

THE SHARP CORNER FORMATION IN 2D EULER DYNAMICS OF PATCHES: INFINITE DOUBLE EXPONENTIAL RATE OF MERGING

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ABSTRACT. For the 2d Euler dynamics of patches, we investigate the convergence to the singular stationary solution in the presence of a regular strain. It is proved that the rate of merging can be double exponential infinitely in time and the estimates we obtain are sharp.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The two-dimensional Euler equation on the plane can be written in the vorticity form as follows

$$\dot{\theta} = \nabla\theta \cdot u, \quad u = \pi\nabla^\perp\Delta^{-1}\theta, \quad \theta(z, 0) = \theta_0, \quad z = (x, y) \in \mathbb{R}^2, \quad \nabla^\perp = (-\partial_y, \partial_x) \quad (1)$$

The constant π in the formula for u can be dropped by time scaling but we write an equation this way on purpose to simplify the later calculations. The global in time regularity for the smooth initial data dates back to the paper by Wolibner [17]. The method of Wolibner was based on the Lagrange formulation of the problem but later other approaches were used (see, e.g., [2]). Whatever method is used to establish the global regularity, one proves the upper bounds for the norms that measure the regularity of the solution and for the 2d Euler all we know is that these norms grow in time not faster than the double exponential. The natural question is: are these estimates sharp? This problem is of course of the same nature as the problem of the possible blow up for the 3d Euler equation which is the central problem in the mathematical theory of fluids. In spite of its importance, very little is known even in dimension two. For the two-dimensional torus, it was proved in [12] that the vorticity gradient can indeed grow as double exponential for arbitrarily long (but fixed) time provided that it is large enough at time $t = 0$. This result, however, (as well as any other known to the author) did not reveal an intrinsic mechanism for the singularity formation but rather only indicated that the standard double exponential estimates can not be dramatically improved.

In this paper, we describe the scenario in which the singularity forms though we consider an easier problem, the problem of the patch evolution. The method we apply is quite general and can be tried for other evolution equations: 2d and 3d Euler equations, surface quasi-geostrophic equation, etc.

The problem of patch evolution deals with the case when the initial data θ_0 in (1) is the characteristic function of some compact set. The initial data $\theta_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ gives rise to globally defined unique solution thanks to the theory of Yudovich [18]. In our case, this compact set will be a centrally symmetric pair of two simply connected domains with smooth boundaries and thus the Yudovich theory assures that the solution will always be

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the centrally symmetric pair of domains because the Euler evolution on the plane preserves the central symmetry. The main question we address here is: how does the geometry of these domains change in time? This problem attracted a lot of attention in both physics and mathematical literature in the last several decades and became a classical one. In [4], Chemin proved that if the boundary of the patch is sufficiently regular then it will retain the same regularity forever; another proof of that fact was given later by Bertozzi and Constantin [1]. We recommend the wonderful books [2, 5] for introduction to the subject and for simplified proofs.

Another model which became quite fashionable lately is the so-called surface quasi-geostrophic equation (SQG for the shorthand). It is different from the Euler equation (1) only in the definition of the velocity: $u = \nabla^\perp \Delta^{-1/2} \theta$ (for SQG) vs. $u = \nabla^\perp \Delta^{-1} \theta$ (for 2d Euler), i.e., the kernel used in convolution is more singular. Only the local in time existence of the regular solutions is known for the smooth initial data and there is a conjecture that the blow up happens in finite time. The patch evolution for the SQG was also extensively studied and there is a numerical evidence and widely accepted belief that the patch with smooth boundary can develop the singularity in finite time (see, e.g., [6] and related [7, 9, 10, 11, 13]). The idea behind it is based on the expectation that the stronger the singularity of the kernel is, the faster the process of the singularity formation is supposed to be and since this process should be nonlinear in nature the blow up can supposedly happen in finite time. That mechanism, however, has never been justified or even explained.

In this paper, we present the scenario in which the singularity of the kernel plays the major role in the speed of the blow up formation however we have the proof only for the case of 2d Euler. As we mentioned already, the patch dynamics for 2d Euler can not go wild in finite time so we are looking for the situation when the geometry goes singular at $t = +\infty$. The latter can be described in various ways: growth in time of the curvature, perimeter, etc. In this paper, we focus on one particular geometric characteristic, the distance between two interacting patches, and show that it can decay as double exponential infinitely in time. The estimates we obtain are sharp so in some sense our results are optimal. That, however, comes with the price as we need to impose the strain whose main purpose is to prevent the bulk of the patches from going into chaotic regime. This will be explained later.

To state the main result we need to introduce some notation first. Let $\Omega^- = \{-z, z \in \Omega\}$, i.e., the image of Ω under the central symmetry. The boundary of Ω will be denoted by Γ . In $\mathbb{R}^2 \sim \mathbb{C}$, we consider the 2d Euler dynamics given by the initial configuration $\Omega(0) \cup \Omega^-(0)$ where $\Omega(0)$ will be defined later. The areas of patches, the distance between them, the value of vorticity – all these quantities are of order one as $t = 0$. The curve $\Gamma(0)$ is smooth and its curvature is of order one as well. As time evolves, the Euler evolution deforms $\Omega(0) \cup \Omega^-(0)$ to a new pair $\Omega(t) \cup \Omega^-(t)$ with $|\Omega(t)| = |\Omega(0)|$ because the flow preserves the area and the central symmetry. These new patches will be separated from each other for all times, i.e. $\text{dist}(\Omega(t), \Omega^-(t)) > 0$, but the question, however, is how small the distance between them can get? To answer this question we need to account for the following well-known fact first. If one considers the model of Euler evolution of two identical point vortices on the the plane, then the dynamics is quite simple: the vortices will rotate with constant angular velocity [2]. That suggests that two symmetric patches will tend to “rotate” until some chaotic regime will homogenize them to the state which is hard to control and there is a numerical evidence that this chaotic regime does occur for many $\Omega(0)$ [15]. So, to avoid this chaotic behavior

of our contours (and we need to control them for all $t > 0!$), one needs to impose the strain which will factor out the intrinsic rotation but which will also be regular enough to not significantly influence the nonlinear mechanism for the singularity formation. Therefore, it is very natural to switch from the original problem to the following one

$$\dot{\theta} = \nabla\theta \cdot (u + S), \quad u = \pi\nabla^\perp\Delta^{-1}\theta, \quad \theta(z, 0) = \chi_{\Omega(0)} + \chi_{\Omega^-(0)} \quad (2)$$

where $\theta(z, t) = \chi_{\Omega(t)} + \chi_{\Omega^-(t)}$ and $S(z, t)$ is sufficiently regular incompressible strain. The analytical perspective (and this is our way to think about the problem) is that the solution θ to (2) we will obtain can be regarded as an approximate solution to the original problem (1). Now we are ready to state the main result of the paper.

Theorem 1.1. *Let $\delta \in (0, 1)$. Then, there is a simply connected domain $\Omega(0)$ with smooth boundary satisfying $\text{dist}(\Omega(0), \Omega^-(0)) \sim 1$, and a time-dependent compactly supported incompressible odd strain*

$$S(z, t) = (P(z, t), Q(z, t))$$

such that

$$\text{dist}(\Omega(t), 0) \sim \text{dist}(\Omega(t), \Omega^-(t)) \lesssim \exp(-e^{\delta t}), \quad t \geq 0$$

where $\Omega(t) \cup \Omega^-(t)$ is the Euler dynamics of $\Omega(0) \cup \Omega^-(0)$ in the presence of the strain $S(z, t)$, i.e. the solution to (2). Moreover, for $S(z, t)$ we have

$$\sup_{t \geq 0, z \neq 0} \frac{|S(z, t)|}{|z|} < \infty \quad (3)$$

and

$$|S(z_2, t) - S(z_1, t)| \lesssim |z_1 - z_2|(1 + |\log|z_1 - z_2||) \quad (4)$$

uniformly in $z_{1(2)} \in \mathbb{C}$ and $t \geq 0$.

Remark 1. As it will be clear from the proof, these contours will touch each other at $t = +\infty$ and the touching point is at the origin. In the local coordinates around the origin the functions parameterizing the contours converge to $\pm|x|$ in a self-similar way which will be described in detail.

Remark 2. The simple modification of the Yudovich theory (see, e.g., [2]) implies that the patch evolution given by (2) is uniquely defined provided that $S(z, t)$ satisfies (4) and is uniformly bounded, the latter is warranted by (3). Since our strain S is divergence-free and odd, this dynamics will also preserve the area and the central symmetry. The latter, in particular, implies that the origin is the stationary point for the dynamics. In the corollary 1.2 below, we will give the lower bound on the distance of any particle trajectory to the origin. This estimate will prove that the theorem 1.1 is essentially sharp.

Now, we need to explain why the strain we add is indeed a small correction from the dynamical perspective. This will be done in the following elementary lemma. We will show that under the S -strain alone no point can approach the origin in the rate faster than exponential as long as assumption (3) is made and so it is the nonlinear term $\nabla\theta \cdot u$ in (2) that produces the “double exponentially” fast singularity formation.

Lemma 1.1. *Let $S(z, t)$ be an odd vector field that satisfies (3) and (4). Consider $\theta(z, t) = \chi_{\Upsilon(t)}(z)$ which solves*

$$\dot{\theta} = \nabla\theta \cdot S(z, t), \quad \theta(z, 0) = \chi_{\Upsilon(0)}(z) \quad (5)$$

where $\text{dist}(\Upsilon(0), 0) > 0$ and $\Upsilon(0)$ is a compact set. Then,

$$\text{dist}(\Upsilon(t), 0) \gtrsim e^{-Ct} \quad (6)$$

Proof. Clearly (5) is a transport equation and we have

$$\dot{z} = -S(z, t), \quad z(0) = z_0$$

for the characteristics $z(z_0, t)$. It has the unique solution (see lemma 3.2, page 67, [14]) due to log-Lipschitz regularity (4). As $S(z, t)$ is odd, $S(0, t) = 0$ and so the origin is a stationary point. The estimate (3) yields

$$|S(z, t)| < C|z|, \quad \forall t \geq 0 \quad \text{and} \quad \forall z \in \mathbb{C}$$

Therefore,

$$|\dot{r}| \leq Cr, \quad r = |z|^2$$

and

$$r(0)e^{-Ct} \leq r(t)$$

This implies (6) as the patch $\Upsilon(t)$ is convected by the flow. \square

Next, we will show that the theorem 1.1 is essentially sharp on the double exponential scale. To explain that, we need the next lemma. As a motivation let us start with two simple questions: if we are given a patch Ω of area ~ 1 such that the velocity generated by Ω is zero at the origin, how large can the radial component of the velocity be in the fixed point close to the origin? How should we choose Ω to get the maximum (or get close to the maximum) of this value? Clearly, we are interested in the radial component since it is the one which will push the points to the origin or away from it. As the problem is invariant under the rotation, we can take $z = (x, 0)$, $x > 0$ without loss of generality.

Lemma 1.2. *Assume that the set Ω is such that $|\Omega| \sim 1$ and*

$$u(0) = 0, \quad u(z) = \pi \nabla^\perp \Delta^{-1} \chi_\Omega = (u_1, u_2)$$

Then,

$$u_1 = x (\Theta_\Omega(|z|) + O(1)), \quad 0 < x < 1, \quad z = (x, 0)$$

and

$$\Theta_\Omega(|z|) = \int_{\Omega \cap \{|\xi| > |z|\}} \frac{\xi_1 \xi_2}{(\xi_1^2 + \xi_2^2)^2} d\xi$$

Proof. This estimate is standard, see lemma 8.1 from [2], pages 315–318. Indeed,

$$u(z) = u(z) - u(0) = \frac{1}{2} \int_\Omega \left(\frac{z - \xi}{|z - \xi|^2} + \frac{\xi}{|\xi|^2} \right)^\perp d\xi$$

We have

$$\int_{\Omega \cap \{|\xi| < 2|z|\}} \left| \frac{z - \xi}{|z - \xi|^2} + \frac{\xi}{|\xi|^2} \right| d\xi \lesssim |z|$$

and

$$\frac{1}{2} \int_{\Omega \cap \{|\xi| > 2|z|\}} \left(\frac{z - \xi}{|z - \xi|^2} + \frac{\xi}{|\xi|^2} \right)^\perp d\xi = (m_1 + m_2, \cdot)$$

where

$$m_1 = -\frac{x^2}{2} \int_{\Omega \cap \{|\xi| > 2|z|\}} \frac{\xi_2}{(\xi_1^2 + \xi_2^2)((x - \xi_1)^2 + \xi_2^2)} d\xi, \quad |m_1| \lesssim |z|$$

and

$$m_2 = x \int_{\Omega \cap \{|\xi| > 2|z|\}} \frac{\xi_1 \xi_2}{(\xi_1^2 + \xi_2^2)((x - \xi_1)^2 + \xi_2^2)} d\xi$$

For m_2 , after scaling by $|z| = x$,

$$m_2 = x \int_{\widehat{\Omega} \cap \{|\widehat{\xi}| > 2\}} \frac{\widehat{\xi}_1 \widehat{\xi}_2}{(\widehat{\xi}_1^2 + \widehat{\xi}_2^2)((1 - \widehat{\xi}_1)^2 + \widehat{\xi}_2^2)} d\widehat{\xi} = x(\Theta_\Omega + O(1))$$

□

Corollary 1.1. *For fixed $z = (x, 0)$, $x \in (0, 1/2)$, we have*

$$\max_{\Omega: |\Omega| \sim 1} \Theta_\Omega(|z|) = -\log |z| + O(1) = \Theta_{E \cup E^-}(|z|) + O(1) \quad (7)$$

where $E = [0, 1]^2$

Proof. The proof is a direct calculation in the polar coordinates. □

In particular, these results give the leading term for the size of radial component of the velocity field generated by the centrally symmetric configuration of patches since for this configuration the velocity is always zero at the origin. We also see that the “cross configuration”, i.e., $E \cup E^-$, gives the optimal value up to an additive constant. This configuration will also play a major role in the proof of theorem 1.1.

We can immediately apply two previous results to the problem of patch dynamics.

Corollary 1.2. *Under the conditions of the theorem 1.1, we have*

$$\text{dist}(\Omega(t), 0) \gtrsim \exp(-Ce^t) \quad (8)$$

where C depends on the constant in (3) and on the initial distance between the contours.

Proof. Indeed, for the trajectory

$$\dot{z} = -(u(z, t) + S(z, t)), \quad z(z_0, 0) = z_0$$

we have

$$\dot{r} \leq -r(\log r + C), \quad r = |z|^2 \ll 1$$

which implies (8). □

The constant δ from the theorem 1.1 as well as the constant one in front of t in (8) can be changed by a simple rescaling (i.e., by multiplying the value of vorticity by a constant which is the same as scaling the time) thus the size of δ is small only when compared to the parameters of the problem.

Remark 3. The optimality up to a constant of E in (7) is the reason why our estimate in the theorem 1.1 is essentially sharp on the double exponential scale. Clearly, one can replace E by any other configuration as long as it forms a corner of $\pi/2$ at the origin.

In \mathbb{R}^2 , in contrast to \mathbb{T}^2 , the kernel of Δ^{-1} is easier to write and Δ^{-1} can be defined on compactly supported L^1 functions so we will address the problem on the whole plane rather than on \mathbb{T}^2 . On the 2d torus, similar results hold.

The interaction of two vortices was extensively studied in the physics literature (see, e.g., [15, 16]). For example, the merging mechanism was discussed in [15] where some justifications (both numerical and analytical) were given. In our paper, we provide rigorous analysis of that process and obtain the sharp bounds.

In [8], the authors study an interesting question of the “sharp front” formation. Loosely speaking, the sharp front forms if, for example, two level sets of vorticity, each represented by a smooth time-dependent curve, converge to a fixed smooth arc as $t \rightarrow \infty$. Let the “thickness” of the front be denoted by $\varrho_{front}(t)$. In [8], the following estimate for 2d Euler dynamics is given (see theorem 3, p. 4312)

$$\varrho_{front}(t) > e^{-(At+B)}$$

with constants A and B depending only on the geometry of the front. The scenario considered in our paper is different as the singularity forms at a point.

The idea of the proof comes from the following very natural question. Consider an active-scalar dynamics

$$\dot{\theta} = \nabla\theta \cdot \nabla^\perp(A\theta) \tag{9}$$

where A is the convolution with a kernel $K_A(\xi)$. In some interesting cases, $A = \Delta^{-\alpha}$ with $\alpha > 0$ so K_A is positive and smooth away from the origin. It also obeys some symmetries inherited from its symbol on the Fourier side, e.g., is radially symmetric. Perhaps, the most interesting cases are $\alpha = 1$ (2d Euler), which is treated in this paper, and $\alpha = 1/2$ (SQG) already mentioned in the text. If one considers the problem (9) with $A = \Delta^{-\alpha}$ on the 2d torus $\mathbb{T}^2 = [-\pi, \pi]^2$, then there is a stationary singular weak solution (the author learned about this solution from [13]), a “cross”

$$\theta_s = \chi_E + \chi_{E^-} - \chi_J - \chi_{J^-} \tag{10}$$

where $E = [0, \pi]^2$ and $J = [-\pi, 0] \times [0, \pi]$. One can think about two patches touching each other at the origin and each forming the right angle. This picture is also centrally symmetric. Now, the question is: is this configuration stable? In other words, can we perturb these patches a little so that they will converge to the stationary solution at least around the origin at $t \rightarrow \infty$? The flow generated by θ_s is hyperbolic and so is unstable. However, if one places a curve into the stationary hyperbolic flow in such a way that its part follows the separatrix of the flow in the attracting direction, then the time evolution of this curve will produce a sharp corner. The problem of course is that the actual flow is induced by the patch itself and so it will be changing in time. That suggests that one has to be very careful with the choice of the initial patch to guarantee that this process is self-sustaining. Nevertheless, that seems possible and thus the mechanism of singularity formation through the hyperbolic flow can probably be justified. We do it here by neglecting the smaller order terms. In general, the application of some sort of fixed point argument seems to be needed. Either way, this scenario is a zero probability event (at least the way the proof goes) if the “random” initial condition is chosen. However, if one wants to see merging for a long but fixed time, then this can be achieved for an open set of initial data so from that perspective our construction is realistic.

We will handle the case $\alpha = 1$ only without trying to make the strain smooth. The question whether one can choose $S(z, t) \in C^\infty$ seems to be the right one to address (rather than trying to make $S(z, t) = 0$) and we formulate it as an

Open problem. *Can one take the strain $S(z, t)$ in the theorem 1.1 to be infinitely smooth?*

If the answer is yes, then one can view this strain as the incompressible flow generated by the some background or another patch far away which is decoupled from the evolution of our symmetric pair (that of course is yet another approximation among many others made in the area of Fluid Dynamics). It is not easy for us to imagine how to get rid of the strain completely for the 2d Euler model if one wants to prove the sharp estimate on the rate of merging. For other models like SQG, this is not ruled out as one has to control the bulk of the patches only for finite time and for this finite time the chaotic regime might not be able to start influencing the picture.

Solving the open problem stated above might require closure of the fixed point argument outlined above. We were able to do that so far only for the model equation where the convolution kernel is a smooth bump. Surprisingly, the analysis needed to justify the self-sustaining process we discussed is reminiscent to what one has to do to prove the homogenization in the 2d Euler for $t \rightarrow \infty$ (see, e.g., [3]). This is technically hard and has never been carried out. There is yet another argument indicating that the answer to the open problem above is positive. In [16], the existence of very interesting V -shapes is mentioned (no proofs though). The V -shape is a patch that rotates with the constant angular velocity under the 2d Euler dynamics. Saffman indicates that there is a continuous parametric curve of these V -shapes, each represented by a pair of centrally symmetric patches with smooth boundary. The endpoint of this curve however is represented by a pair of centrally symmetric patches that touch each other at the origin and form a sharp corner there (a singularity we want to produce dynamically). It would be interesting to prove existence of this curve and show that the dynamics representing the evolution of our patch can move across these “invariant sets” (i.e., the particular V -shapes) thus approaching the endpoint of this parametric curve locally around the origin. That would be consistent with the self-sustaining scenario we want to understand.

For $\alpha > 1$, one can show that the merging happens but the attraction to the origin is only exponential. This case is much easier as the convolution kernel is not singular anymore so, for example, one can take the strain to be exponentially decaying in time.

For $\alpha < 1$, we expect our technique to show that the contours can touch each other in finite time thus proving the outstanding problem of blow up for $\alpha = 1/2$. This, however, will require serious refinement of the method.

There are other stationary singular weak solutions known for 2d Euler dynamics and for other problems in fluid mechanics. It would be interesting to perform analogous stability analysis for each of them with the goal of, e.g., solving the problems of blow up. From that perspective the idea is quite general: find the singular stationary solution which generates the “hyperbolic dynamics” and construct the stable manifold around it which will belong to the functional space of high regularity. For the “true” 2d Euler or SQG one can try some modification of the same “cross configuration”.

The structure of the paper is as follows. In the sections 2, 3, and 4, we prove some auxiliary results. The section 5 contains the construction of $\Omega(t)$, $S(z, t)$ and the proof of the theorem 1.1. In appendix, we find an approximate self-similar solution to the local equation of curve’s evolution.

2. VELOCITY FIELD GENERATED BY THE LIMITING CONFIGURATION

In this section, we consider the velocity field generated by one particular pair of patches which resembles around the origin the limiting ($t = \infty$) configuration of the dynamics described in theorem 1.1. In \mathbb{R}^2 , take $\theta_s(z) = \chi_E(z) + \chi_{E^-}(z)$, where $E = [0, 1]^2$. Recall that in the case of \mathbb{T}^2 analogous configuration (10) is a steady state. Let $x, y > 0$, the other cases can be treated using the symmetry of the problem. We have

$$\pi \nabla \Delta^{-1} \theta_s = \frac{1}{2} \int_{E \cup E^-} \frac{(x - \xi_1, y - \xi_2)}{(x - \xi_1)^2 + (y - \xi_2)^2} d\xi_1 d\xi_2$$

Integrating, we have for the first component

$$\begin{aligned} & \frac{1}{4} \int_0^1 \left(\log(x^2 + (y + \xi_2)^2) - \log(x^2 + (y - \xi_2)^2) \right) d\xi_2 + r_1(x, y) \\ &= \frac{1}{2} \int_0^y \log(x^2 + \xi^2) d\xi + r_2(x, y) \end{aligned}$$

where $r_{1(2)}(x, y)$ are odd and smooth around the origin. Integrating by parts, we get the following expression for the integral

$$y \log(x^2 + y^2) - 2y + 2x \arctan(y/x)$$

By symmetry, for the second component of the gradient we have

$$x \log(x^2 + y^2) - 2x + 2y \arctan(x/y)$$

Thus, the velocity is

$$u_s(z) = \pi \nabla^\perp \Delta^{-1} \theta_s = \frac{1}{2} (-x, y) \log(x^2 + y^2) + r(x, y) \quad (11)$$

around the origin, where $|r(z)| \lesssim |z|$. The correction $r(x, y)$ has a bounded gradient in $0 < x, y < 1$ and so it belongs to the Lipschitz class. This ensures that the first term in (11) is log-Lipschitz.

Consider a positive function $\Phi(\phi)$ on the unit circle $\phi \in \mathbb{T}$, $\pi/2$ -periodic, smooth, $\Phi(\phi) = \Phi(\pi/2 - \phi)$, and such that

$$\Phi(\phi) = \cos \phi, |\phi| < \phi_0 = \arctan 0.5; \quad \Phi(\phi) = \sin \phi, \pi/2 - \phi_0 < \phi < \pi/2 + \phi_0$$

Define the following potential:

$$\Lambda_s(z) = xy \log Q(x, y), \quad Q(x, y) = |z| \Phi(\phi), \quad \text{where } z = |z| e^{i\phi} \quad (12)$$

Clearly, Q is homogeneous of order one and

$$Q(x, y) = x \text{ if } |\phi| < \phi_0; \quad Q(x, y) = y \text{ if } |\phi - \pi/2| < \phi_0 \quad (13)$$

Lemma 2.1. *Around the origin, we have*

$$u_s(z) = \nabla^\perp \Lambda_s(z) + r(z) \quad (14)$$

where $r(z)$ is log-Lipschitz and

$$|r(z)| \lesssim |z| \quad (15)$$

Proof. It is sufficient to handle $x, y > 0$. One can write

$$\nabla^\perp \Lambda_s(z) = \frac{1}{2}(-x, y) \log(x^2 + y^2) + r_3(z)$$

Then, it is a direct calculation to check that $|r_3(x, y)| \lesssim |z|$ and that $\nabla r_3(z)$ is bounded. \square

Remark 1. This particular choice of Λ_s was made only to simplify the calculations below.

We will focus later on the first term in (14) as the second one has a smaller size and will be absorbed into the strain later (see the formulation of the main theorem). The level sets of Λ_s around the origin are hyperbolas asymptotically. Indeed, take $x = \epsilon \widehat{x}, y = \epsilon \widehat{y}$ and consider

$$\Lambda_s(z) = \epsilon^2 \log \epsilon$$

or

$$\widehat{x}\widehat{y} \left(1 + \frac{\log |\widehat{z}|}{\log \epsilon} + \frac{\log \Phi(\phi)}{\log \epsilon} \right) = 1 \quad (16)$$

and so the second and the third terms are small as $\epsilon \rightarrow 0$ provided that $|\widehat{z}| \sim 1$.

For the velocity,

$$\nabla^\perp \Lambda_s(z) = \begin{cases} (-x(\log y + 1), y \log y), & \phi \in (\pi/2 - \phi_0, \pi/2) \\ (-x \log x, y(\log x + 1)), & \phi \in (0, \phi_0) \end{cases} \quad (17)$$

Within the sector $\phi \in (\phi_0, \pi/2 - \phi_0)$, the formula for the gradient is more complicated but we will see later that the part of the patch's boundary that belongs to this sector will not have a significant contribution to the velocity and so very rough estimates will suffice.

These formulas show that the flow generated by Λ_s is hyperbolic around the origin. Moreover, the attraction and repelling is double exponential. For example, the point $(0, y_0)$ will have a trajectory $(0, y_0 e^t)$. In the next section, we will study the Cauchy problem associated to this flow.

Remark 2. Later on we will also need to handle the vector field generated by the "smoothed cross configuration". Suppose we have a small parameter ϵ and

$$E(\epsilon) = E \setminus \{|z| < \epsilon\} \quad (18)$$

Then, for $\nabla^\perp \Delta^{-1} \chi_{E(\epsilon)}$, the direct calculation analogous to the one done in lemma 1.2 yields

$$\pi \nabla^\perp \Delta^{-1} \chi_{E(\epsilon) \cup E^-(\epsilon)} = \nabla^\perp (xy \log \epsilon) + O(|z|), \quad |z| \lesssim \epsilon \quad (19)$$

3. THE CAUCHY PROBLEM FOR THE DYNAMICS GENERATED BY THE LIMITING CONFIGURATION

The construction of the patches and the strain in the main theorem will be based on the calculations done in this section. We consider the following Cauchy problem where t is a parameter:

$$\dot{z}(\tau, t) = \nabla^\perp \Lambda_s(z(\tau, t)), \quad \tau \geq 0$$

the formula for the right hand side is given in (17), and the initial position is

$$z(0, t) = (\epsilon(t), e^{-1}), \quad \epsilon(t) = \exp(-e^{ot}) \quad (20)$$

For now, ϱ is some fixed positive number. So, at each time t , a point (let us say it has an **index** t) with initial position $(\epsilon(t), e^{-1})$ starts moving under the flow so that at time $t + \tau$ (i.e., time τ spent since the start of the motion) we will see the arc built by points that are indexed by $t + \tau_1, \tau_1 \in [0, \tau]$. Let us call this arc $\Gamma_1(t)$ (rotated by $\pi/4$ in the anticlockwise direction, it will be a part of the $\Gamma(t) = \partial\Omega(t)$ from the main theorem).

Our goal in this section is to study the $\Gamma_1(t)$ around the origin when $t \rightarrow +\infty$. This is a straightforward and rather tedious calculation which we have to perform.

For each t , consider the trajectory $z(\tau, t)$ within the fixed neighborhood of the origin: $\{[0, e^{-1}] \times [0, e^{-1}]\}$. This trajectory will be an invariant set for potential Λ_s that corresponds to $\Lambda_s = -e^{-1}\epsilon(t)$. The parameter t , again, can be regarded as the tag of this trajectory. For $t \rightarrow \infty$, these trajectories will look more and more like hyperbolas around zero due to (16).

There will be three important events for each trajectory: the first one is when it crosses the ray $\phi = \pi/2 - \phi_0$ (i.e., $y = 2x$) at time $\tau = T_1(t)$, the second one is when it crosses the ray $\phi = \phi_0$ (i.e., $y = x/2$) at time $T_2(t)$, and the third one is when it crosses the line $x = e^{-1}$ and thus leaves the domain of interest $0 < x < e^{-1}, 0 < y < e^{-1}$. We will denote this time by $T_3(t)$. Since the trajectory is an invariant set for Λ_s ,

$$z(t, T_1(t)) = (\widehat{\epsilon}(t), 2\widehat{\epsilon}(t)) \quad (21)$$

and $\widehat{\epsilon}(t)$ can be found from the equation (see (13))

$$2\widehat{\epsilon}^2(t) \log(2\widehat{\epsilon}(t)) = -e^{-1}\epsilon(t) \quad (22)$$

This gives

$$\widehat{\epsilon}(t) = \sqrt{\frac{\epsilon(t)}{e \log \epsilon^{-1}(t)}} (1 + \bar{o}(1)), \quad t \rightarrow \infty \quad (23)$$

One can write an asymptotical expansion up to any order but we are not going to need it.

Within $\phi > \pi/2 - \phi_0$, we have $\dot{y} = y \log y$, $y(0, t) = e^{-1}$ and so

$$y(\tau, t) = e^{-e^\tau}, \quad \tau < T_1(t)$$

Then, the value of $T_1(t)$ can be found from

$$e^{-e^{T_1(t)}} = 2\widehat{\epsilon}(t)$$

or (due to (22))

$$2e^{T_1} - T_1 = 1 - \log 2 + e^{gt}$$

That gives us an asymptotics

$$T_1(t) = \varrho t - \log 2 + e^{-gt}(1 - 2 \log 2 + \varrho t) + O(t^2)e^{-2gt} \quad (24)$$

Therefore, for the actual time $w_1 = t + T_1(t)$, we have

$$w_1 = t(1 + \varrho) - \log 2 + e^{-gt}(1 - 2 \log 2 + \varrho t) + O(t^2)e^{-2gt}$$

the curve we study will intersect the line $y = 2x$ at the point $(\widehat{\epsilon}(t), 2\widehat{\epsilon}(t))$. Then, assuming that t is a function of w_1 , we have

$$t(w_1) = \frac{w_1}{1 + \varrho} + \frac{\log 2}{1 + \varrho} - \frac{2^{-\delta}(\delta w_1 + 1 - 2 \log 2 + \delta \log 2)}{1 + \varrho} e^{-\delta w_1} + O(e^{-(1+\epsilon_3)\delta w_1})$$

$$\delta = \varrho/(1 + \varrho) < 1, \quad \epsilon_3 > 0 \quad (25)$$

and

$$f_1(w_1) = \widehat{e}(t(w_1)) = F_1(\varrho) \exp \left(-e^{\delta w_1} 2^{-1/(\varrho+1)} + \frac{\delta(\delta-1)}{2} w_1 \right) (1 + \bar{o}(1)) \quad (26)$$

where $F_1(\varrho)$ can be computed explicitly.

3.1. The asymptotical form of the curve within $e^{-1} > y > 2x > 0$. We first compute the scaling limit of the curve in the $\{|z| < (f_1(w_1))^\kappa\}$ neighborhood of zero where κ is any small positive number. Fix the time w_1 and solve the system backward in time using auxiliary functions $\alpha(\tau, w_1), \beta(\tau, w_1)$:

$$\dot{\alpha} = \alpha(\log \beta + 1), \quad \dot{\beta} = -\beta \log \beta \quad (27)$$

and $\alpha(0) = f_1(w_1), \beta(0) = 2f_1(w_1)$. We get then

$$\beta(\tau, w_1) = (2f_1(w_1))^{e^{-\tau}}, \quad \alpha(\tau, t) = e^\tau (f_1(w_1))^{2-e^{-\tau}} \quad (28)$$

Take T , the moment at which the curve is studied, and let

$$T = w_1 - \tau, \quad \alpha = f_1(T)\widehat{\alpha}, \quad \beta = 2f_1(T)\widehat{\beta}$$

Given $\widehat{\alpha}$ and $\widehat{\beta}$, the parameters w_1 and τ are now the functions of $T, \widehat{\alpha}$, and $\widehat{\beta}$. We rescale the variables (x, y) as

$$x = f_1(T)\widehat{x}, \quad y = f_1(T)\widehat{y}$$

The curve we study will go through the point $\widehat{x} = 1, \widehat{y} = 2$ after this rescaling and this is the normalization we need. Will the rescaled curve have any limiting behavior? To answer this question, we consider equations

$$e^\tau (f_1(w_1))^{2-e^{-\tau}} = f_1(w_1 - \tau)\widehat{\alpha}, \quad (2f_1(w_1))^{e^{-\tau}} = 2f_1(w_1 - \tau)\widehat{\beta}$$

Denoting $u = \log \widehat{\alpha}, v = \log \widehat{\beta}$, we get

$$\frac{v}{u} = \frac{(e^{-\tau} - 1) \log 2 + e^{-\tau} \log f_1(w_1) - \log f_1(w_1 - \tau)}{(2 - e^{-\tau}) \log f_1(w_1) + \tau - \log f_1(w_1 - \tau)} \quad (29)$$

Now, let $\tau = e^{-\delta w_1} \widehat{\xi}$ with $\widehat{\xi} = \bar{o}(e^{\delta w_1})$. From (26), we immediately get

$$\frac{v}{u} \rightarrow -\omega, \quad \omega = \frac{1 - \delta}{1 + \delta}, \quad \text{as } T \rightarrow \infty \quad (30)$$

and thus the rescaled curve will converge uniformly to the graph of the function $\widehat{y} = 2\widehat{x}^{-\omega}$ on any interval $\widehat{x} \in [\widehat{a}, 1]$ with fixed \widehat{a} (this is the regime of fixed ξ). If $\widehat{\xi} = \bar{o}(e^{\delta w_1})$, then the corresponding interval is $\widehat{x} \in [f_1^{\varpi(w_1)}(w_1), 1]$ (here $\varpi(w_1)$ is a positive function converging to zero arbitrarily slowly) and on that interval we have (30) uniformly. This implies $\widehat{y} = 2\widehat{x}^{-\omega + \bar{o}(1)}$ on that interval for the rescaled curve we study and the convergence $\bar{o}(1) \rightarrow 0$ is uniform.

If one fixes $\tau > 0$ in (29) instead (the regime of $\widehat{\xi} \sim e^{\delta w_1}$), then

$$\frac{v}{u} \rightarrow \frac{e^{-\tau} - e^{-\delta\tau}}{2 - e^{-\tau} - e^{-\delta\tau}} = -s_-(\tau, \delta), \quad \text{as } T \rightarrow \infty$$

First, notice that

$$s_-(\tau, \delta) > 0$$

and $s_-(\tau, \delta) \rightarrow \omega, \tau \rightarrow 0$. Now, we see that, depending on $\tau \geq 0$, we have different zones for x and different asymptotical regimes

$$\widehat{y} = 2\widehat{x}^{-s_-(\tau, \delta) + \bar{o}(1)}$$

For example, if $\tau \rightarrow 0$, the limiting behavior is again

$$\widehat{y} = 2\widehat{x}^{-\omega + \bar{o}(1)} \quad (31)$$

Notice now that $\delta \rightarrow 0$ as $\varrho \rightarrow 0$ and $\omega \rightarrow 1$ if $\delta \rightarrow 0$ so in the $\varrho \rightarrow 0$ limit the limiting shape for $\tau \ll 1$ is hyperbola, which is approximately the invariant set for Λ_s . This makes the perfect sense as the initial point $(e^{-e^{2t}}, e^{-1})$ approaches the separatrix OY slower and slower when $\varrho \rightarrow 0$. Had it been constant in t , the curve would exactly follow the invariant set for Λ_s .

Remark 1. For the distance to the origin we have

$$\min_{\widehat{x} > 0} (\widehat{x}^2 + 4\widehat{x}^{-2\omega}) = \widehat{x}_m^2 + 4\widehat{x}_m^{-2\omega} < \sqrt{1+4} = \sqrt{5}$$

where

$$x_m = (4\omega)^{1/(2\omega+2)}$$

For any fixed $\tau > 0$, we computed the asymptotical shape of the curve but we will also need the bounds on the curve for $\tau(T) \rightarrow +\infty$. In the next section we will need to control the ratio x/y for the points on the curve. From (28), we get

$$\frac{x}{y} = e^\tau 2^{-e^{-\tau}} (f_1(w_1))^{2(1-e^{-\tau})} < (f_1(T))^{2(1-e^{-\tau_0})} \quad (32)$$

for $\tau > 2\tau_0 > 0$. Indeed, $w_1 = T + \tau$ and so $e^\tau (f_1(w_1))^{\epsilon_3} < e^\tau (f_1(\tau))^{\epsilon_3} \lesssim 1$ for any $\epsilon_3 > 0$.

3.2. The behavior of the curve in the sector $x/2 < y < 2x$. Similarly to (21), we have

$$z(t, T_2(t)) = (2\widehat{\epsilon}(t), \widehat{\epsilon}(t))$$

since the level sets for Λ_s are symmetric with respect to the line $y = x$.

Let us compute the asymptotics of $T_2(t) - T_1(t)$, i.e. the time it takes for the trajectory with index t to pass through the sector. Notice that inside this sector $x \sim y$ and so we have the following system of equations

$$\begin{cases} \dot{\alpha} = -\alpha(\log \alpha + H_1(\alpha, \beta)), & \alpha(0) = \widehat{\epsilon}(t) \\ \dot{\beta} = \beta(\log \beta + H_2(\alpha, \beta)), & \beta(0) = 2\widehat{\epsilon}(t) \end{cases}$$

where $H_{1(2)} \sim 1$. We can find $T_2 - T_1$ then from

$$T_2 - T_1 = \int_0^{T_2 - T_1} \frac{\dot{\alpha} d\tau}{\alpha(\log \alpha^{-1} - H_1(\alpha, \beta))}$$

we have

$$\log \left(\frac{\log \widehat{\epsilon}^{-1}(t) + C}{\log \widehat{\epsilon}^{-1}(t) + C - \log 2} \right) < T_2 - T_1 < \log \left(\frac{\log \widehat{\epsilon}^{-1}(t) - C}{\log \widehat{\epsilon}^{-1}(t) - C - \log 2} \right) \quad (33)$$

and

$$T_2 - T_1 = \frac{\log 2}{\log \widehat{\epsilon}^{-1}(t)} + O(\log^{-2} \widehat{\epsilon}(t)) = (2 \log 2) e^{-\varrho t} + O(t e^{-2\varrho t})$$

So, at the actual time $w_2 = t + T_2(t)$,

$$w_2 = t(1 + \varrho) - \log 2 + e^{-\varrho t}(1 + \varrho t) + O(t^2 e^{-2\varrho t})$$

the trajectory intersects the ray $y = x/2$ at the point $(2\widehat{\epsilon}(t), \widehat{\epsilon}(t))$. If one writes t as a function in w_2 , then

$$(2\widehat{\epsilon}(t), \widehat{\epsilon}(t)) = (2f_2(w_2), f_2(w_2))$$

with

$$f_2(w_2) = F_2(\varrho) \exp\left(-e^{\delta w_2} 2^{-1/(\varrho+1)} + \frac{\delta(\delta-1)}{2} w_2\right) (1 + \bar{o}(1))$$

(compare it to (26)). The simple calculation yields that

$$f_2(T) \sim f_1(T) \quad (34)$$

One can be more precise here: repeating the calculations (28) and doing the estimation similar to (33), one can prove that the curve scaled to go through the point $\widehat{x} = 1, \widehat{y} = 2$ (i.e., we scale by $f_1(T)$ at any time T) converges uniformly to the graph of the function $\widehat{y} = 2\widehat{x}^{-\omega}$. This is achieved again by scaling the local time τ as $\tau = e^{-\delta w_1} \widehat{\xi}$.

3.3. The case $0 < y < x/2 < e^{-1}/2$. This part of the curve is more complicated but the analysis is nearly identical to what we did in the first subsection. Consider equations

$$\dot{\alpha}(\tau) = -\alpha(\tau) \log \alpha(\tau), \quad \dot{\beta}(\tau) = \beta(\tau)(\log \alpha(\tau) + 1)$$

with initial conditions $\alpha(0) = 2f_2(w_2), \beta(0) = f_2(w_2)$. We are interested in the shape of the curve at time $T = w_2(T) + \tau$ where $w_2(T)$ is the actual time in the past when the trajectory intersected the ray $y = x/2$.

$$\alpha(\tau) = (2f_2(w_2))^{e^{-\tau}}, \quad \beta(\tau) = (f_2(w_2))^{2-e^{-\tau}} e^\tau \quad (35)$$

We again scale by $f_2(T)$ as follows

$$\alpha = 2f_2(T)\widehat{\alpha}, \quad \beta = f_2(T)\widehat{\beta}$$

and then rescale the variables as $x = f_2(T)\widehat{x}$ and $y = f_2(T)\widehat{y}$. If $u = \log \widehat{\alpha}, v = \log \widehat{\beta}$ then

$$\frac{v}{u} = \frac{(2 - e^{-\tau}) \log f_2(w_2) + \tau - \log f_2(w_2 + \tau)}{(e^{-\tau} - 1) \log 2 + e^{-\tau} \log f_2(w_2) - \log f_2(w_2 + \tau)}$$

and, again, if $\tau = \widehat{\xi} e^{-\delta w_2}$, then

$$\frac{v}{u} \rightarrow -\omega \quad (36)$$

and thus the curve, rescaled to go through the point $\widehat{x} = 2, \widehat{y} = 1$, will converge to the graph of the function $\widehat{y} = (\widehat{x}/2)^{-\omega}$ uniformly on any interval $[2, \widehat{b}]$ where \widehat{b} is fixed. For the general case of $\widehat{\xi} = \bar{o}(e^{\delta w_2})$ we have $\widehat{y} = (\widehat{x}/2)^{-\omega + \bar{o}(1)}$ for the rescaled curve uniformly on the interval $\widehat{x} \in [2, f_2^{-\varpi(w_2)}(w_2)]$ at time $T = w_2 + \tau$.

For $\tau > 0$ fixed, we have

$$\frac{v}{u} \rightarrow \frac{2 - e^{-\tau} - e^{\delta\tau}}{e^{-\tau} - e^{\delta\tau}} = -s_+(\tau, \delta) \quad (37)$$

and we again have

$$s_+(\tau, \delta) \rightarrow \omega, \quad \text{as } \tau \rightarrow 0$$

For any fixed $\tau \in (0, \tau_{cr})$, we have

$$s_+(\tau, \delta) = -\frac{2 - e^{-\tau} - e^{\delta\tau}}{e^{-\tau} - e^{\delta\tau}} > 0$$

The critical $\tau_{cr}(\delta) = \log \xi_{cr}(\delta)$ where $\xi_{cr}(\delta)$ is the solution of the equation

$$2 - \frac{1}{\xi} - \xi^\delta = 0, \quad \xi > 1$$

Notice that $\xi_{cr}(\delta) \rightarrow 1$ as $\delta \rightarrow 1$ and $\xi_{cr}(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$. Consequently, for the critical value

$$\tau_{cr}(\delta) \rightarrow 0, \quad \delta \rightarrow 1$$

and

$$\tau_{cr}(\delta) \rightarrow +\infty, \quad \delta \rightarrow 0$$

As before, for the subcritical value $\tau < \tau_{cr}$, the curve will have a limiting shape given by

$$\hat{y} = (\hat{x}/2)^{-s_+(\tau, \delta) + \bar{o}(1)}$$

however $s_+(\tau, \delta)$ decays in τ and $s_+(\tau_{cr}, \delta) = 0$.

The (x, y) -coordinates of part of the curve corresponding to each $\tau < \tau_{cr}$ is easy to find. We again have

$$f_2(T) = F_2(\varrho) \exp\left(-e^{\delta T} 2^{-1/(\varrho+1)} + \frac{\delta(\delta-1)}{2} T\right) (1 + \bar{o}(1))$$

and

$$\begin{aligned} x &\sim (f_2(T))^{d_2(\tau) + \bar{o}(1)}, & d_2(\tau) &= e^{-\tau(1+\delta)} < 1 \\ y &\sim (f_2(T))^{l_2(\tau) + \bar{o}(1)}, & l_2(\tau) &= 2e^{-\delta\tau} - e^{-\tau(1+\delta)} \end{aligned} \quad (38)$$

and $l_2(\tau) > 1$ for $\tau < \tau_{cr}$. In the x -coordinate, the domain corresponding to $\tau < \tau_{cr}$ will be characterized by $2f_2(T) < x < f_2^{d_2(\tau_{cr}-\epsilon_4)}(T)$ where $\epsilon_4 < \tau_{cr} - \tau$. Moreover,

$$d_2(\tau_{cr}) \rightarrow 0, \quad \text{as } \delta \rightarrow 0$$

What can be said about the curve for the region with $\tau \geq \tau_{cr} - \epsilon_4$ where ϵ_4 is small? For the analysis that follows in the next section we will only need very rough bounds.

Starting at time w_2 , the point with coordinates $(2f_2(w_2), f_2(w_2))$ will go along the trajectory which will then cross the line $x = e^{-1}$ in time

$$\tau_{full} = \log \log \frac{1}{2f_2(w_2)} \sim \delta w_2 - \frac{\log 2}{1 + \varrho} + \bar{o}(1)$$

and so $T_3(t) = w_2 + \tau_{full} = (1 + \delta)(1 + \rho)t + (1 + \delta - (\rho + 1)^{-1}) \log 2 + \bar{o}(1)$. Therefore, for the part of the curve that corresponds to $\tau > \tau_{cr} - \epsilon_4$ at time T , we have

$$x > (f_2(T))^{d_2(\tau_{cr}-2\epsilon_4)} \quad (39)$$

from (38) and $d_2(\tau_{cr} - 2\epsilon_4) < 1$. In the next section, we will be interested in the ratio y/x for every point on this part of the curve. We use (35) to get

$$\frac{y}{x} = e^\tau 2^{-e^{-\tau}} (f_2(w_2(T)))^{2-2e^{-\tau}} < e^\tau (f_2(w_2(T)))^\zeta, \quad \zeta = 2 - 2e^{-(\tau_{cr}-\epsilon_4)}, \quad \epsilon_4 > 0$$

and

$$\zeta > 0$$

However, the formulas for τ_{full} and T_3 indicate that $T < (1 + \delta)w_2(T)$. Therefore, part of the curve corresponding to $\tau > \tau_{cr} - \epsilon_4$ will have

$$\frac{y}{x} < e^\tau (f_2(T/(1 + \delta)))^\zeta < (f_2(T/(1 + 2\delta)))^\zeta \quad (40)$$

for large T since $f_2(T)$ decays as double exponential.

3.4. Self-similar behavior around the origin. At any given time T , denote the point where the curve intersects the line $y = x$ by $(f(T), f(T))$. The calculations given above show that

$$f(T) \sim f_1(T) \sim f_2(T) \quad (41)$$

We also saw that the curve rescaled by this $f(T)$ (and thus normalized to go through the point $\hat{x} = 1, \hat{y} = 1$) will converge uniformly to the graph of the function $\hat{y} = \hat{x}^{-\omega}$. How large is the domain of convergence? If the curve is calculated at time T and if $\varpi(T)$ is arbitrary positive function such that $\varpi(T) \rightarrow 0$ as $T \rightarrow \infty$, we have

$$\hat{y} = \hat{x}^{-\omega + \bar{o}(1)} \quad (42)$$

for the curve at time T uniformly on

$$\left(f(T)\right)^{\varpi(T)} < \hat{x} < \left(f(T)\right)^{-\varpi(T)}$$

where $f(T) \sim f_1(T) \sim f_2(T)$ is the rescaling parameter. Outside this window, we have different scaling limits for different values of the parameter $\tau > 0$.

Remark 2. What is the nature of the scaling law for $\tau \sim 0$ that we have got? The Euler dynamics is defined by the convolution with the kernel

$$K(z) \sim \log |z|$$

and $|\log(s\epsilon)| = |\log \epsilon| + O(1)$ for $s \in [1, M]$ with any fixed M . Thus, the strength of the created hyperbolic flow is more or less the same within any annulus $\epsilon < |z| < M\epsilon$. Outside this annulus, say, for $|z| = \sqrt{\epsilon}$, the size of the kernel is quite different.

Now, assume that we are given the hyperbolic flow in the whole plane defined by the following equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

One obtains these equations after rotating the phase space in, e.g., (27) by $\pi/4$ degrees.

Let us find the evolution of the contour $\mathcal{C}(t)$ under this flow which would be self-similar in the sense that $\mathcal{C}(t) = \mathcal{C}(0)e^{-\delta t}$. Assume that $\mathcal{C}(t)$ is given by the graph of the function $y(x, t)$ and then one gets nonhomogeneous Burgers equation for $y(x, t)$

$$y_t(x, t) = -y_x(x, t)y(x, t) + x$$

By our assumptions,

$$y(x, t) = e^{-\delta t} H(e^{\delta t} x)$$

so we have

$$\delta(\xi H'(\xi) - H(\xi)) = \xi - H'(\xi)H(\xi) \quad (43)$$

$$H' = \frac{\delta H + \xi}{\delta \xi + H}$$

If $H = \xi z(\xi)$, then

$$|z - 1|^A |z + 1|^B = C |\xi|^{-1}, A = \frac{1 + \delta}{2}, B = \frac{1 - \delta}{2}$$

and so

$$|H - \xi|^A |H + \xi|^B = C$$

Due to scaling, we can assume $C = 1$. In the coordinates $H - \xi = \beta, H + \xi = \alpha$, we have

$$\beta = \alpha^{-(1-\delta)/(1+\delta)}$$

(compare with (42)).

Is it possible to find the initial data such that the evolution of the curve is self-similar on the larger interval? The answer to this question is yes. For example, in (20), one can take

$$\epsilon(t) = \exp\left(-e^{t^{\tilde{\gamma}}}\right)$$

with $\tilde{\gamma} \in (0, 1)$. Then, the distance from the curve to the origin will be $\lesssim \exp\left(-e^{t^{\tilde{\gamma}_1}}\right)$, $\tilde{\gamma}_1 \in (0, 1)$ but the self-similar behavior will take place in a wider relative range (but yet not on the whole ball $B_{e^{-1}}(0)$!). This is a general rule: the slower the Cauchy data jumps from one invariant set of Λ_s to another, the more regular the curve is around the origin.

4. COMPARISON OF THE VELOCITY FIELDS GENERATED BY $\Gamma_1(t)$ AND BY THE LIMITING CONFIGURATION.

In this section, let the symbol t denote the actual time at which the curve is considered and let $\epsilon = f(t)$ where $f(t)$ was introduced earlier. Recall that we denoted by $\Gamma_1(t)$ the arc constructed in the previous section. Let us take $\Gamma_1(t)$ and close it in the smooth and arbitrary way to produce the simply-connected domain $\Omega_1(t)$ (see Figure 1). Then, $\text{dist}(\Omega_1(t), 0) \sim f(t)$. We will compare now the velocity fields generated by two patches $E(0.1\epsilon) \cup E^-(0.1\epsilon)$ and $\Omega_1(t) \cup \Omega_1^-(t)$. Recall that $E(\epsilon)$ was introduced in (18).

We will need to use the following lemma in which the picture above will be rotated by $\pi/4$ degrees in the anticlockwise direction. Let us denote by $\Gamma_2(t), \Omega_2(t)$, and $E_2(0.1\epsilon)$ the resulting sets. Clearly, the calculations done above indicate that the lower part of $\Gamma_2(t)$ converges uniformly to the graph of the function $|x|$ however it is the precise form of this convergence that will play the crucial role in comparing the velocity fields.

Fix t and assume that some arc (let us call it Γ_2 as we will later apply this lemma to the part of $\Gamma_2(t)$) lies above the graph of $|x|$ and below the graph of a certain function $g(x)$ defined on $[-a, a]$ where, e.g., $a = 1/(e\sqrt{2})$.

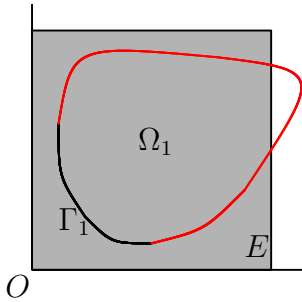


Figure 1

Lemma 4.1. *Rescale $g(x)$ as follows*

$$\widehat{g}(\widehat{x}) = \epsilon^{-1}g(\epsilon\widehat{x}), \quad |\widehat{x}| < a/\epsilon$$

Assume that \widehat{g} satisfies the following properties:

- (a) $0 < C_1 < \widehat{g}(\widehat{x}) < C_2$, $\widehat{x} \in [-1, 1]$ with some absolute constants $C_{1(2)}$.
- (b) For $|\widehat{x}| > 1$,

$$|\widehat{x}| \leq \widehat{g} \leq |\widehat{x}| + h(\widehat{x}), \quad h(\widehat{x}) = \bar{o}(|\widehat{x}|), \quad \widehat{x} \rightarrow \infty \quad (44)$$

Consider

$$D(z, t) = \nabla^\perp \Delta^{-1}(\chi_{\Omega_2(t) \cup \Omega_2^-(t)} - \chi_{E_2(0.1\epsilon) \cup E_2^-(0.1\epsilon)})$$

Then, we have the following estimate

$$|D(z, t)| \lesssim |z| \left(1 + \int_{1 < |\widehat{x}| < a/\epsilon} \frac{h(\widehat{x}) + h(-\widehat{x})}{|\widehat{x}| \cdot (|\widehat{x} - \widehat{z}_1| + 1)} d\widehat{x} \right), \quad z = \epsilon\widehat{z}, \widehat{z} = (\widehat{z}_1, \widehat{z}_2)$$

Proof. Take $z \in \mathbb{C}^+$. From the central symmetry, we have

$$\nabla^\perp \Delta^{-1}(\chi_{\Omega_2 \cup \Omega_2^-} - \chi_{E_2(0.1\epsilon) \cup E_2^-(0.1\epsilon)})(0) = 0$$

Now, let us use the following formula

$$\left| \frac{\xi - z}{|\xi - z|^2} - \frac{\xi}{|\xi|^2} \right| = \frac{|z|}{|\xi - z| \cdot |\xi|}$$

to get

$$|D(z, t)| \lesssim |z| \left(1 + \int_{(E_2(0.1\epsilon) \setminus \Omega_2) \cap \{|\xi| < a/10\}} \frac{d\xi}{|z - \xi| \cdot |\xi|} + \int_{(E_2(0.1\epsilon) \setminus \Omega_2)^- \cap \{|\xi| < a/10\}} \frac{d\xi}{|z - \xi| \cdot |\xi|} \right) \quad (45)$$

for any $|z| < a/10$. Consider the first integral in (45). We split it into two as follows

$$\int_{(E_2(0.1\epsilon) \setminus \Omega_2) \cap \{|\xi_1| < 2\epsilon\}} \frac{d\xi}{|z - \xi| \cdot |\xi|} + \int_{(E_2(0.1\epsilon) \setminus \Omega_2) \cap \{2\epsilon < |\xi_1|, |\xi| < a/10\}} \frac{d\xi}{|z - \xi| \cdot |\xi|} \quad (46)$$

where $\xi = (\xi_1, \xi_2)$. Scale by ϵ : $z = \epsilon\widehat{z}$, $\xi = \epsilon\widehat{\xi}$. Then, for the first term we have

$$\int_{(E_2(0.1\epsilon) \setminus \Omega_2) \cap \{|\xi_1| < 2\epsilon\}} \frac{d\xi}{|z - \xi| \cdot |\xi|} \leq \int_{\{|\widehat{\xi}_1| < 2, 0.01 < \widehat{\xi}_2 < C\}} \frac{d\widehat{\xi}}{|\widehat{z} - \widehat{\xi}| \cdot |\widehat{\xi}|}$$

as the set $E_2(0.1\epsilon)$ is at the distance 0.1ϵ from the origin by (18).

For the last expression we readily have

$$\sup_{\widehat{z} \in \mathbb{C}} \int_{\{|\widehat{\xi}_1| < 2, 0.01 < \widehat{\xi}_2 < C\}} \frac{d\widehat{\xi}}{|\widehat{z} - \widehat{\xi}| \cdot |\widehat{\xi}|} < C$$

Then, consider the second term in (46), do the scaling again, and use a trivial bound $|\widehat{z} - \widehat{\xi}| \geq |\widehat{z}_1 - \widehat{\xi}_1|$ (here $\widehat{z} = (\widehat{z}_1, \widehat{z}_2)$). If $|\widehat{z}_1| < 1$, this term can be bounded by

$$C \int_{1 < |\widehat{x}| < a/\epsilon} \frac{h(\widehat{x})}{|\widehat{x}| \cdot (|\widehat{x} - \widehat{z}_1| + 1)} d\widehat{x}$$

due to (44) (here we used \widehat{x} instead of $\widehat{\xi}_1$). Let $|\widehat{z}_1| > 1$. Then, the integral over the subset with $|\widehat{\xi}_1 - \widehat{z}_1| > 0.5$ is again at most

$$C \int_{1 < |\widehat{x}| < a/\epsilon} \frac{h(\widehat{x})}{|\widehat{x}| \cdot (|\widehat{x} - \widehat{z}_1| + 1)} d\widehat{x}$$

The integral over the subset $|\widehat{\xi}_1 - \widehat{z}_1| < 0.5$ is estimated by

$$C \int_{2 < |\widehat{\xi}_1| < a/\epsilon, |\widehat{\xi}_1 - \widehat{z}_1| < 0.5} d\widehat{\xi}_1 \int_{|\widehat{\xi}_1|}^{|\widehat{\xi}_1| + h(\widehat{\xi}_1)} \frac{d\widehat{\xi}_2}{|\widehat{\xi}| \cdot |\widehat{\xi} - \widehat{z}|}$$

by (44). Then, we have a bound

$$\int_{2 < |\widehat{\xi}_1| < a/\epsilon, |\widehat{\xi}_1 - \widehat{z}_1| < 0.5} d\widehat{\xi}_1 \int_{|\widehat{\xi}_1|}^{|\widehat{\xi}_1| + h(\widehat{\xi}_1)} \frac{d\widehat{\xi}_2}{|\widehat{\xi}| \cdot |\widehat{\xi} - \widehat{z}|} < C \frac{1}{|\widehat{z}_1|} (1 + \log |\widehat{z}_1|) < C$$

since $|\widehat{z}_1| > 1$.

For the second integral in (45), we have a similar bound but with $h(-\widehat{x})$ instead of $h(\widehat{x})$. \square

We now immediately get

Corollary 4.1. *If the function h in (44) satisfies*

$$|h(\widehat{x})| \lesssim |\widehat{x}|^{-\gamma}, \quad \gamma > 0 \tag{47}$$

for $|\widehat{x}| < \epsilon^{-\alpha}$, $\alpha \in (0, 1)$ and

$$|\ln \epsilon| \max_{\epsilon^{-\alpha} < |\widehat{x}| < a/\epsilon} \left| \frac{h(\widehat{x})}{\widehat{x}} \right| < C \tag{48}$$

then

$$|D(z, t)| \lesssim |z|, \quad |z| \gtrsim \epsilon$$

Proof. From (47)

$$\int_{1 < |\widehat{x}| < \epsilon^{-\alpha}} \frac{h(\widehat{x})}{|\widehat{x}| \cdot (|\widehat{x} - \widehat{z}_1| + 1)} d\widehat{x} \lesssim (|\widehat{z}_1| + 1)^{-1}$$

and (48) yields

$$\int_{\epsilon^{-\alpha} < |\hat{x}| < a/\epsilon} \frac{h(\hat{x})}{|\hat{x}| \cdot (|\hat{x} - \hat{z}_1| + 1)} d\hat{x} \lesssim |\ln \epsilon| \max_{\epsilon^{-\alpha} < |\hat{x}| < a/\epsilon} \left| \frac{h(\hat{x})}{\hat{x}} \right| < C$$

□

Now, let us apply this corollary to our situation. For this, we will use the results from the previous section.

Remark 1. Notice that for each ϱ , the arc $\Gamma_1(t)$ satisfies the conditions of the corollary with some $\alpha(\varrho)$. For $|\hat{x}| > \epsilon^{-\alpha}$, we can use (32) and (40) to get

$$\left| \frac{h(\hat{x})}{\hat{x}} \right| \lesssim \left(f(t/(1+2\delta)) \right)^\zeta, \quad \zeta > 0$$

and

$$\left| \left(\log f(t) \right) f(t/(1+2\delta))^\zeta \right| \rightarrow 0$$

Thus, we have

$$\sup_z \left| \frac{D(z, t)}{|z|} \right| < C \quad (49)$$

Remark 2. If one repeats the estimates in lemma above, we have

$$|\nabla^\perp \Delta^{-1} (\chi_{E_2 \cup E_2^-} - \chi_{E_2(0.1\epsilon) \cup E_2^-(0.1\epsilon)})| \lesssim |z|, \quad \text{if } |z| \gtrsim \epsilon \quad (50)$$

5. CONSTRUCTION OF THE VORTEX PATCH DYNAMICS AND PROOF OF THE MAIN THEOREM.

We first construct an incompressible strain Ψ which satisfies the following properties (see Figure 2 for the upper part of the actual picture):

1. Ψ is odd and is compactly supported.
2. Around the points $(4, 4)$ and $(-2, 2)$ it is the standard hyperbolic time-independent flow (these are the domain D_1 and D_6). At $(-2, 2)$, we choose the separatrices to be $y_1 = -x$ and $y_2 = 2 + (x + 2)$. The flow is attracting along y_2 and is repelling along y_1 . To define the curve $\Gamma(t)$ in D_1 we need the following result which will later guarantee necessary initial conditions for the dynamics around $(0, 0)$. Let the local coordinates near $(-2, 2)$ be denoted by (ξ, η) .

Lemma 5.1. *Fix any $\varrho > 0$ and consider the standard hyperbolic dynamics around the origin*

$$\begin{cases} \dot{\xi} = \xi, & \xi(0) = \xi_0 \\ \dot{\eta} = -\eta, & \eta(0) = \eta_0 \end{cases}$$

Let $G(\xi) = 0$ for $\xi \leq 0$ and $G(\xi) = \xi^{-1} \exp(-\xi^{-\varrho})$ for $\xi > 0$. Consider the evolution of the smooth curve $\Gamma(0) = \{(\xi, G(\xi)), |\xi| < 1\}$ under this flow. Call it $\Gamma(t) = \{(\xi, G(\xi, t)), |\xi| < 1\}$. Then,

$$G(1, t) = e^{-e^{\varrho t}} \quad (51)$$

Proof. This is a straightforward calculation. The point (ξ_0, η_0) moves to $(\xi_0 e^t, \eta_0 e^{-t})$ in time t . Thus, the point $(e^{-t}, G(e^{-t})) \in \Gamma(0)$ will move to $(1, e^{-e^{\varrho t}})$ in time t . □

Remark 1. Notice that the part of $\Gamma(t)$ that belongs to the left half-plane does not change in time. Within the window $|\xi| < 1$, the curve $\Gamma(t)$ is always smooth and converges to the coordinate axis.

3. Around the origin (domain D_2), we choose

$$\Psi(z, t) = \nabla^\perp \Lambda_s^{(t)} \left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}, t \right) \quad (52)$$

where $\Lambda_s^{(t)}$ is Λ_s modified in the $0.1f(t)$ -neighborhood of zero:

$$\Lambda_s^{(t)}(z) = xy \log \left(0.1f(t) Q_1 \left(\frac{z}{0.1f(t)} \right) \right)$$

where $Q_1(z)$ is smooth, positive, and $Q_1(z) = |z|\Phi(\phi)$ for $|z| > 1$. Clearly, $\Lambda_s^{(t)} = \Lambda_s$ for $|z| > 0.1f(t)$ so the dynamics of the curve considered in the section 3 would be the same had we studied the flow generated by the potential $\Lambda_s^{(t)}$ instead. One can also easily check that

$$\nabla^\perp \Lambda_s^{(t)}(z) = (-x, y) \log f(t) + O(|z|) \quad (53)$$

for $|z| \lesssim f(t)$. We changed the coordinates in (52) as we want to rotate the picture described in section 3 by $\pi/4$ in the positive direction. We also modified the value of the potential in the $0.1f(t)$ neighborhood of the origin to get rid of the artificial singularity generated by the sharp corner in the limiting configuration.

4. Between D_1 and D_2 the potential can be smoothly interpolated.

5. In $D_{3(5)}$, the flow is laminar with direction perpendicular to the black segments and in the north-eastern direction.

6. The potential between zones D_2 and D_3 can be smoothly interpolated, as well as the potential between D_5 and D_6 . In the zone D_7 , the potential is zero so the curve is frozen. This zone again is interpolated smoothly between D_1 and D_6 .

7. In the zone D_4 , we construct non-stationary potential in the following way (only in this zone the flow is essentially time-dependent!). We need an argument that allows an interpolation between two laminar flows and guarantees the prescribed evolution of the curve $\Gamma(t)$ in these laminar zones. What we want is to define dynamics in the regions D_3, D_4, D_5 right after the points on the curve leave D_2 . We need to define this dynamics in such a way that the motion of $\Gamma(t)$ is localized to these regions and, moreover, that it does not move in D_5 . Once again, in D_3 and D_5 we postulate the flow to be laminar and then we want to define it in D_4 . We will do that in the local coordinates.

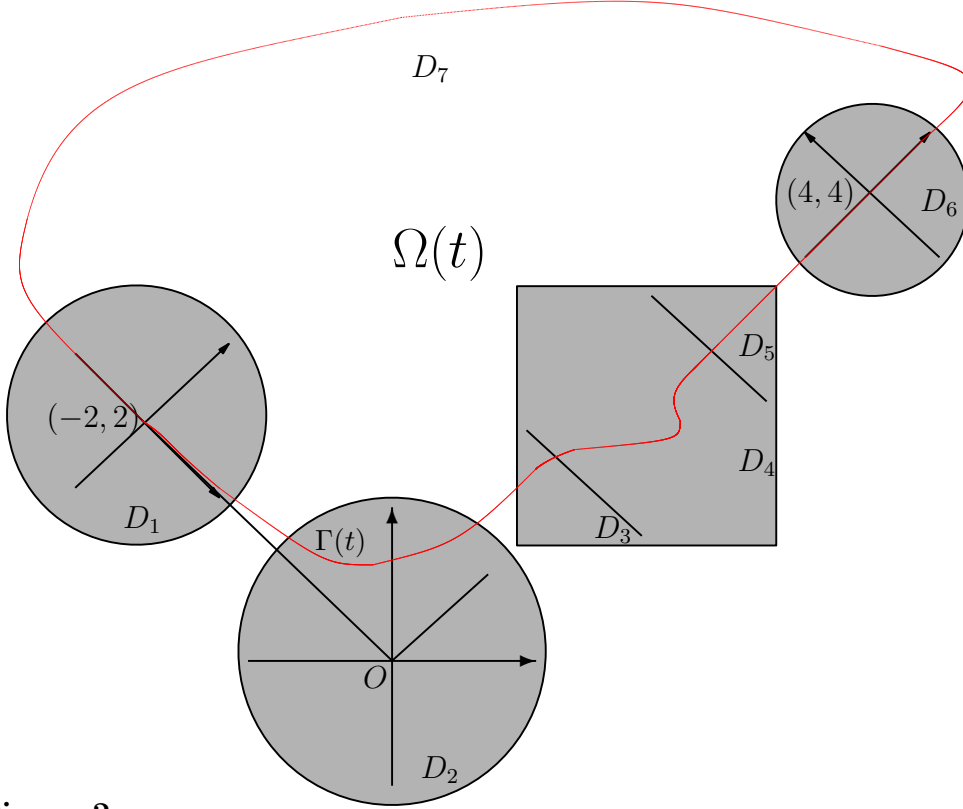


Figure 2

Assume that potential $\Lambda(z) = -y$ in $B = \{z : -1 < x < 0\} \cup \{z : 1 < x < 2\} \sim D_3 \cup D_5$. This potential generates the laminar flow

$$\dot{\theta} = \nabla \theta \cdot \nabla^\perp \Lambda$$

where $\nabla^\perp \Lambda(z) = (-1, 0)$. We want to define smooth $\Lambda(z, t)$ in $D_4 = \{z : 0 < x < 1\}$ such that the resulting $\Lambda(z, t)$ is smooth globally on $D_3 \cup D_4 \cup D_5$. Moreover, given smooth decaying $\nu(t)$ (e.g., $\nu \in L^1(\mathbb{R}^+)$) is enough for decay condition), we need to define a curve $\Gamma(0) = \{(x, \gamma(x, 0))\}$ that evolves under this flow $\Gamma(t) = \{(x, \gamma(x, t))\}$ such that $\gamma(0, t) = \nu(t)$ and $\gamma(1, t) = 0$. This function $\nu(t)$ is determined by $\Gamma(t)$ in the zone D_2 where it approaches the separatrix in the double exponential rate. To be more precise, ν is proportional to the distance from $\Gamma(t)$ to this separatrix in the area where D_2 and D_3 meet.

We will look for

$$\Lambda(z, t) = -y - g_1(x)g_2(x - t)$$

where $g_{1(2)}$ are smooth. Then, to guarantee the global smoothness, we need $g_1(x) = 0$ around $x = 0$ and $x = 1$. Now, take a point $(0, \nu(T))$ and trace its trajectory for $t > T$. We have

$$x(t, T) = t - T, t \in [T, T + 1]$$

and

$$y(t, T) = \nu(T) - \int_T^t \left(g_1'(\tau - T)g_2(\tau - T - \tau) + g_1(\tau - T)g_2'(\tau - T - \tau) \right) d\tau, \\ t \in [T, T + 1]$$

Since we want $y(T+1, T) = 0$ and g_1 to vanish on the boundary,

$$\nu(T) = g_2'(-T) \int_T^{T+1} g_1(\tau - T) d\tau = g_2'(-T) \int_0^1 g_1(x) dx$$

and this identity should hold for all $T > 0$. Take any g_1 with mean one, this defines g_2 on the negative half-line as long as we set $g_2(-\infty) = 0$. We can continue it now to the whole line in a smooth fashion to have g_2 globally defined. How do we define the initial curve at $t = 0$? We extend smooth $\nu(t)$ to $t \in [-1, 0]$ arbitrarily and apply the procedure explained above to $t \in [-1, \infty)$. The curve that we see at $t = 0$ will be the needed initial value for the dynamics that starts at $t = 0$. It is only left to mention that to localize the picture in the vertical direction we can multiply $\Lambda(z, t)$ by a suitable cut-off in the y direction.

The part of the curve that is in D_5, D_6, D_7 , and the north-western part of D_1 is stationary, it does not move at all (this is easy to ensure by making this part of the curve the level set of the stationary potential $\Lambda(z), \Psi = \nabla^\perp \Lambda$). For the rest of the curve, it does change in time and the flow is directed along it in the anti-clockwise direction.

Now that the explicit $\Psi(z, t)$ and the curve $\Gamma(t) = \partial\Omega(t)$ evolving under this flow are defined, we are ready to prove theorem 1.1.

Proof. (Theorem 1.1). We have by construction

$$\dot{\theta} = \nabla\theta \cdot \Psi(z, t)$$

and $\theta(z, t) = \chi_{\Omega(t)}(z) + \chi_{\Omega^-(t)}(z)$. Let us define $S(z, t)$ by

$$S(z, t) = \Psi(z, t) - \pi\nabla^\perp\Delta^{-1}\theta$$

To show that this difference satisfies (3) we only need to consider the behavior of θ around $z = 0$ since the contribution from $(\Omega(t) \cup \Omega^-(t)) \setminus B_{0.1a}(0)$ to $\nabla^\perp\Delta^{-1}\theta$ is of order $O(|z|)$ as immediately follows from the central symmetry. We write

$$\Psi(z, t) - \pi\nabla^\perp\Delta^{-1}\theta = \Psi(z, t) - I + I - \pi\nabla^\perp\Delta^{-1}\theta$$

where I is obtained by replacing $\Omega(t)$ around the origin by $E_2(0.1f(t))$ configuration. Then, $|I - \pi\nabla^\perp\Delta^{-1}\theta| \lesssim |z|$ follows from (49). In the $f(t)$ -neighborhood of the origin, $\Psi(z, t) - I$ is at most $C|z|$ due to (19) and (53). For $|z| \gtrsim f(t)$, we can use (15) and (50) to show that $|\Psi(z, t) - I| \lesssim |z|$. Thus, we have (3). The uniform log-Lipschitz condition (4) immediately follows as well since the velocity generated by any patch does satisfy it. The error $S(z, t)$ will in fact be much smaller than $\Psi(z, t)$ around the origin and so can be considered as a small error or correction. It is odd as θ is even and is also divergence free as the difference of two divergence free vector fields. We get

$$\dot{\theta} = \nabla\theta \cdot \left(\pi\nabla^\perp\Delta^{-1}\theta + S(z, t) \right) \tag{54}$$

and the theorem is proved as the dynamics of $\Omega(t)$ satisfies

$$\text{dist}(\Omega(t), \Omega^-(t)) = 2\text{dist}(\Omega(t), 0) \sim f(t)$$

For $f(t)$ we have a double exponential decay due to (26) and (41). □

Remark 2. Strictly speaking, the Cauchy data for the evolution in the domain D_2 will not be given by (51) as the flow will distort it when moving between zones D_1 and D_2 . However, this leads only to a minor change (a fixed time increment, in fact), so the same argument goes through.

Remark 3. The parameter δ in the formulation of the theorem 1.1 corresponds to δ in (25) and can be chosen arbitrarily from the interval $(0, 1)$. The size of this interval is determined by the parameters of the problem: the value of vorticity and initial size of the patches.

6. APPENDIX: APPROXIMATE SELF-SIMILAR SOLUTION TO THE CONTOUR DYNAMICS

In this section, we address the following question: is it possible to construct an approximate solution to the Euler dynamics of patches such that the self-similarity will persist on the larger set? We will do that in a rather artificial way as we already can make the right guess about what the solution should be. The construction presented here gives an independent (and even shorter) proof of the double-exponential merging but it does not explain the mechanism of the singularity formation and is less illuminating in our opinion.

Assume that the boundary of the simply-connected patch is parameterized by $\gamma(s, t)$. Then, the velocity at every point of the contour can be computed by ([6], formula (1))

$$u(\gamma(\xi, t), t) = C \int_0^{2\pi} \log |\gamma(\xi, t) - \gamma(s, t)| \gamma'_s(s, t) ds$$

This is a simple corollary of the Gauss integration formula.

If we have a centrally symmetric pair of vortices interacting with each other and, like before, the part of $\Gamma(t)$ close to the origin can be parameterized by the function $y(x, t)$, then the equation for evolution reads

$$\dot{y}(x, t) = C \int_{-0.5}^{0.5} (y'(x, t) - y'(\xi, t)) \log \left(\frac{(x - \xi)^2 + (y(x, t) - y(\xi, t))^2}{(x + \xi)^2 + (y(x, t) + y(\xi, t))^2} \right) d\xi + r(x, y, t) \quad (55)$$

where $r(x, y, t)$ is a contribution from those parts of $\Gamma(t)$ and $\Gamma^-(t)$ that are away from the origin. This $r(x, y, t)$ is therefore smooth and $r(0, 0, t) = 0$ by symmetry. Let us drop $r(x, y, t)$ and try to find an approximate self-similar solution? In other words, we want $y(x, t) = \epsilon(t)\phi(x/\epsilon(t))$ to satisfy (55) up to some smaller order correction. Substitution into (55) gives

$$\frac{\dot{\epsilon}}{\epsilon} (\phi(\hat{x}) - \phi'(\hat{x})\hat{x}) = C \int_{-0.5\epsilon^{-1}}^{0.5\epsilon^{-1}} (\phi'(\hat{x}) - \phi'(\hat{\xi})) \log \left(\frac{(\hat{x} - \hat{\xi})^2 + (\phi(\hat{x}) - \phi(\hat{\xi}))^2}{(\hat{x} + \hat{\xi})^2 + (\phi(\hat{x}) + \phi(\hat{\xi}))^2} \right) d\hat{\xi} + E(\hat{x}, t) \quad (56)$$

where E is an error we will control later on. Let us rewrite the integral as follows (up to a constant multiple)

$$\int_{-0.5\epsilon^{-1}}^{0.5\epsilon^{-1}} (\phi'(\hat{x}) - \phi'(\hat{\xi})) (\hat{x}\hat{\xi} + \phi(\hat{x})\phi(\hat{\xi})) K(\hat{x}, \hat{\xi}) d\hat{\xi}$$

with

$$K(\widehat{x}, \widehat{\xi}) = \frac{1}{b-a} \int_a^b \frac{d\eta}{\eta},$$

$$a = (\widehat{x} - \widehat{\xi})^2 + (\phi(\widehat{x}) - \phi(\widehat{\xi}))^2, \quad b = (\widehat{x} + \widehat{\xi})^2 + (\phi(\widehat{x}) + \phi(\widehat{\xi}))^2$$

and so we get

$$k_1(\widehat{x}, t)\phi'(\widehat{x})\widehat{x} + k_2(\widehat{x}, t)\phi'(\widehat{x})\phi(\widehat{x}) + k_3(\widehat{x}, t)\widehat{x} + k_4(\widehat{x}, t)\phi(\widehat{x}) \quad (57)$$

where the coefficients k_j are defined correspondingly. Assume now that ϕ satisfies the following assumptions:

- (a) $\phi(\widehat{x})$ is smooth
- (b) $0 < C_1 < \phi(\widehat{x}) < C_2$ for $\widehat{x} \in [-1, 1]$
- (c) $\phi(\widehat{x}) = |\widehat{x}| + \rho(\widehat{x})$ where $|\widehat{x}| > 1$ with

$$|\rho(\widehat{x})| < |\widehat{x}|^{-\gamma}, \quad |\rho'(\widehat{x})| < |\widehat{x}|^{-\gamma}, \quad \gamma > 0$$

Let us estimate the coefficients k_j now. We will handle k_3 , the analysis for k_2 is similar.

$$k_3(\widehat{x}) = -\frac{1}{4} \int_{-0.5\epsilon^{-1}}^{0.5\epsilon^{-1}} \frac{\xi\phi'(\xi)}{\xi\widehat{x} + \phi(\xi)\phi(\widehat{x})} \left(\int_a^b \frac{d\eta}{\eta} \right) d\xi \quad (58)$$

Consider $\widehat{x} \in [1, 0.5\epsilon^{-1}]$, the other values can be treated similarly. For the integral over the positive ξ we have (after the change of variables $\xi = \widehat{x}\xi_1$, recall that $x = \epsilon\widehat{x}$)

$$I_1 = - \int_0^{0.5x^{-1}} \frac{\xi_1(1 + \rho'(\xi_1\widehat{x}))}{\xi_1 + (1 + \rho(\widehat{x})/\widehat{x})(\xi_1 + \rho(\widehat{x}\xi_1)/\widehat{x})} \cdot A d\xi_1$$

$$A = \frac{1}{4} \int_{(\xi_1-1)^2 + (1-\xi_1 + \rho(\widehat{x})/\widehat{x} - \rho(\widehat{x}\xi_1)/\widehat{x})^2}^{(1+\xi_1)^2 + (1+\xi_1 + \rho(\widehat{x})/\widehat{x} + \rho(\widehat{x}\xi_1)/\widehat{x})^2} \frac{d\eta}{\eta}$$

For A , we have a representation

$$A = \frac{1}{\xi_1} + \frac{\rho(\widehat{x})}{2\widehat{x}\xi_1} + O(\xi_1^{-2}), \quad \xi_1 > 1$$

so

$$I_1 = -\frac{1}{2} \int_1^{0.5x^{-1}} \frac{d\xi_1}{\xi_1} \left(1 + O\left(\frac{\rho(\widehat{x}\xi_1)}{\widehat{x}\xi_1}\right) \right) (1 + \rho'(\xi_1\widehat{x})) + \dots$$

$$= 0.5 \log x + O(1)$$

For the other integral, changing the sign in integration

$$I_2 = - \int_0^{0.5\epsilon^{-1}} \xi(1 + \rho'(-\xi)) \frac{1}{b-a} \int_a^b \frac{d\eta}{\eta} d\xi$$

Here, we have

$$b = (\widehat{x} - \xi)^2 + (\widehat{x} + \xi + \rho(\widehat{x}) + \rho(-\xi))^2, \quad a = (\widehat{x} + \xi)^2 + (\widehat{x} - \xi + \rho(\widehat{x}) - \rho(-\xi))^2$$

As both \widehat{x} and ξ are large in the interesting regime, we are in the situation when

$$a, b > (\widehat{x}^2 + \xi^2)/2$$

so we can use the mean-value formula

$$\frac{1}{b-a} \int_a^b \frac{d\eta}{\eta} = \frac{1}{b} + \frac{b-a}{2\eta_1^2}, \quad \eta_1 \in (a, b)$$

Substituting, we have two terms: $I_2 = -(T_1 + T_2)$.

$$T_1 = \int_0^{0.5x^{-1}} \xi_1(1 + \rho'(-\widehat{x}\xi_1))B^{-1}d\xi_1$$

where

$$B = 2(1 + \xi_1^2) + 2(1 + \xi_1)(\rho(\widehat{x})/\widehat{x} + \rho(-\widehat{x}\xi_1)/\widehat{x}) + (\rho(\widehat{x})/\widehat{x} + \rho(-\widehat{x}\xi_1)/\widehat{x})^2$$

Thus,

$$T_1 = \int_0^{0.5x^{-1}} \frac{\xi_1(1 + \rho'(-\widehat{x}\xi_1))}{2(1 + \xi_1^2)} d\xi_1 + O(1) = -0.5 \log x + O(1)$$

For the other term, we have

$$\begin{aligned} |T_2| &\lesssim \int_0^{0.5\epsilon^{-1}} \xi \frac{\widehat{x}|\rho(-\xi)| + \xi|\rho(\widehat{x})| + |\rho(\widehat{x})\rho(-\xi)|}{(\widehat{x}^2 + \xi^2)^2} d\xi \\ &\lesssim \int_1^{0.5\epsilon^{-1}} \frac{\xi d\xi}{(\widehat{x}^2 + \xi^2)^2} \left(\frac{\widehat{x}}{\xi^\gamma} + \frac{\xi}{\widehat{x}^\gamma} + 1 \right) d\xi < C \end{aligned}$$

Combining all terms, we have

$$k_3 = \log x + O(1), \quad x > \epsilon$$

For $x \sim 0$, we get $I_{1(2)} = 0.5 \log \epsilon + O(1)$. These calculations show that

$$k_3 = \begin{cases} \log |x| + O(1), & |x| > \epsilon \\ \log \epsilon + O(1), & |x| < \epsilon \end{cases} \quad (59)$$

Analogous estimates can be obtained for k_2 . They yield

$$k_2 = - \begin{cases} \log |x| + O(1), & |x| > \epsilon \\ \log \epsilon + O(1), & |x| < \epsilon \end{cases} \quad (60)$$

The estimates for other terms are

$$|k_{1(4)}| = O(1)$$

Indeed,

$$k_1 = \int_{-0.5\epsilon^{-1}}^{0.5\epsilon^{-1}} \xi K(\widehat{x}, \xi) d\xi, \quad k_4 = - \int_{-0.5\epsilon^{-1}}^{0.5\epsilon^{-1}} \phi'(\xi)\phi(\xi)K(\widehat{x}, \xi) d\xi$$

and if one does the same analysis as we did for k_3 in (58), we will get the sum of two integrals: one over positive ξ and the other one over negative ξ . Each will have the same large logarithmic leading term but they will come with different signs now and so will cancel each other in the sum leaving us with the uniformly bounded error terms.

Thus (57) can be written as

$$- \log \epsilon (\phi' \phi - \widehat{x}) + E$$

where

$$E = (k_2 + \log \epsilon)\phi' \phi + (k_3 - \log \epsilon)\widehat{x} + k_1 \widehat{x} \phi'(\widehat{x}) + k_4 \phi(\widehat{x})$$

Going back to (56) and choosing C appropriately ($C < 0$), one wants to make the following choice for ϵ and ϕ :

$$\epsilon' = \epsilon \delta \log \epsilon, \quad x - \phi' \phi = -\delta(\phi - x\phi'), \quad \delta \in (0, 1)$$

Take for $\epsilon(t)$ one particular solution

$$\epsilon(t) = \exp(-e^{\delta t})$$

The equation for ϕ we had before (see (43)) and so we have

$$|\phi - x|^A |\phi + x|^B = 1, \quad A = (1 + \delta)/2, \quad B = (1 - \delta)/2$$

and its solution ϕ trivially satisfies assumptions (a), (b), (c) mentioned above.

For the original equation (55), the error one gets after substituting $\epsilon\phi(x/\epsilon)$ amounts to ϵE where

$$\epsilon E \lesssim \epsilon \quad \text{if} \quad |x| \lesssim \epsilon$$

and

$$|\epsilon E| \lesssim \epsilon + \epsilon |\log |x| - \log \epsilon| (\phi' \phi - \widehat{x})| + |x \phi'(\widehat{x}) k_1| + |\epsilon \phi(\widehat{x}) k_4| \quad \text{if} \quad |x| \gtrsim \epsilon$$

Therefore, for $|x| \gtrsim \epsilon$,

$$|\epsilon E| \lesssim |z| + \epsilon \left| \frac{x}{\epsilon} \right|^{-\gamma(\delta)} \log \left| \frac{x}{\epsilon} \right|, \quad \gamma(\delta) > 0$$

and $z = (x, \epsilon\phi(\widehat{x}))$. Thus, we see that the error is small again so it is possible to find the approximate solution with the self-similar scaling that holds on the ball of size ~ 1 . We are not trying to make this picture global and define the incompressible strain on the whole \mathbb{R}^2 which corresponds to the error ϵE but we believe it is possible.

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