^{-2(n+j)} g_{(n+j)}(t_n) \cdot (g_{(n+j)}(t_n) + e^{-\frac{1}{2}4^{(n+j)}}) \cdot c_6 4^{3n}\,, \end{aligned}$$

where we used (21) to bound t_n in the last inequality.

For shorthand, let's denote $a_{n,j} := g_{n+j}(t_n)$ for $j \geqslant 0, n \in \mathbb{N}$. Then, we can write

$$a_{n,(j+1)} \leqslant \min\Big(c_2 4^{-2(n+j)} a_{n,j} \cdot (a_{n,j} + e^{-\frac{1}{2}4^{(n+j)}}) \cdot c_6 4^{3n}, c_2 4^{-(n+j+1)}\Big)\,.$$

Notice also that $a_{n,0} = c_2 4^{-n}$. The induction argument gives $a_{n,j} \leqslant b_{n,j}$ where $\{b_{n,j}\}$ are introduced in (22) a few lines below. Since $g_{n+j}(t) \leqslant a_{n,j}$ for all $t \leqslant t_n$ and $t_n \geqslant c_5 4^{3n}$, the estimate (18) now follows from Proposition 5.1. The proof is finished.

□

Let c_2, c_6 be positive constants and $c_2 \geqslant 2$. For each $n \in \mathbb{N}$, define $\{b_{n,j}\}_{j=0}^{\infty}$ recursively by $b_{n,0} := c_2 4^{-n}$ and

$$b_{n,(j+1)} = \min\left(c_2 4^{-2(n+j)} b_{n,j} \cdot (b_{n,j} + e^{-\frac{1}{2}4^{(n+j)}}) \cdot c_6 4^{3n}, c_2 4^{-(n+j+1)}\right) \quad \text{for } j \geqslant 0. \tag{22}$$

Proposition 5.1. *There are positive constants c_3, c_4 and $n_0 \in \mathbb{N}$, such that*

$$b_{n,j} \leqslant c_3 4^{-n-c_4 2^j} \quad \text{for all } n \geqslant n_0, j \in \mathbb{Z}^+\,.$$

Proof. For a later use, take $n_0 \in \mathbb{N}$ so large that

$$2c_2c_6 4^{3n_0} \geqslant 4. \tag{23}$$

From now on, let $n \geqslant n_0$. We first claim that

$$b_{n,j} \geqslant e^{-\frac{1}{2}4^{(n+j)}} \tag{24}$$

for all $j \geqslant 0$ and for all $n \geqslant n_0$. Indeed, it can be shown by an induction in j: we know

$$b_{n,0} = c_2 4^{-n} \geqslant 2 \cdot 4^{-n} \geqslant e^{-\frac{1}{2}4^n}$$

because $c_2 \geqslant 2$. Suppose $b_{n,j} \geqslant e^{-\frac{1}{2}4^{(n+j)}}$ for some $j \geqslant 0$. We need to show

$$b_{n,(j+1)} \geqslant e^{-\frac{1}{2}4^{(n+j+1)}}.$$

Recall that $b_{n,(j+1)}$ is either $\Big(c_2 4^{-2(n+j)} b_{n,j} \cdot (b_{n,j} + e^{-\frac{1}{2}4^{(n+j)}}) \cdot c_6 4^{3n}\Big)$ or $\big(c_2 4^{-(n+j+1)}\big)$. In the former case,

$$\begin{aligned} b_{n,(j+1)} &\geqslant 2c_2c_6 4^{3n}(4^{-(n+j)})^2(e^{-\frac{1}{2}4^{(n+j)}})^2 \geqslant 2c_2c_6 4^{3n_0}(4^{-(n+j)})^2(e^{-\frac{1}{2}4^{(n+j)}})^2 \\ &\geqslant 4(4^{-(n+j)})^2(e^{-\frac{1}{2}4^{(n+j)}})^2 \geqslant (e^{-\frac{1}{2}4^{(n+j)}})^2(e^{-\frac{1}{2}4^{(n+j)}})^2 = e^{-\frac{1}{2}4^{(n+j+1)}}, \end{aligned}$$

where we use (23) for the third inequality. If, however, $b_{n,(j+1)} = c_2 4^{-(n+j+1)}$, then $b_{n,(j+1)} \geqslant 2 \cdot 4^{-(n+j+1)} \geqslant e^{-\frac{1}{2}4^{(n+j+1)}}$. Thus, (24) is proved.

By the claim, for all $n \geqslant n_0$ and $j \geqslant 0$, we get

$$b_{n,(j+1)} \leqslant \min\Big(2c_2 4^{-2(n+j)}(b_{n,j})^2 \cdot c_6 4^{3n}, c_2 4^{-(n+j+1)}\Big).$$

To get the needed bound on $b_{n,j}$, we again argue by comparison to exact recursion. Define $\{c_{n,j}\}_{j=0}^{\infty}$ by $c_{n,0} := b_{n,0} = c_2 4^{-n}$ and by the following iteration:

$$c_{n,(j+1)} = \min\Big(2c_2 4^{-2(n+j)}(c_{n,j})^2 \cdot c_6 4^{3n}, c_2 4^{-(n+j+1)}\Big) \quad \text{for } j \geqslant 0.$$

Then, we clearly have $c_{n,j} \geqslant b_{n,j}$ for $j \geqslant 0$ and for $n \geqslant n_0$.

To iterate the formula for $c_{n,j}$, it is convenient to rewrite it in the following form

$$c_{n,(j+1)} = \min\Big(4^{\beta} 4^{n-2j}(c_{n,j})^2, 4^{\alpha} 4^{-(n+j+1)}\Big) \quad \text{for } j \geqslant 0,$$

where real α and β are defined by $2c_2c_6 = 4^{\beta}$ and $c_2 = 4^{\alpha}$. If we represent $c_{n,j}$ as $c_{n,j} = 4^{-p_{n,j}}$, then $p_{n,0} = n - \alpha$ and

$$p_{n,(j+1)} = \max\Big(- \beta - n + 2j + 2p_{n,j}, -\alpha + n + j + 1\Big) \quad \text{for } j \geqslant 0.$$

We further write $p_{n,j} = n + q_{n,j}$ and notice that

$$q_{n,(j+1)} = \max\Big(- \beta + 2(j + q_{n,j}), -\alpha + j + 1\Big) \quad \text{for } j \geqslant 0.$$

Take the smallest $j_0 \in \mathbb{N}$ for which $j_0 \geqslant \alpha + 1$ and $2j_0 \geqslant \beta$. Then, we have

$$q_{n,j_0} \geqslant -\alpha + (j_0 - 1) + 1 \geqslant 1$$

and, for any $j \geqslant j_0$,

$$q_{n,(j+1)} \geqslant -\beta + 2(j + q_{n,j}) \geqslant -\beta + 2j_0 + 2q_{n,j} \geqslant 2q_{n,j}.$$

It implies that, for any $j \geqslant j_0$, we get

$$q_{n,j} \geqslant 2^{j-j_0}$$

and then

$$p_{n,j} \geqslant n + 2^{j-j_0}.$$

In other words, for any $n \geqslant n_0$ and for any $j \geqslant j_0$, we have

$$c_{n,j} \leqslant 4^{-(n+2^{j-j_0})} = 4^{-(n+(2^{-j_0})2^j)}. \tag{25}$$

We claim now that for any $n \geqslant n_0$ and for any $j \geqslant 0$, one has

$$c_{n,j} \leqslant 4^{\alpha+(2^{-j_0})} 4^{-(n+(2^{-j_0})2^j)}. \tag{26}$$

Indeed, for $j \geqslant j_0$, this follows from $\alpha \geqslant 0$ and (25). From the definition of $c_{n,j}$, we get $c_{n,j} \leqslant 4^\alpha 4^{-(n+j)}$ for any $j \geqslant 0$. Thus, the case $j = 0$ is trivial. For $1 \leqslant j \leqslant j_0$, we use an elementary bound $4^{-j} \leqslant 4^{-(2^{-j_0})2^j}$.

Taking $c_3 := 4^{\alpha+(2^{-j_0})}$ and $c_4 := 2^{-j_0}$ in (26), we finish the proof of the proposition. □

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References

[1] A. Afendikov, A. Mielke, Dynamical properties of spatially non-decaying 2D Navier-Stokes flows with Kolmogorov forcing in an infinite strip. J. Math. Fluid. Mech. 7 (2005), suppl. 1, S51-S67.

[2] J. Bedrossian, N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes Etudes Sci. 122 (2015), 195–300.

[3] J. Beichman, S. Denisov, 2D Euler equation on the strip: stability of a rectangular patch. Comm. Partial Differential Equations 42 (2017), no. 1, 100-120.

[4] S. Caprino, C. Marchioro, On nonlinear stability of stationary planar Euler flows in an unbounded strip. Nonlinear Anal.-Theory 10 (1986), no. 11, 1263–1275.

[5] Th. Gallay, S. Slijepcevic, Energy bounds for the two-dimensional Navier-Stokes equations in an infinite cylinder. Comm. Partial Differential Equations 39 (2014), 1741–1769.

[6] Th. Gallay, S. Slijepcevic, Uniform boundedness and long-time asymptotics for the two-dimensional Navier-Stokes equations in an infinite cylinder. Journal of Mathematical Fluid Mechanics 17 (2015), 23–46.

[7] D. Iftimie, M.C. Lopes Filho, H.J. Nussenzveig Lopes, Confinement of vorticity in two-dimensional ideal incompressible exterior flow. Quart. Appl. Math. 65 (2007), 499–521.

[8] D. Iftimie, T. Sideris, P. Gamblin, On the evolution of compactly supported planar vorticity. Comm. Partial Differential Equations 24 (1999), no. 9-10, 1709–1730.

[9] J.P. Kelliher, A characterization at infinity of bounded vorticity, bounded velocity solutions to the 2D Euler equations. Indiana Univ. Math. J. 64 (2015), no. 6, 1643–1666.

[10] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow. Cambridge University Press, 2003.

[11] C. Marchioro, Bounds on the growth of the support of a vortex patch. Comm. Math. Phys. 164 (1994), 507–524.

[12] C. Marchioro, On the growth of the vorticity support for an incompressible non-viscous fluid in a two-dimensional exterior domain. Math. Meth. Appl. Sci. 19 (1996), 53–62.

[13] P. Serfati, Borne en temps des caractéristiques de l'équation d'Euler 2D à tourbillon positif et localisation pour le modèle point-vortex. Preprint.

[14] V.I. Yudovič, Non-stationary flows of an ideal incompressible fluid. Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 1032–1066.

Kyudong Choi: kchoi@unist.ac.kr

Ulsan National Institute of Science and Technology
Department of Mathematical Sciences
UNIST-gil 50, Ulsan, 44919, Republic of Korea

Sergey Denisov: denissov@wisc.edu

University of Wisconsin–Madison
Mathematics Department
480 Lincoln Dr., Madison, WI, 53706, USA

and

Keldysh Institute for Applied Mathematics, Russian Academy of Sciences
Miusskaya pl. 4, 125047 Moscow, RUSSIA