WAVE PACKET DECOMPOSITION FOR SCHRÖDINGER EVOLUTION WITH ROUGH POTENTIAL AND GENERIC VALUE OF PARAMETER

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ABSTRACT. We develop the wave packet decomposition to study the Schrödinger evolution with rough potential. As an application, we obtain the improved bound on the wave propagation for the generic value of parameter.

Keywords: Schrödinger evolution, perturbation theory, wave packet decomposition.

CONTENTS

1. INTRODUCTION.

That work aims to develop a new technique in perturbation theory for dispersive equations. As a model case, we take Schrödinger evolution with time-dependent real-valued potential $V(x,t)$:

$$iu_t(x,t,k) = -k\Delta u(x,t,k) + V(x,t)u(x,t,k), \quad u(x,0,k) = f(x), \quad t \in \mathbb{R} \quad \Delta := \partial^2_{xx}\,, \tag{1.1}$$

where $k \in \mathbb{R}$ is a real-valued parameter. For a large value of $T \gg 1$, we consider the problem (1.1) either on the real line $x \in \mathbb{R}$ or on the torus $x \in \mathbb{R}/2\pi T\mathbb{Z}$ and make the following assumptions about the real-valued potential V and the initial data f when studying u for $t \in [0, 2\pi T]$:

(A) if $x \in \mathbb{R}$, then

$$\|V(x,t)\|_{L^\infty(\mathbb{R}\times[0,2\pi T])} \leqslant T^{-\gamma}, \gamma > 0\,, \tag{1.2}$$

$$\|f\|_2 < \infty, \tag{1.3}$$

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(B) if $x \in \mathbb{R}/2\pi T\mathbb{Z}$, then

$$\|V(x,t)\|_{L^\infty(\mathbb{R}/2\pi T\mathbb{Z}\times[0,2\pi T])} \leqslant T^{-\gamma}, \gamma > 0\,, \tag{1.4}$$

$$\|f\|_2 < \infty\,. \tag{1.5}$$

The behavior of free evolution $e^{i\Delta t}$ is well-understood (the presence of k only scales the time t). It is governed by the dispersion relation for which the higher Fourier modes propagate with higher speed, giving rise to ballistic transport. In the current work, we study how the presence of V changes the free evolution $e^{ik\Delta}f$ for the time interval $t \in [0, 2\pi T]$.

The solution u in (1.1) is understood as the solution to the Duhamel integral equation

$$u(\cdot,t,k) = e^{ik\Delta t}f - i\int_0^t e^{ik\Delta(t-\tau)}V(\cdot,\tau)u(\cdot,\tau,k)d\tau, \quad t \in [0,2\pi T] \tag{1.6}$$

in the class $u \in C([0,2\pi T], L^2)$. Its existence and uniqueness are immediate from the contraction mapping principle. More generally, for $0 \leqslant t_1 \leqslant t \leqslant 2\pi T$, the symbol $U(t_1,t,k)$ denotes the operator $f \mapsto U(t_1,t,k)f$ where $U(t_1,t,k)f$ solves

$$i\partial_t U(t_1,t,k)f = -k\Delta U(t_1,t,k)f + V(x,t)U(t_1,t,k)f, \quad U(t_1,t_1,k)f = f\,.$$

The propagator U satisfies the standard group property: $U(t_1,t_1,k) = I, U(t_1,t_2,k)U(t_0,t_1,k) = U(t_0,t_2,k)$, and, since V is real-valued, U is a unitary operator in $L^2 : \|Uf\|_2 = \|f\|_2$. To set the stage, we start with an elementary perturbative result (below, the symbol $\|O\|$ indicates the operator norm of the operator O in the space L^2).

Lemma 1.1. *Suppose* $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant 2\pi T$ *and* $\|V\|_\infty \lesssim T^{-\gamma}$. *Then,*

$$U(t_1,t,k) = e^{ik\Delta(t-t_1)} - i\int_{t_1}^t e^{ik\Delta(t-\tau)}V(\cdot,\tau)e^{ik\Delta(\tau-t_1)}d\tau + Err \tag{1.7}$$

and the operator norm of Err *allows the estimate* $\|Err\| \lesssim T^{-2\gamma}(t_2-t_1)^2$.

Proof. That follows from the representation

$$U(t_1,t,k) = e^{ik\Delta(t-t_1)} - i\int_{t_1}^t e^{ik\Delta(t-\tau)}V(\cdot,\tau)e^{ik\Delta(\tau-t_1)}d\tau + Err\,, \tag{1.8}$$

$$Err := -\int_{t_1}^t e^{ik\Delta(t-\tau_1)}V(\cdot,\tau_1)\int_{t_1}^{\tau_1} e^{ik\Delta(\tau_1-\tau_2)}V(\cdot,\tau_2)U(t_1,\tau_2,k)d\tau_2 d\tau_1$$

and two bounds: $\|V\|_{L^\infty} \lesssim T^{-\gamma}$ and $\|U(t_1,\tau_2,k)\| = 1$. □

That lemma has an immediate application.

Corollary 1.2. *Suppose* $\gamma > 1$, *then*

$$\|U(0,t,k) - e^{ik\Delta t}\| = O(T^{1-\gamma}),\ t \in [0,2\pi T],\ k \in \mathbb{R}\,. \tag{1.9}$$

Hence, for $\gamma > 1$, the propagated wave is asymptotically close to the free dynamics irrespective of the value of k and of the initial vector f. In the current paper, we study the problem for generic k and show that the analogous result holds for some γ below the elementary threshold 1. We also give examples which show that (1.9) cannot hold for $\gamma < 1$ for all k, in general.

The Schrödinger evolution with smooth V and $T \sim 1$ was studied in [1, 3, 13] where the upper and lower estimates for the Sobolev norms of the solution were obtained. In the context of the general evolution equations, similar questions were addressed in [7, 9] (see [8] for the motivation to study them) where the technique from the complex analysis was employed. The asymptotics of the 2×2 systems of ODE were studied for generic value of parameters using the harmonic analysis methods (see, e.g., [4, 5] and references therein). Below, we develop a different approach. It is based on writing the linear in V term in (1.8)

$$\int_{t_1}^t e^{ik\Delta(t-\tau)}V(\cdot,\tau)e^{ik\Delta(\tau-t_1)}d\tau \tag{1.10}$$

in the convenient basis of wave packets (see, e.g., [10]) which allows a detailed and physically appealing treatment of the multi-scattering situation at hand. We will mostly focus on problem **(A)**; however, our analysis is valid for the problem **(B)**, too. Denote (see Figure 1)

$$\Upsilon_T := \{x \in [-\pi T, \pi T], t \in [cT, 2\pi T], 0 < c < 2\pi\}\,. \tag{1.11}$$

The following theorem is one application of our perturbative technique.

Theorem 1.3. *There is $\gamma_0 \in (0,1)$ such that the following statement holds. Suppose V satisfies*

$$\|V\|_{L^\infty(\Upsilon_T^c)} \lesssim T^{-\alpha}, \alpha > 1, \tag{1.12}$$

$$\|V\|_{L^\infty(\Upsilon_T)} \lesssim T^{-\gamma}, \gamma > \gamma_0\,, \tag{1.13}$$

$$\operatorname{supp} \widehat{V}(\xi_1, \xi_2) \subset \{\xi : \rho_1 < |\xi| < \rho_2\} \tag{1.14}$$

and $\rho_2 > \rho_1 > 0$. Then, for each interval $I \subset \mathbb{R}^+$, there is a positive parameter $\delta(\rho_1, \rho_2, I)$ such that for every function f that satisfies

$$\|f\|_2 = 1, \tag{1.15}$$

$$\|f\|_{L^2(|x|>2\pi T)} = o(1),\ T \to \infty \tag{1.16}$$

and

$$\operatorname{supp} \widehat{f} \subset (-\delta, \delta)\,, \tag{1.17}$$

there exists a set $Nres \subset I$ such that $\lim_{T\to\infty} |Res| = 0, Res := I \backslash Nres$ and

$$\lim_{T\to\infty} \|u(x, 2\pi T, k) - e^{ik\Delta 2\pi T} f\|_2 \to 0 \tag{1.18}$$

for $k \in Nres$.

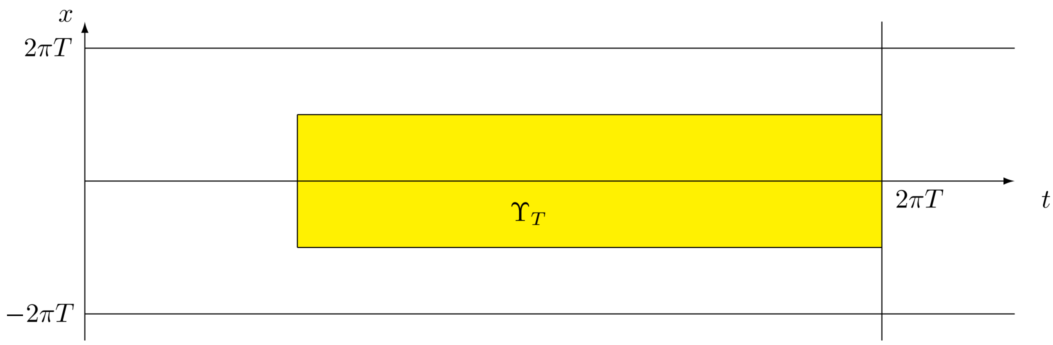

Figure 1

Remark. In contrast to Corollary 1.2, the theorem says that the propagation is asymptotically free even for some $\gamma < 1$ if we consider a generic value of k. It is crucial that we can handle essentially arbitrary initial data f. In fact, for f that is well-localized on the frequency side, a simple perturbative argument can be used to get an analogous result. The conditions (1.16), (1.17), and (1.12)-(1.14) on the f and V are included to guarantee that the bulk of the wave u given by f hits Υ_T. On the other hand, assumption (1.17) is essential for the statement to hold and cannot be dropped as will be illustrated in Section 8 by example.

Remark. For given f, such set $Nres$ (which can depend on f) will be called the set of *non-resonant* parameters within the interval I. Later, for every $\gamma < 1$, we will present f and potential V so that the resonant set Res is nonempty.

Remark. Simple modification of Lemma 1.1 shows that the potential V from Theorem 1.3 is negligible outside Υ_T due to (1.12). Inside Υ_T it satisfies the "weak decay condition" if $\gamma < 1$, and it oscillates on scale ~ 1 there due to (1.14). For the problem we consider, the assumption (1.14) is actually necessary since dropping it can introduce a well-known WKB-type correction in the dynamics and (1.18) would fail.

Lemma 1.4. *Let $\widehat{\phi}(\xi_1, \xi_2)$ be a smooth centrally symmetric real-valued bump function supported on $B_1(0)$ in $\mathbb{R}^2$. Denote $\kappa := \sqrt{T}$ and let $\phi_\kappa(x,t) = \phi(x/\kappa, t/\kappa)$. Then, the function*

$$V = T^{-\gamma} \cdot \sum_{|2\pi n\kappa|<T/4, |2\pi m\kappa - \pi T|<T/20} c_{n,m} \cos(x\lambda_{n,m} + t\mu_{m,n}) \phi_\kappa(x - 2\pi n\kappa, t - 2\pi m\kappa),$$

$$(n,m) \in \mathbb{Z}^2,\ c_{n,m} \in [-1,1], \gamma > \gamma_0, \lambda_{n,m}^2 + \mu_{n,m}^2 \sim 1 \tag{1.19}$$

satisfies conditions of the previous theorem.

Proof. Indeed, $\widehat{\phi}\in\mathbb{S}(\mathbb{R}^2)$ so $\phi\in\mathbb{S}(\mathbb{R}^2)$ and ϕ_κ is real-valued. Then, the Fourier transform of $\cos(x\lambda_{n,m}+t\mu_{m,n})\phi_\kappa(x-2\pi n\kappa,t-2\pi m\kappa)$ is supported on two balls centered at $(\pm\lambda_{n,m},\pm\mu_{n,m})$ with radii κ^{-1}. Moreover,

$$|V|\leqslant C_bT^{-\gamma}\sum_{|2\pi n\kappa|<T/4,|2\pi m\kappa-\pi T|<T/20}\Big(1+|x\kappa^{-1}-2\pi n|+|t\kappa^{-1}-2\pi m|\Big)^{-b}$$

with any positive b. Hence, taking c small enough in the definition of Υ_T, we get $|V|\lesssim T^{-\gamma}$ on Υ_T and $|V|\leqslant C_bT^{-b}$ with any positive b outside of Υ_T. The condition $\lambda_{n,m}^2+\mu_{n,m}^2\sim 1$ guarantees that the function $\widehat{V}$ is supported on the annulus $\rho_1<|\xi|<\rho_2$ with some positive ρ_1 and ρ_2. ☐

The random potentials V that model Anderson localization/delocalization phenomenon exhibit strong oscillation (see, e.g., [2, 6, 11, 12]) similar to the one considered in Theorem 1.3.

The structure of the paper is as follows. The second section contains the formulation of our central perturbation result, we develop the suitable wave packet decomposition there. The third and fourth sections frame the problem in this new setup. The fifth and sixth sections study two important classes of potentials. We finalize the proofs of the main theorems in section seven. The section eight contains the examples of resonance formation. In the appendix, we collect some auxiliary results.

Some notation:

- If $(x_1,x_2)\in\mathbb{R}^2$ and $r>0$, then $B_r(x_1,x_2)$ denotes the closed ball of radius r around the point (x_1,x_2).
- We will denote the Fourier transform in $\mathbb{R}^d$ of function f by

$$\mathcal{F}f=\widehat{f}=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}}fe^{-i\langle x,\xi\rangle}dx\,.$$

We will use both symbols $\widehat{\cdot}$ and $\mathcal{F}$ in the text depending on what is more convenient.

- Suppose $f_T\in L^2(\mathbb{R}),\|f_T\|_{L^2(\mathbb{R})}=1$ and $T\gg 1$ is a large parameter. We will say that f_T is concentrated around a_T at scale ℓ_T ($a_T\in\mathbb{R},\ell_T>0$) if

$$\lim_{q\to+\infty}\limsup_{T\to\infty}\int_{|x-a_T|>q\ell_T}|f_T|^2dx=0\,.$$

- Let $\kappa:=T^{\frac12}$. We introduce the lattice $2\pi\kappa\mathbb{Z}\times 2\pi\kappa\mathbb{Z}$ and call the cubes $\{B_{j,\ell}\}=[2\pi\kappa j,2\pi\kappa(j+1)]\times[2\pi\kappa\ell,2\pi\kappa(\ell+1)],j,\ell\in\mathbb{Z}$ the characteristic cubes in $(x,t)\in\mathbb{R}^2$.
- If B is a cube on the plane, then c_B denotes its center and αB denotes the α-dilation of B around $c_B,\alpha>0$.
- Given an interval $I=[k_1,k_2]\subset\mathbb{R}^+$, the symbol I^{-1} denotes $[k_2^{-1},k_1^{-1}]$.
- Symbol $C_c^\infty(\mathbb{R}^d)$ denotes the set of smooth compactly supported functions on $\mathbb{R}^d$ and $\mathbb{S}(\mathbb{R}^d)$ stands for the Schwartz class of functions.
- Given any real number x, we write $\langle x\rangle=(x^2+1)^{\frac12}$.
- We will use the following notation standard in the modern harmonic analysis, i.e., given two positive quantities A and B and a large parameter T, we write $A\lessapprox B$ if $A\leqslant C_\epsilon T^\epsilon B$ for all $T>1$ and $\epsilon>0$. Given a quantity f, the symbol $O(f)$ will indicate another quantity that satisfies $|O(f)|\leqslant C|f|$ with a constant C.
- For a function $g\in L^2(\mathbb{R})$ and a measurable set $E\subset\mathbb{R}$, the symbol P_Eg will indicate the Fourier orthogonal projection to the set E, i.e., $P_Eg=\mathcal{F}^{-1}(\mathcal{F}g\cdot\chi_E)$.
- Symbol κ is used for $T^{\frac12}$ and η for k^{-1}. In many estimates below, the letter R will indicate a quantity that satisfies $|R|\lessapprox 1$ where $\kappa>1$ (and T) are large parameters.

2. The main estimate and the wave packet decomposition.

We start with a result that illustrates the Fourier restriction phenomenon of the Duhamel operator $\int_0^T e^{ik(T-t)\Delta}dt$.

Lemma 2.1. *Suppose* $g(x,t) \in \mathbb{S}(\mathbb{R}^2)$. *Then,*

$$\int_{\mathbb{R}} \left\| \int_0^{2\pi T} e^{ik\Delta(2\pi T-t)} \partial_x g(\cdot,t)dt \right\|_2^2 dk \lesssim \|g\|^2_{L^2(\mathbb{R}^2)}$$

and

$$\int_{\mathbb{R}} \left\| P_{|\xi|>\delta} \int_0^{2\pi T} e^{ik\Delta(2\pi T-t)} g(\cdot,t)dt \right\|_2^2 dk \lesssim \delta^{-2}\|g\|^2_{L^2(\mathbb{R}^2)}$$

for all $\delta > 0$.

Proof. Taking the Fourier transform in variable x, we get

$$\mathcal{F}\left(\int_0^{2\pi T} e^{-ik\Delta t}\partial_x g(\cdot,t)dt\right) = C\xi \int_0^{2\pi T} e^{ikt\xi^2}(\mathcal{F}g)(\xi,t)dt = C\xi\mathcal{F}(\chi_{0<t<2\pi T}\cdot g)(\xi,-k\xi^2),$$

where $\mathcal{F}$ in the right-hand side is a two-dimensional Fourier transform, which is restricted to a k-dependent parabola. The simple change of variables and Plancherel's formula gives us

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\xi\mathcal{F}(\chi_{0<t<2\pi T}\cdot g)(\xi,-k\xi^2)|^2 d\xi dk \lesssim \int_{\mathbb{R}}\int_{0<t<2\pi T} |g(x,t)|^2 dxdt \leqslant \|g\|_2^2\,.$$

The second bound in the lemma follows immediately from the first. □

In our paper, we will be estimating expression (the one-collision operator)

$$\int_0^{2\pi T} e^{ik\Delta(2\pi T-t)} V(\cdot,t)e^{ik\Delta t} f dt\,,$$

which corresponds to the linear in V term in (1.7). Notice that $V(\cdot,t)e^{ikt\Delta}f$ depends on k and the previous lemma is not applicable. However, for f localized on low frequencies on the Fourier side, the dependence on k is so weak that a variant of that lemma can be used to obtain an estimate better than the general bound we get. To treat the general f, we will apply the different technique standard in the harmonic analysis. That approach uses the decomposition of f into a different "basis" of the so-called wave packets and each element of such basis has a particular localization both on the physical and on the Fourier side. Such a decomposition is a standard tool in modern harmonic analysis (see, e.g., [10] for one application).

2.1. **k-dependent wave packet decomposition.** In the proof of Theorem 1.3, we can take for I the finite union of dyadic intervals $[2^j, 2^{j+1}), j \in \mathbb{Z}$ and, therefore, it is enough to prove the claim for dyadic intervals only. Now, notice that given $k \in [2^j, 2^{j+1})$, we can rescale the space variable $\widetilde{x} = 2^{(j-1)/2}x$ to reduce the problem to $k \in [2,4)$. The support of $\widehat{V}$ (parameters ρ_1 and ρ_2) and the domain Υ_T will change, too, but the proof can be easily adjusted. Let $I = [2,4]$. Take a nonnegative function $h \in C_c^\infty(\mathbb{R})$ supported on $[0,4\pi]$ such that the partition of unity

$$1 = \sum_{n\in\mathbb{Z}} h^2(s-2\pi n) \tag{2.1}$$

holds for $s \in \mathbb{R}$. Since $k \in I$, we get $k \geqslant 2$. Then, for every $f \in \mathbb{S}(\mathbb{R})$ and every $s \in [2\pi n, 2\pi(n+k)]$, we can write

$$f(s)h(s-2\pi n) = (2\pi k)^{-1}\sum_{\ell\in\mathbb{Z}}\left(\int_{2\pi n}^{2\pi(n+k)} f(\tau)h(\tau-2\pi n)e^{-i\ell k^{-1}(\tau-2\pi n)}d\tau\right)e^{i\ell k^{-1}(s-2\pi n)},$$

and, if we introduce an auxiliary variable $\eta = k^{-1}$,

$$f(s) \overset{(2.1)}{=} \sum_{n,\ell\in\mathbb{Z}^2} \widetilde{f}_{n,\ell}(\eta)e^{i\ell\eta s}h(s-2\pi n), \quad \widetilde{f}_{n,\ell}(\eta) := \eta(2\pi)^{-1}\int_{2\pi n}^{2\pi(n+k)} f(\tau)h(\tau-2\pi n)e^{-i\ell\eta\tau}d\tau$$

$$\overset{k>2,\,\mathrm{supp}\,h\subset[0,4\pi]}{=} \eta(2\pi)^{-1}\int_{2\pi n}^{2\pi(n+2)} f(\tau)h(\tau-2\pi n)e^{-i\ell\eta\tau}d\tau\,.$$

We dilate by κ to write

$$f(x) = \sum_{n,\ell\in\mathbb{Z}^2} \omega_{n,\ell}(x,k)f_{n,\ell}(\eta),\ \omega_{n,\ell}(x,k) = h_n(x)e^{i\ell\eta\kappa^{-1}x}\,, \tag{2.2}$$

where

$$h_n(x) := \kappa^{-\frac{1}{2}} h\left(\frac{x - 2\pi\kappa n}{\kappa}\right),\ f_{n,\ell}(\eta) := \eta(2\pi)^{-1} \int_{2\pi n\kappa}^{2\pi(n+2)\kappa} f(x)h_n(x)e^{-i\ell\eta\kappa^{-1}x}dx \tag{2.3}$$

and $f_{n,\ell}(\eta)$ can be written as follows

$$f_{n,\ell}(\eta) = e^{-2\pi i\eta n\ell} f^*_{n,\ell}(\eta),$$

$$f^*_{n,\ell}(\eta) := \frac{\eta}{2\pi\sqrt{\kappa}} \int_0^{4\pi\kappa} f(s + 2\pi n\kappa)h(s\kappa^{-1})e^{-i\ell\eta s\kappa^{-1}}ds\,. \tag{2.4}$$

By Plancherel's identity,

$$\sum_{\ell\in\mathbb{Z}} |f^*_{n,\ell}|^2 = \frac{\eta}{2\pi}\int_{\mathbb{R}} |f(s)h(s\kappa^{-1} - 2\pi n)|^2 ds$$

and

$$\sum_{n,\ell\in\mathbb{Z}^2} |f^*_{n,\ell}|^2 \sim \|f\|_2^2\,, \tag{2.5}$$

because $\sum_{n\in\mathbb{Z}} |h(x - 2\pi n)|^2 = 1$ and $\eta \in I^{-1}$. By the standard approximation argument, we can extend (2.5) from $f \in \mathbb{S}(\mathbb{R})$ to $f \in L^2(\mathbb{R})$.

Recall that $\eta = k^{-1}$. Using the known evolution of $\omega_{0,0}$ for $t \in [0, 2\pi T]$ (check (9.3)), we can use the modulation property (a) from Appendix (check (9.1)) to get

$$e^{ik\Delta t}\omega_{n,\ell} = \kappa^{-\frac{1}{2}}\Omega_{T_{n,\ell}}(x,t,k)e^{i\eta(\alpha(T^{\rightarrow})x-\alpha^2(T^{\rightarrow})t)}, \quad \alpha(T^{\rightarrow}) := \ell/\kappa\,,$$

where the function

$$\Omega_{T_{n,\ell}}(x,t,k) = \mathcal{U}((x - 2\pi(n+1)\kappa - 2t\alpha(T^{\rightarrow}))/\kappa, tk/\kappa^2)$$

oscillates slowly in $x \in \mathbb{R}$ and $t \in [-T, T]$, i.e.,

$$\left|\frac{\partial^j\Omega_{T_{n,\ell}}(x,t,k)}{\partial x^j}\right| \leqslant_j \kappa^{-j}, \left|\frac{\partial^j\Omega_{T_{n,\ell}}(x,t,k)}{\partial t^l}\right| \leqslant_j \kappa^{-j}\,,$$

provided $|\ell| \lesssim \kappa$, and its derivatives in k are bounded as follows

$$\left|\frac{\partial^j\Omega_{T_{n,\ell}}(x,t,k)}{\partial k^j}\right| \leqslant_j 1,\ k \in I\,. \tag{2.6}$$

Moreover, $|\Omega_{T_{n,\ell}}(x,t)|$ is negligible away from $\{|x - 2\pi(n+1)\kappa - 2t\alpha(T^{\rightarrow})| < \kappa^{1+\delta}, |t| < 2\pi T\}, \delta > 0$. It will be convenient to work with tubes

$$T^{\rightarrow} = T_{n,\ell} := \{|x - 2\pi(n+1)\kappa - 2t\alpha(T^{\rightarrow})| < 2\pi\kappa,\ |t| < 2\pi T\}$$

that are k-independent. Such $T^{\rightarrow}$ has base as an interval $x \in [2\pi n\kappa, 2\pi(n+2)\kappa]$ and $2\alpha(T^{\rightarrow})$ is its slope in variable t. Given $T^{\rightarrow}$, we denote the corresponding n as $n(T^{\rightarrow})$. Notice that $T_{n,\ell} \cap T_{n+j,\ell} = \varnothing$ for $|j| > 1$.

One can be more specific about $\Omega_{T_{n,\ell}}(x,t,k)$: if $\mathcal{U}$ is a function generated by h in (9.2) then (9.4) gives

$$\Omega_{T_{n,\ell}}(x,t) = \sum_{\lambda\in\mathbb{Z}} \Omega^{(\lambda)}_{T_{n,\ell}}(x,t,k), \quad \Omega^{(\lambda)}_{T_{n,\ell}}(x,t,k) := \mathcal{U}_\lambda((x - 2\pi(n+1)\kappa)/\kappa - 2t\alpha(T^{\rightarrow})/\kappa^2, tk/\kappa^2)\,. \tag{2.7}$$

Notice that $\mathcal{U}_0((x - 2\pi(n+1)\kappa)/\kappa - 2t\alpha(T^{\rightarrow})/\kappa^2, tk/\kappa^2)$ is supported inside the $T_{n,\ell}$ and all other terms $\mathcal{U}_\lambda((x - 2\pi(n+1)\kappa)/\kappa - 2t\alpha(T^{\rightarrow})/\kappa^2, tk/\kappa^2), \lambda \neq 0$ are supported in the tubes obtained by its vertical translations by $2\pi\kappa\lambda$. Although the supports of all terms in (2.7) are parallel tubes, the contributions from large λ are negligible due to (9.5). Hence, we can rewrite the formula for evolution in a slightly different notation

$$e^{ik\Delta t}f = \sum_\lambda\sum_{T^{\rightarrow}} f^*_{T^{\rightarrow}}(\eta)\cdot\left(\kappa^{-\frac{1}{2}}\Omega^{(\lambda)}_{T^{\rightarrow}}(x,t,k)\right)\cdot e^{i\eta(\alpha(T^{\rightarrow})(x-2\pi n(T^{\rightarrow})\kappa)-\alpha^2(T^{\rightarrow})t)}\,. \tag{2.8}$$

We now discuss the properties of the coefficients $f_{n,\ell}$ in (2.3). Clearly,

$$f_{n,\ell}(\eta) = \frac{\eta}{(2\pi)^{3/2}}(\widehat{f} * \widehat{h}_n)(\ell\eta/\kappa),\ \widehat{h}_n(\xi) = \sqrt{\kappa}e^{-2\pi i\kappa n\xi}\widehat{h}(\xi\kappa)\,.$$

Hence, conditions (1.16) and (1.17) imply that

$$\sum_{|n|\geqslant 2\kappa}\sum_{\ell\in\mathbb{Z}}|f_{n,\ell}|^2=o(1),\qquad \sum_{|n|\leqslant 2\kappa}\sum_{|\ell|\geqslant C\delta\kappa}|f_{n,\ell}|^2=o(1)\,,$$

where $T\to\infty$, $\eta\in I^{-1}$ and C is a suitable positive constant. Since the evolution U is unitary, that indicates that, when studying Uf, we can restrict our attention to wave packets $\omega_{n,\ell}$ with $|n|\leqslant 2\kappa$ and $|\ell|\leqslant C\delta\kappa$ where the last condition is equivalent to $|\alpha(T^{\rightarrow})|\lesssim\delta$.

The expression in (1.10) will be central for our study and the following theorem is the key for proving Theorem 1.3.

Theorem 2.2 (The main estimate for *"one-collision operator"*)**.** *Suppose V and f satisfy assumptions of the Theorem 1.3, except that V is not necessarily real-valued. Define*

$$f_o=\sum_{|n|\leqslant 2\kappa,|\ell|\leqslant C\delta\kappa}\omega_{n,\ell}(x,k)f_{n,\ell}(\eta) \tag{2.9}$$

and

$$Q=\int_0^{2\pi T}e^{ik\Delta(2\pi T-t)}V(\cdot,t)e^{ik\Delta t}dt\,. \tag{2.10}$$

Then,

$$\left(\int_{I^{-1}}\|Qf_o\|^2d\eta\right)^{\frac{1}{2}}\lessapprox T^{1-\gamma-\frac{1}{832}}\,. \tag{2.11}$$

We call the operator Q from (2.10) the one-collision operator. The proof of this theorem will be split into several statements to be discussed in the next sections. This proof will be finished in Section 7.

3. Interaction of wave packets with V on characteristic cubes.

Consider the characteristic cubes $B_{p,q}=[2\pi p\kappa,2\pi(p+1)\kappa]\times[2\pi q\kappa,2\pi(q+1)\kappa]$, $p,q\in\mathbb{Z}^+$. We split the time interval $[0,2\pi T]$ into κ equal intervals of length $2\pi\kappa$ so that

$$[0,2\pi T]=\bigcup_{q=1}^{\kappa}[2\pi\kappa(q-1),2\pi\kappa q]$$

and Υ_T is covered by $\sim\kappa^2$ characteristic cubes. Recall that we study the operator

$$Q=\int_0^{2\pi T}e^{ik\Delta(2\pi T-\tau)}V(\cdot,\tau)e^{ik\Delta\tau}d\tau$$

from (2.10). We will do that by first taking the partition of unity

$$1=\sum_{p,q}\phi(x/(2\pi\kappa)-p,t/(2\pi\kappa)-q), \tag{3.1}$$

where smooth $\phi=1$ on $0.9\cdot([0,1]\times[0,1])$ and is zero outside $1.1\cdot([0,1]\times[0,1])$. If B is a characteristic cube with parameters (p,q), we might use notation V_B instead of $V_{p,q}$. Let

$$V=\sum_{p,q}V_{p,q}(x-2\pi\kappa p,t-2\pi\kappa q),V_{p,q}:=V(x+2\pi p\kappa,t+2\pi q\kappa)\cdot\phi(x/(2\pi\kappa),t/(2\pi\kappa)). \tag{3.2}$$

Later, we will need the following lemma which shows that each $V_{p,q}$ shares the main properties of V but it is localized to $1.1B_{0,0}$ instead of Υ_T.

Lemma 3.1. *Given assumptions of Theorem 2.2, we get*

$$\operatorname{supp}V_{p,q}\subset 1.1B_{0,0}, \tag{3.3}$$

$$\|V_{p,q}\|_{L^\infty(\mathbb{R}^2)}\lesssim T^{-\gamma},\gamma>\gamma_0\,, \tag{3.4}$$

$$\|\widehat{V}_{p,q}(\xi_1,\xi_2)\|_{(L^\infty\cap L^1)(\{\xi:\rho_1-\epsilon<|\xi|<\rho_2+\epsilon\}^c)}\leqslant_{\epsilon,N}T^{-N} \tag{3.5}$$

for every small $\epsilon>0$ and every $N\in\mathbb{N}$.

Proof. The first two bounds are immediate and the last one follows from the properties of convolution. □

Later in the text, we will use the following observation several times. Since $V_{p,q}$ is supported on the ball of radius $C\kappa$, we can write $V_{p,q} = V_{p,q} \cdot \rho(x/(C_1\kappa), t/(C_1\kappa))$ where ρ a smooth bump function which is equal to 1 on $B_1(0)$ and C_1 is large enough. Then,

$$\widehat{V}_{p,q}(\xi) = \int_{\mathbb{R}^2} \widehat{V}_{p,q}(s) D_\kappa(\xi - s) ds, \quad D_\kappa(s) := (C_1\kappa)^2 \widehat{\rho}(C_1\kappa s), \quad \widehat{\rho} \in \mathbb{S}(\mathbb{R}^2)\,, \tag{3.6}$$

i.e., $\widehat{V}_{p,q}$ can be written as a convolution of $\widehat{V}_{p,q}$ with a function well-localized on scale κ^{-1} around the origin. Hence, applying the Cauchy-Schwarz inequality, we get

$$|\widehat{V}_{p,q}(\xi)|^2 \leqslant \int_{\mathbb{R}^2} |\widehat{V}_{p,q}(s)|^2 |D_\kappa(\xi - s)| ds \cdot \int_{\mathbb{R}^2} |D_\kappa(\xi - s)| ds \lesssim \int_{\mathbb{R}^2} |\widehat{V}_{p,q}(s)|^2 |D_\kappa(\xi - s)| ds\,. \tag{3.7}$$

That bounds the value of function $|\widehat{V}_{p,q}|^2$ at any point ξ by, essentially, the average of that function on κ^{-1}-ball around that point (up to a contribution from the fast-decaying tail).

3.1. Contribution from each characteristic cube $B_{p,q}$.

We first focus on the contribution to scattering picture coming from $V_{p,q}(x - 2\pi\kappa p, t - 2\pi\kappa q)$. Notice that in the expression

$$\int_0^{2\pi T} e^{ik\Delta(2\pi T - \tau)} V_{p,q}(\cdot - 2\pi\kappa p, \tau - 2\pi\kappa q) e^{ik\Delta\tau} f_o d\tau$$

the function $V_{p,q}(\cdot - 2\pi\kappa p, \tau - 2\pi\kappa q) e^{ik\Delta\tau} f_o$ satisfies

$$\left\| P_{|\xi|>C}\left(V_{p,q}(\cdot - 2\pi\kappa p, \tau - 2\pi\kappa q) e^{ik\Delta\tau} f_o \right) \right\|_2 \leqslant_N T^{-N},\ N \in \mathbb{N} \tag{3.8}$$

for suitable C due to (3.5).

We apply wave packet decomposition for f_o and for Qf_o. The wave packets based on $t = 0$ will be denoted $\omega^{\rightarrow}_{n,\ell}$ and those corresponding to $t = 2\pi T$ will be denoted $\omega^{\leftarrow}_{m,j}$. Similarly, the corresponding tubes are $T^{\rightarrow}_{n,\ell}$ and $T^{\leftarrow}_{m,j}$. Recall that, given any forward tube $T^{\rightarrow}$, we denote the corresponding wave packet $\omega^{\rightarrow}_{n,\ell}$ as $\omega(T^{\rightarrow})$, the corresponding coordinate $n = n(T^{\rightarrow})$ and the frequency $\ell(T^{\rightarrow}) = \kappa\alpha(T^{\rightarrow})$. The notation for the backward tube is similar. The set of forward tubes relevant to us is

$$\mathcal{T}^{\rightarrow} = \{T^{\rightarrow} : n(T^{\rightarrow}) \leqslant 2\kappa, |\alpha(T^{\rightarrow})| \leqslant C\delta\} \tag{3.9}$$

and, thanks to (3.8), the set of backward tubes of interest satisfies $|n(T^{\leftarrow})| \leqslant C\kappa, |\alpha(T^{\leftarrow})| \leqslant C\}$. From now on, we will take only these tubes into account. In estimating Qf_o, we will use wave packet decomposition and the bound (2.5).

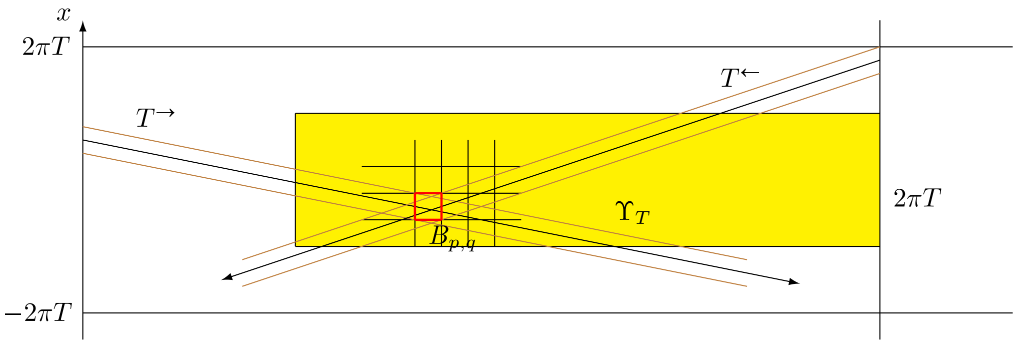

Figure 2

The southwestern corner of the cube $B_{p,q}$ has coordinates $2\pi\kappa p, 2\pi\kappa q$ and, if it is intersected by forward tube $T^{\rightarrow}$ and backward tube $T^{\leftarrow}$, then their parameters satisfy (see Figure 2)

$$2\alpha(T^{\rightarrow})q = p - n(T^{\rightarrow}) + O(1), \quad 2\alpha(T^{\leftarrow})(\kappa - q) = n(T^{\leftarrow}) - p + O(1) \tag{3.10}$$

and $O(1)$ indicates a real-valued quantity which depends only on $p, q, T^{\leftarrow}, T^{\rightarrow}$ and satisfies a uniform estimate $|O(1)| < C$.

Then, taking f_o, applying Q, and using (2.2) and (2.8), we compute the coordinate with respect to backward tube $T^{\leftarrow}$ by the formula (we suppress summation in λ here for a moment)

$$\langle Q_{p,q}f_o,\omega(T^{\leftarrow})\rangle=\sum_{T^{\rightarrow}\in\mathcal{T}^{\rightarrow}}f_{T^{\rightarrow}}(\eta)\int_0^{2\pi T}\langle V_{p,q}(x-2\pi\kappa p,t-2\pi\kappa q)e^{ik\Delta\tau}\omega(T^{\rightarrow}),e^{-ik\Delta(2\pi T-\tau)}\omega(T^{\leftarrow})\rangle d\tau=$$

$$\sum_{T^{\rightarrow}\in\mathcal{T}^{\rightarrow}}f^*_{T^{\rightarrow}}(\eta)\int_{\mathbb{R}^2}\Big(V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\kappa^{-\frac12}\Omega_{T^{\rightarrow}}(x,\tau,k)e^{i\eta(\alpha(T^{\rightarrow})(x-2\pi n(T^{\rightarrow})\kappa)-\alpha^2(T^{\rightarrow})\tau)}$$

$$\times\quad\kappa^{-\frac12}\overline{\Omega_{T^{\leftarrow}}(x,\tau-2\pi T,k)}e^{-i\eta(\alpha(T^{\leftarrow})x-\alpha^2(T^{\leftarrow})(\tau-2\pi T))}\Big)dxd\tau= \tag{3.11}$$

$$\Big(\sum_{T^{\rightarrow}\in\mathcal{T}^{\rightarrow}}f^*_{T^{\rightarrow}}(\eta)\int_{\mathbb{R}^2}\Big(V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\kappa^{-\frac12}\Omega_{T^{\rightarrow}}(x,\tau,k)e^{i\eta(\alpha(T^{\rightarrow})(x-2\pi n(T^{\rightarrow})\kappa)-\alpha^2(T^{\rightarrow})\tau)}$$

$$\times\quad\kappa^{-\frac12}\overline{\Omega_{T^{\leftarrow}}(x,\tau-2\pi T,k)}e^{-i\eta(\alpha(T^{\leftarrow})(x-2\pi n(T^{\leftarrow})\kappa)-\alpha^2(T^{\leftarrow})(\tau-2\pi T))}\Big)dxd\tau\Big)e^{-i\eta\alpha(T^{\leftarrow})2\pi n(T^{\leftarrow})\kappa}\,.$$

If the tubes $T^{\leftarrow}$ and $T^{\rightarrow}$ both intersect $B_{p,q}$, then (3.10) holds and the integral above can be written as

$$\exp\Big(2\pi i\eta\kappa\Big((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q+\alpha^2(T^{\leftarrow})\kappa+O(1)\Big)\Big)F_{T^{\rightarrow},T^{\leftarrow}}(\eta),$$

where

$$F_{T^{\rightarrow},T^{\leftarrow}}(\eta):=\int_{\mathbb{R}^2}\Big(V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\Big(\kappa^{-1}\Omega_{T^{\rightarrow}}(x,\tau,k)\overline{\Omega_{T^{\leftarrow}}(x,\tau-2\pi T,k)}\Big)$$

$$e^{i\eta((\alpha(T^{\rightarrow})-\alpha(T^{\leftarrow}))(x-2\pi p\kappa)-(\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))(\tau-2\pi q\kappa))}\Big)dxd\tau\,. \tag{3.12}$$

Notice that the formula (2.7) can be applied to $\Omega_{T^{\rightarrow}}(x,\tau,k)$ and $\Omega_{T^{\leftarrow}}(x,\tau-2\pi T,k)$ which gives

$$V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\Omega_{T^{\rightarrow}}(x,\tau,k)\overline{\Omega_{T^{\leftarrow}}(x,\tau-2\pi T,k)}=$$

$$\sum_{\lambda,\lambda'\in\mathbb{Z}^2}V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\Omega^{(\lambda)}_{T^{\rightarrow}}(x,\tau,k)\cdot\overline{\Omega^{(\lambda')}_{T^{\leftarrow}}(x,\tau-2\pi T,k)}$$

and, accordingly,

$$F_{T^{\rightarrow},T^{\leftarrow}}(\eta)=\sum_{\lambda,\lambda'\in\mathbb{Z}^2}F^{(\lambda,\lambda')}_{T^{\rightarrow},T^{\leftarrow}}(\eta)\,. \tag{3.13}$$

Now, $\Omega^{(\lambda)}_{T^{\rightarrow}}(x,\tau,k)\cdot\overline{\Omega^{(\lambda')}_{T^{\leftarrow}}(x,\tau-2\pi T,k)}$ is compactly supported inside the intersection of corresponding tubes and its supnorm decays fast in $|\lambda|$ and $|\lambda'|$ due to (9.5). In all estimates that follow, we will only handle the term that corresponds to $\lambda=\lambda'=0$, i.e.

$$\langle Q^{(0,0)}_{p,q}f_o,\omega(T^{\leftarrow})\rangle=e^{2\pi i\eta(-\kappa\alpha(T^{\leftarrow})n(T^{\leftarrow})+\kappa^2\alpha^2(T^{\leftarrow}))}\times$$

$$\sum_{T^{\rightarrow}:T^{\rightarrow}\cap B_{p,q}\neq\varnothing}f^*_{T^{\rightarrow}}(\eta)e^{2\pi i\eta\kappa((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q+O(1))}F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta) \tag{3.14}$$

where the backward tube $T^{\leftarrow}$ intersects $B_{p,q}$. The other terms in (3.13) lead to the same bounds except that the resulting estimates will involve a strong decay in $|\lambda|$ and $|\lambda'|$. The first factor $e^{2\pi i\eta(-\kappa\alpha(T^{\leftarrow})n(T^{\leftarrow})+\kappa^2\alpha^2(T^{\leftarrow}))}$ in the formula above only depends on $T^{\leftarrow}$ and it is unimodular. It will play no role in our estimates.

Now, $\Omega^{(0)}_{T^{\rightarrow}}(x,t,k)$ is supported inside $T^{\rightarrow}$ and we consider

$$F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)=\int_{\mathbb{R}^2}\Big(V_{p,q}(x-2\pi\kappa p,\tau-2\pi\kappa q)\Big(\kappa^{-1}\Omega^{(0)}_{T^{\rightarrow}}(x,\tau,k)\overline{\Omega^{(0)}_{T^{\leftarrow}}(x,\tau-2\pi T,k)}\Big)\times$$

$$e^{i\eta((\alpha(T^{\rightarrow})-\alpha(T^{\leftarrow}))(x-2\pi\kappa p)-(\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))(\tau-2\pi\kappa q))}\Big)dxd\tau\ = \tag{3.15}$$

$$\int_{\mathbb{R}^2}\Big(V_{p,q}(x,\tau)\Big(\kappa^{-1}\Omega^{(0)}_{T^{\rightarrow}}(x+2\pi\kappa p,\tau+2\pi\kappa q,k)\overline{\Omega^{(0)}_{T^{\leftarrow}}(x+2\pi\kappa p,\tau+2\pi\kappa q-2\pi T,k)}\Big)\times$$

$$e^{i\eta((\alpha(T^{\rightarrow})-\alpha(T^{\leftarrow}))x-(\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))\tau)}\Big)dxd\tau\,.$$

By (2.7), (3.1), and (9.5), the function $\phi(x/(2\pi\kappa)-p,t/(2\pi\kappa)-q)\Omega^{(0)}_{T^\to}(x,\tau,k)\overline{\Omega^{(0)}_{T^\leftarrow}(x,\tau-2\pi T,k)}$ is smooth and is supported on $2B_{p,q}$. Arguing like in (3.6), we can write an estimate

$$|F^{(0,0)}_{T^\to,T^\leftarrow}(\eta)|\lesssim \kappa^{-1}\Big(|\widehat{V}_{p,q}|*|\psi_\kappa|\Big)(-(\alpha(T^\to)-\alpha(T^\leftarrow))\eta,(\alpha^2(T^\to)-\alpha^2(T^\leftarrow))\eta)\,, \tag{3.16}$$

where $\psi_\kappa(\xi_1,\xi_2)=\kappa^2\Psi(\kappa\xi_1,\kappa\xi_2)$, $\Psi\in\mathbb{S}(\mathbb{R}^2)$, and Ψ does not depend on $p,q,T^\to,T^\leftarrow$ and $k\in I$. Consider the function $|\widehat{V}_{p,q}|*|\psi_\kappa|$. We can use (3.6) to conclude that

$$|\widehat{V}_{p,q}|*|\psi_\kappa|\lesssim|\widehat{V}_{p,q}|*P_\kappa,\ P_\kappa(\xi)=\kappa^2P(\xi\kappa) \tag{3.17}$$

and P is a nonnegative function decaying fast at infinity. Applying Cauchy-Schwarz to $|\widehat{V}_{p,q}|*P_\kappa$ as

$$(|\widehat{V}_{p,q}|*P_\kappa)^2(\xi)\leqslant\left(\int_{\mathbb{R}^2}|\widehat{V}_{p,q}(s)|^2*P_\kappa(\xi-s)ds\right)\cdot\int_{\mathbb{R}^2}P_\kappa(\xi-s)ds\lesssim\int_{\mathbb{R}^2}|\widehat{V}_{p,q}(s)|^2*P_\kappa(\xi-s)ds\,, \tag{3.18}$$

we get the analog of (3.7). Hence, the function $|\widehat{V}_{p,q}|*|\psi_\kappa|$ satisfies (check (3.5))

$$\|\Big(|\widehat{V}_{p,q}|*|\psi_\kappa|\Big)\|_{(L^\infty\cap L^1)(\{\xi:\rho_1-\epsilon<|\xi|<\rho_2+\epsilon\}^c)}\lesssim_{\epsilon,N}T^{-N} \tag{3.19}$$

for every small $\epsilon>0$ and every $N\in\mathbb{N}$. Moreover,

$$\|\Big(|\widehat{V}_{p,q}|*|\psi_\kappa|\Big)\|_2\lesssim T^{\frac12-\gamma}\,. \tag{3.20}$$

Formula (3.16) shows that $F^{(0,0)}_{T^\to,T^\leftarrow}(\eta)$ is bounded by the restriction of $|\widehat{V}_{p,q}|*|\psi_\kappa|$ to the "curve":

$$(-(\alpha(T^\to)-\alpha(T^\leftarrow))\eta,(\alpha^2(T^\to)-\alpha^2(T^\leftarrow))\eta),\quad \eta\in I^{-1}\,.$$

For fixed $\alpha(T^\to)$ and $\alpha(T^\leftarrow)$, that is a straight segment or a point 0 when $\eta\in I^{-1}$. For fixed $\eta\in I^{-1}$ and $\alpha(T^\to)$, that is a piece of parabola when $\alpha(T^\leftarrow)$ spans a segment. By (3.19), $F^{(0,0)}_{T^\to,T^\leftarrow}$ is negligible unless (recall that $\eta\in[\frac14,\frac12]$)

$$|\alpha(T^\to)-\alpha(T^\leftarrow)|\sim_{\rho_1,\rho_2}1,\ |\alpha(T^\to)+\alpha(T^\leftarrow)|\sim_{\rho_1,\rho_2}1\,. \tag{3.21}$$

For given ρ_1 and ρ_2, we can choose positive δ small enough to guarantee that $|\alpha(T^\to)|\leqslant\delta$ implies

$$|\alpha(T^\leftarrow)|\sim 1\,. \tag{3.22}$$

Without loss of generality, we can therefore assume that $\alpha(T^\leftarrow)\sim 1$. Hence, the set of backward tubes relevant to us is

$$\mathcal{T}^\leftarrow:=\{T^\leftarrow:|n(T^\leftarrow)|<C(\rho_1,\rho_2,\delta)\kappa,\ 0<C_1(\rho_1,\rho_2,\delta)\leqslant\alpha(T^\leftarrow)\leqslant C_2(\rho_1,\rho_2,\delta)\}$$

so that (3.21) holds for each $T^\to\in\mathcal{T}^\to$ and $T^\leftarrow\in\mathcal{T}^\leftarrow$. Notice that each $B_{p,q}\in\Upsilon_T$ is intersected by $\sim\delta\kappa$ tubes $T^\to\in\mathcal{T}^\to$ and by $\sim\kappa$ tubes $T^\leftarrow\in\mathcal{T}^\leftarrow$.

3.2. **Sparsifying V.** For $T^\to\in\mathcal{T}^\to$ and $T^\leftarrow\in\mathcal{T}^\leftarrow$, each set $T^\to\cap T^\leftarrow$ can be covered by at most C cubes $B_{p,q}$ (see Figure 2 above).

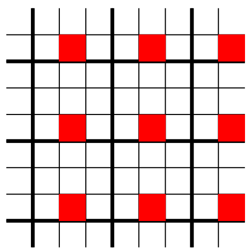

Figure 3

Notice that, for each $P\in\mathbb{N}$, the potential V can be written (by "sparsifying" periodically) as

$$V(x,t)=\sum_{p,q\in\mathbb{Z}^2}V_{p,q}(x-2\pi\kappa p,t-2\pi\kappa q)=\sum_{\alpha\in\{0,\dots,P-1\},\beta\in\{0,\dots,P-1\}}V^{(\alpha,\beta)}(x,t)\,, \tag{3.23}$$

where each $V^{(\alpha,\beta)}$, defined by

$$V^{(\alpha,\beta)}(x,t):=\sum_{n,m\in\mathbb{Z}^2}V_{nP+\alpha,mP+\beta}(x-2\pi\kappa(nP+\alpha),t-2\pi\kappa(mP+\beta))\,,$$

satisfies $\mathrm{dist}\,(\mathrm{supp}\,V_{nP+\alpha,mP+\beta},\mathrm{supp}\,V_{n'P+\alpha,m'P+\beta})\gtrsim P\kappa$ for all $(n,m)\neq(n',m')$. Figure 3 has $P=3$ and the red boxes correspond to choosing $\alpha=0$ and $\beta=1$.

Therefore, for P large enough, $T^{\rightarrow}\in\mathcal{T}^{\rightarrow}$ and $T^{\leftarrow}\in\mathcal{T}^{\leftarrow}$, the set $T^{\rightarrow}\cap T^{\leftarrow}$ intersects at most one characteristic cube out of the set $B_{nP+\alpha,mP+\beta}$ when α and β are fixed.

Hence, going from one function V to any of P^2 functions $V^{(\alpha,\beta)}$, we can always guarantee that *each* $T^{\rightarrow}\cap T^{\leftarrow}$ *intersects at most one* $B_{p,q}$. Going forward, we can assume without loss of generality that this property holds for the original V.

3.3. **Contribution from all of V.** In the general case, given $T^{\leftarrow}$, we need to account for all characteristic cubes that intersect it. Hence, we need to control

$$D_{T^{\leftarrow}}(\eta):=e^{2\pi i\eta(-\kappa\alpha(T^{\leftarrow})n(T^{\leftarrow})+\kappa^2\alpha^2(T^{\leftarrow}))}\times \sum_{B_{p,q}:B_{p,q}\cap T^{\leftarrow}\neq\varnothing}\ \sum_{T^{\rightarrow}:B_{p,q}\cap T^{\rightarrow}\neq\varnothing} f^*_{T^{\rightarrow}}(\eta)e^{2\pi i\eta\kappa\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q+O(1)\right)}F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)\,. \tag{3.24}$$

Take $\mu(\eta)$, any smooth non-negative bump function supported on I^{-1} and write

$$\int_{\mathbb{R}}\mu(\eta)\left|D_{T^{\leftarrow}}(\eta)\right|^2d\eta=\int_{\mathbb{R}}\mu(\eta)\times \sum_{B_{p,q}\cap T^{\leftarrow}\neq\varnothing}\ \sum_{B_{p',q'}\cap T^{\leftarrow}\neq\varnothing}\ \sum_{\substack{T^{\rightarrow}:\\ B_{p,q}\cap T^{\rightarrow}\neq\varnothing}}\ \sum_{\substack{T'^{\rightarrow}:\\ B_{p',q'}\cap T'^{\rightarrow}\neq\varnothing}} e^{2\pi i\eta\kappa\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q-(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow}))q'+O(1)\right)}$$
$$\times\ f^*_{T^{\rightarrow}}(\eta)\overline{f^*_{T'^{\rightarrow}}(\eta)}F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)\overline{F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}(\eta)}d\eta=\Sigma^{(nr)}_{T^{\leftarrow}}+\Sigma^{(r)}_{T^{\leftarrow}}\,, \tag{3.25}$$

where $\Sigma^{(r)}_{T^{\leftarrow}}$ is the sum that corresponds to the resonance indexes given by the following *global resonance conditions:*

$$\left|\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q-(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow}))q'\right)\right|\lessapprox 1\,, \tag{3.26}$$

that do not depend on the choice of V. For definiteness, we can assume that the bound $\lessapprox 1$ takes the form: $|\cdot|<C_{\epsilon'}\kappa^{\epsilon'}$ with a fixed positive ϵ' that can be chosen arbitrarily small. Having mentioned that, we define $\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})$ as a set of backward tubes $T^{\leftarrow}$ that form the global resonance with $T^{\rightarrow}$ and $T'^{\rightarrow}$.

Lemma 3.2. *We have* $|\Sigma^{(nr)}_{T^{\leftarrow}}|\leqslant C_j\kappa^{-j}$ *for every* $j\in\mathbb{N}$.

Proof. Recall that (2.9) restricts $|\ell|\leqslant C\delta\kappa$ in (2.4). Next, one can apply a non-stationary phase argument. In the integral

$$\int_{\mathbb{R}}\mu f^*_{T^{\rightarrow}}\overline{f^*_{T'^{\rightarrow}}}e^{2\pi i\eta\kappa\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q-(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow}))q'\right)}e^{2\pi i\kappa\eta O(1)}F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}\overline{F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}}d\eta\,,$$

we use (2.4), (3.12), and the bound (2.6) to integrate in η by parts consecutively to get

$$\left|\int_{\mathbb{R}}\mu f^*_{T^{\rightarrow}}\overline{f^*_{T'^{\rightarrow}}}e^{2\pi i\eta\kappa\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q-(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow}))q'\right)}e^{2\pi i\kappa\eta O(1)}F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}\overline{F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}}d\eta\right|\lessapprox$$
$$\leqslant_{j,\epsilon}\kappa^{-j},\ j\in\mathbb{N},\epsilon>0\,,$$

provided that $\left|\left((\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow}))q-(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow}))q'\right)\right|\geqslant\kappa^{\epsilon}$. Since the quadruple sum in (3.25) contains at most $C\kappa^4$ terms, we get the statement of the lemma. □

This lemma shows that we only need to focus on the resonant terms which constitute a small proportion of all possible combinations. We will study these resonant configurations next.

3.4. **Geometry and combinatorics of global resonances.** In that subsection, we list some properties of global resonance configurations that will play a crucial role later. As before, we consider only those cubes $B_{p,q}$ that intersect Υ_T and hence

$$|p|<c_1\kappa,\quad c_2\kappa<q\leqslant\kappa,\ c_2>0\,. \tag{3.27}$$

1. The first observation about global resonances. Recall that they are determined by the condition (3.26) and it can be rewritten as

$$\left|\frac{\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow})}{\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow})}q-q'\right|\lessapprox\frac{1}{|\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow})|}\,.$$

Given (3.21), we get

$$\left|\frac{\alpha^2(T^{\rightarrow})-\alpha^2(T^{\leftarrow})}{\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\leftarrow})}q-q'\right|\lessapprox 1\,. \tag{3.28}$$

Then, $T^{\rightarrow}, T^{\leftarrow}$ and q define the set of q' of cardinality $\lessapprox 1$.

2. The second observation. Suppose $T^{\rightarrow}$, $T^{\leftarrow}$, $B=B_{p,q}$, and $B'=B_{p',q'}$ are given. Here, B is the cube intersecting $T^{\rightarrow}\cap T^{\leftarrow}$ and B' is the cube intersecting $T'^{\rightarrow}\cap T^{\leftarrow}$. Let us count the number of possible $T'^{\rightarrow}$ that can create a global resonance. Denote $s:=q'-q$. It will be convenient to introduce a symbol R that indicates a quantity that satisfies $|R|\lessapprox 1$ when $T\to\infty$. We can rewrite (3.26) as $(\alpha^2(T'^{\rightarrow})-\alpha^2(T^{\rightarrow}))q'=s(\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow}))+R$. One has $\alpha^2(T'^{\rightarrow})\overset{q'\sim\kappa}{=}\alpha^2(T^{\rightarrow})+sq'^{-1}(\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow}))+O(R\kappa^{-1})$, which gives the following restriction on s:

$$s\geqslant s_0(\alpha(T^{\rightarrow}),\alpha(T^{\leftarrow}),q')+O(R),\qquad s_0:=-\frac{\alpha^2(T^{\rightarrow})q'}{\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow})}\,.$$

Therefore,

$$\alpha(T'^{\rightarrow})=\pm\Big(\alpha^2(T^{\rightarrow})+sq'^{-1}(\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow}))+O(R\kappa^{-1})\Big)^{\frac12}= \tag{3.29}$$
$$\pm\left(\frac{\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow})}{q'}\right)^{\frac12}|s-s_0+O(R)|^{\frac12}\,.$$

We write

$$\left(\frac{\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow})}{q'}\right)^{\frac12}|s-s_0+O(R)|^{\frac12}=\left(\frac{\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow})}{q'}\right)^{\frac12}|s-s_0|^{\frac12}+\Delta \tag{3.30}$$

and

$$|\Delta|\lessapprox\begin{cases}(\kappa|s-s_0|)^{-\frac12}, & |s-s_0|>|R|,\\ \kappa^{-\frac12}, & |s-s_0|<|R|\end{cases}\,. \tag{3.31}$$

Hence, we get at most $\#(s)$ possible choices for $T^{\rightarrow}$, where

$$\#(s)\lessapprox\kappa^{\frac12}|s-s_0|^{-\frac12},\, s\geqslant s_0+|R| \tag{3.32}$$

and $\#(s)\lessapprox\kappa^{\frac12}$ for $s\leqslant s_0+|R|$. Notice that

$$\sum_{s\geqslant s_0+O(R)}\#(s)\lessapprox\kappa^{\frac12}+\sum_{s\geqslant s_0+|R|}^{\kappa}\frac{\sqrt{\kappa}}{|s-s_0|^{\frac12}}\sim\kappa\,. \tag{3.33}$$

Hence, each $T^{\rightarrow}$ and $T^{\leftarrow}$ define $\sim\kappa$ "globally resonant" tubes $T'^{\rightarrow}$. Moreover, for given $T^{\leftarrow}$ and B, prescribing parameter $\alpha(T^{\rightarrow})$ or $-\alpha(T^{\rightarrow})$ define the same sets of possible $\alpha(T'^{\rightarrow})$. If we take $\alpha(T^{\rightarrow})\geqslant 0$ branch, then a simple analysis shows that the function

$$\alpha'_{pr}:=\left(\frac{\alpha^2(T^{\leftarrow})-\alpha^2(T^{\rightarrow})}{q'}\right)^{\frac12}|s-s_0|^{\frac12} \tag{3.34}$$

(which defines the "principal direction" in (3.30)) is increasing in $s\geqslant s_0$, vanishes at $s=s_0$, and it is equal to $\alpha(T^{\rightarrow})$ when $s=0$. The largest "indeterminacy" $\#(s)\sim\kappa^{\frac12}$ corresponds to $s=s_0+O(R)$ as follow from the formula (3.31). Also, notice that (check (3.29))

$$\alpha'_{pr}-|\alpha(T^{\rightarrow})|\sim\frac{s}{\kappa|\alpha(T^{\rightarrow})|+\sqrt{s\kappa}}\gtrsim\frac{s}{\kappa},\qquad s\geqslant CR\,.$$

As a consequence, we get the following lemma.

Lemma 3.3. *Suppose $\tau \in (0, \frac{1}{2})$ is a parameter. If $s \in (\kappa^{1-\tau}, \kappa)$, then*

$$\alpha'_{pr} - |\alpha(T^{\rightarrow})| \gtrsim \kappa^{-\tau}, \tag{3.35}$$

$$|\alpha(T'^{\rightarrow})| - |\alpha(T^{\rightarrow})| \gtrsim \kappa^{-\tau}. \tag{3.36}$$

Proof. The first estimate is immediate from the previous bound. The second one follows from the first and from (3.31). □

3. The third observation. In a similar way, we obtain a lemma.

Lemma 3.4. *For given $T^{\rightarrow}$, $T'^{\rightarrow}$ and $B : B \cap T^{\rightarrow} \neq \varnothing$, we can have $\lessapprox 1$ cubes $B' : B' \cap T'^{\rightarrow} \neq \varnothing$ that define the global resonance through some $T^{\leftarrow}$ such that $B \cap T^{\leftarrow} \neq \varnothing$ and $B' \cap T^{\leftarrow} \neq \varnothing$. The total number of such resonances (choices of $T^{\leftarrow}$) is at most $\lessapprox T(dist(B, B') + \kappa)^{-1}$.*

Proof. The first claim is proved by analyzing (3.26) with the argument similar to the one given above. For the second one, observe that any two cubes B and B' define at most $CT(dist(B, B') + \kappa)^{-1}$ tubes that intersect both of them. □

4. The fourth observation. The following lemma says that any two forward tubes that are not nearly parallel cannot form too many global resonances. Recall that, when working with backward tubes $T^{\leftarrow}$, we assume that $\alpha(T^{\leftarrow}) \sim 1$.

Lemma 3.5 (**No Lattice Lemma**). *Let υ and ϵ be two small positive parameters satisfying $2(\upsilon + \epsilon) < 1$. Assume $T_0^{\rightarrow}$, $T_1^{\rightarrow}$ form global resonances with both $T_1^{\leftarrow}$ and $T_2^{\leftarrow}$ where*

$$q(B^{(0,2)}) - q(B^{(0,1)}) \gtrsim \kappa^{1-\upsilon} \tag{3.37}$$

and $B^{(j,l)}$ denotes a cube whose intersection with $T_j^{\rightarrow} \cap T_l^{\leftarrow}$ is nonempty. Suppose

$$|\alpha(T_0^{\rightarrow}) - \alpha(T_1^{\rightarrow})| \gtrsim \kappa^{-\epsilon}, \tag{3.38}$$

then there are at most $N : N \lessapprox \kappa^{2(\epsilon+\upsilon)}$ tubes $T_j^{\rightarrow}, j \in \{1, \ldots, N\}$ that can form a global resonance with $T_0^{\rightarrow}$ through both $T_1^{\leftarrow}$ and $T_2^{\leftarrow}$ and that satisfy

$$|\alpha(T_0^{\rightarrow}) - \alpha(T_j^{\rightarrow})| \gtrsim \kappa^{-\epsilon}. \tag{3.39}$$

Proof. In this proof, we measure the distance in "units" $2\pi\kappa$, understanding that this quantity is proportional to the distance between the centers of two adjacent cubes. Assume that the center lines of $T_j^{\rightarrow}, j \in \{0, \ldots, n\}$ are given (in units) by linear functions $a_j + \alpha_j \tau$ and those of $T_l^{\leftarrow}$ are given by $\tau\beta_l - s_l$, where $\tau \in [0, \kappa]$. Hence, $\alpha_j = 2\alpha(T_j^{\rightarrow}), \beta_l = 2\alpha(T_l^{\leftarrow})$ and $|a_j|, |s_l| \lesssim \kappa$ as we measure the distance in units. We can assume that $a_0 = 0$ without loss of generality as the global resonance condition is invariant with respect to vertical translations. The assumption (3.37) guarantees that

$$s_2\beta_1 - s_1\beta_2 \gtrsim \kappa^{1-\upsilon}. \tag{3.40}$$

The global resonance conditions (3.26) imply that

$$\beta_l a_0 + \alpha_0 a_0 + \alpha_0 s_l = \ldots = \beta_l a_n + \alpha_n a_n + \alpha_n s_l = O(R), \quad |R| \lessapprox 1 \tag{3.41}$$

for all $l \in \{1, 2\}$. We define $\delta a_j = a_j - a_0, \delta\alpha_j = \alpha_j - \alpha_0$, $\delta\beta = \beta_2 - \beta_1, \delta s = s_2 - s_1$. Hence, by subtracting the first equality from the others, one has

$$(\beta_l + \alpha_0) \cdot \frac{\delta a_j}{\delta \alpha_j} + (s_l + a_j) = \frac{O(R)}{\delta\alpha_j}. \tag{3.42}$$

Subtracting equations (3.41) with different l, and then subtracting the first equation from all others, we obtain

$$\delta\beta \cdot \delta a_j + \delta s \cdot \delta\alpha_j = O(R) \tag{3.43}$$

and, since $|\delta\alpha_j| \overset{(3.39)}{\gtrsim} \kappa^{-\epsilon}$, we get

$$\delta s = -\frac{\delta a_j}{\delta\alpha_j}\delta\beta + O(R\kappa^{\epsilon}). \tag{3.44}$$

The assumptions we made and a simple geometric reasoning (recall that all forward and backward tubes intersect $\sim \kappa$ units away from the vertical Ox axis) gives

$$s_l + a_j \sim \kappa \tag{3.45}$$

for all l and j. The formula (3.42) and the bounds $\beta_l + \alpha_0 \sim 1, |\delta\alpha_j| \overset{(3.39)}{\gtrsim} \kappa^{-\epsilon}$ yield

$$\left|\frac{\delta a_j}{\delta \alpha_j}\right| \sim \kappa \tag{3.46}$$

and, therefore, $|a_j| = |\delta a_j| \gtrsim \kappa^{1-\epsilon}$. Recall that $a_0 = 0$ and $|\alpha_0| \leqslant C\delta$. By choosing δ small, we can make $|\alpha_0| < 0.1\beta_1$, which implies that $s_1 \sim \kappa$ and (3.40) can be rewritten as

$$\delta s \cdot \beta_1 - s_1 \cdot \delta\beta \gtrsim \kappa^{1-\upsilon}\,. \tag{3.47}$$

Therefore, since $\beta_1 \sim 1$, we get $\kappa^{1-\upsilon} \overset{(3.44)+(3.46)}{\lesssim} \kappa \cdot |\delta\beta| + O(R\kappa^{\epsilon})$. Hence, $|\delta\beta| \gtrsim \kappa^{-\upsilon}$ and $|\delta s| \overset{(3.44)}{\gtrsim} \kappa^{1-\upsilon}$. Next, (3.44) yields $\delta a_j/\delta\alpha_j = -\delta s/\delta\beta + O(R\kappa^{\upsilon+\epsilon})$. From (3.42) and $s_l + a_j \overset{(3.45)}{\sim} \kappa$ we get $-\delta a_j/\delta\alpha_j \sim \kappa$ so the previous formula yields $\delta s/\delta\beta \sim \kappa$. Thus,

$$\frac{\delta a_j}{\delta\alpha_j} = -C_1\kappa + O(R\kappa^{\upsilon+\epsilon}),\ C_1 \sim 1 \tag{3.48}$$

and C_1 is independent of j. Substituting this into (3.42) gives

$$a_j = -(\beta_l + \alpha_0)\cdot\Big(-C_1\kappa + O(R\kappa^{\upsilon+\epsilon})\Big) - s_l + \frac{O(R)}{\delta\alpha_j}\,,$$

which we can write as (fixing $l \in \{1,2\}$)

$$a_j \overset{a_0=0}{=} \delta a_j = C_2\kappa - s_l + O(R\kappa^{\upsilon+\epsilon}), \quad C_2 \in \mathbb{R}$$

with j-independent C_2. The inverse of (3.48) provides

$$\frac{\delta\alpha_j}{\delta a_j} = -C_3\kappa^{-1} + O(R\kappa^{-2+\upsilon+\epsilon}), \quad \delta\alpha_j = C_4 + O(R\kappa^{-1+\upsilon+\epsilon}), \quad C_3 > 0,\, C_4 \in \mathbb{R}\,.$$

The numbers C_1, C_2, C_3, C_4 are j-independent and they have absolute values $\lesssim 1$. So, if we have two different $n, m \geqslant 1$, then $\alpha_n - \alpha_m = \delta\alpha_n - \delta\alpha_m = O(R\kappa^{-1+\upsilon+\epsilon})$. Hence, we have restrictions on the base of the tube $T_j^{\rightarrow}$: $a_j = C_2\kappa - s_l + O(R\kappa^{\upsilon+\epsilon})$ and on the slope: $|\alpha_j - \alpha_1| = O(R\kappa^{-1+\upsilon+\epsilon})$. That gives at most $R^2\kappa^{2(\upsilon+\epsilon)}$ choices for such tubes (see Figure 4). □

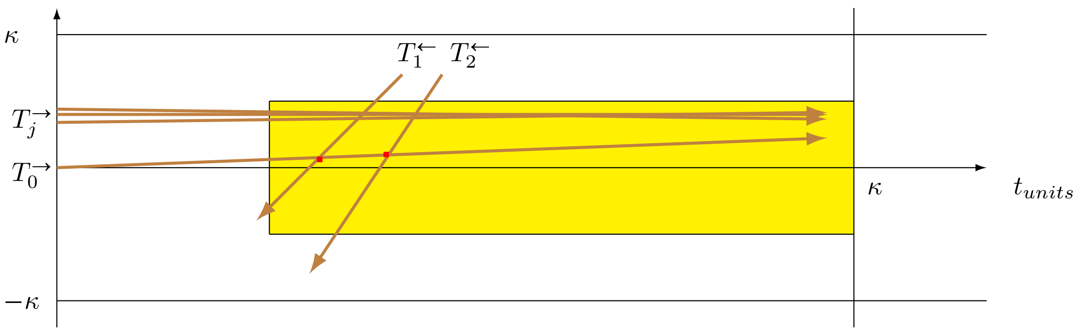

Figure 4 (in units)

4. Resonant terms and the decomposition of V.

For the generic $\eta \in I^{-1}$, our goal is to estimate $\sum_{T^{\leftarrow}} |D_{T^{\leftarrow}}(\eta)|^2$ where $D_{T^{\leftarrow}}$ is from (3.24). The contribution from non-resonant indexes is negligible, see Lemma 3.2. Consider the sum of the resonant terms. It can be rewritten in the following form

$$A := \sum_{T^\leftarrow} \widehat{\sum_{T^\to, T'^\to}} f^*_{T^\to} \bar f^*_{T'^\to} \times \tag{4.1}$$

$$e^{2\pi i\eta\kappa\left((\alpha^2(T^\to)-\alpha^2(T^\leftarrow))q-(\alpha^2(T'^\to)-\alpha^2(T^\leftarrow))q'\right)} e^{2\pi i\kappa\eta O(1)} F^{(0,0)}_{T^\to,T^\leftarrow} \overline{F^{(0,0)}_{T'^\to,T^\leftarrow}}$$

$$= \sum_{T^\to, T'^\to} f^*_{T^\to} \bar f^*_{T'^\to} \times \tag{4.2}$$

$$\sum_{T^\leftarrow \in \mathcal{G}(T^\to, T'^\to)} e^{2\pi i\eta\kappa\left((\alpha^2(T^\to)-\alpha^2(T^\leftarrow))q-(\alpha^2(T'^\to)-\alpha^2(T^\leftarrow))q'\right)} e^{2\pi i\kappa\eta O(1)} F^{(0,0)}_{T^\to,T^\leftarrow} \overline{F^{(0,0)}_{T'^\to,T^\leftarrow}},$$

where $T^\leftarrow \in \mathcal{T}^\leftarrow$ and $T^\to, T'^\to \in \mathcal{T}^\to$. The symbol $\widehat{\sum}$ in the first formula indicates that the summation is done over the resonant configurations, i.e., those for which (3.26) holds. Then, by Cauchy-Schwarz inequality,

$$|A|^2 \leqslant \left(\sum_{T^\to, T'^\to} |f^*_{T^\to} \bar f^*_{T'^\to}|^2\right) \times \tag{4.3}$$

$$\left(\sum_{T^\to, T'^\to} \left|\sum_{T^\leftarrow \in \mathcal{G}(T^\to, T'^\to)} e^{2\pi i\kappa\eta O(1)} e^{2\pi i\eta\kappa\left((\alpha^2(T^\to)-\alpha^2(T^\leftarrow))q-(\alpha^2(T'^\to)-\alpha^2(T^\leftarrow))q'\right)} F^{(0,0)}_{T^\to,T^\leftarrow} \overline{F^{(0,0)}_{T'^\to,T^\leftarrow}}\right|^2\right).$$

The first factor is bounded as $\sum_{T^\to, T'^\to} |f^*_{T^\to} \bar f^*_{T'^\to}|^2 \lesssim \|f\|_2^4$ by (2.5). The second one is estimated by the following quantity

$$\mathcal{I} := \sum_{T^\to, T'^\to} \left(\sum_{T^\leftarrow \in \mathcal{G}(T^\to, T'^\to)} |F^{(0,0)}_{T^\to,T^\leftarrow}(\eta) F^{(0,0)}_{T'^\to,T^\leftarrow}(\eta)|\right)^2. \tag{4.4}$$

In the rest of the argument, we will study $\mathcal{I}$.

4.1. **Initial data f of low complexity.** Before considering the general case, it is instructive to sketch what our method gives for some special choices of initial data f.

Example 1: the plane wave as initial profile. That corresponds to $f(x) = \kappa^{-1}\phi(x/T)$ where ϕ is a smooth bump function compactly supported around the origin. In that case, (4.2) has a double sum in $T^\to$ and $T'^\to$ containing only κ^2 terms since $|\alpha(T^\to)| \lessapprox \kappa^{-1}$ for every forward tube that provides a nontrivial contribution. Moreover, we have $|f^*_{T^\to}| \lesssim \kappa^{-\frac12}$ for every such tube. Formula (3.28) says that we have resonance of $T^\to$, $T'^\to$, and $T^\leftarrow$ only for $|q-q'| \lessapprox 1$ which means that $T^\to$ and $T'^\to$ are essentially neighbors. When estimating A from (4.1), we do not use Cauchy-Schwarz but instead write

$$|A| \lessapprox \kappa^{-1} \sum_{T^\to} \sum_{T^\leftarrow} |F^{(0,0)}_{T^\to,T^\leftarrow}(\eta)|^2. \tag{4.5}$$

Taking (3.16) into account, one obtains

$$\sum_{T^\leftarrow} |F^{(0,0)}_{T^\to,T^\leftarrow}(\eta)|^2 \lesssim \tag{4.6}$$

$$\sum_{B_{p,q}: B_{p,q}\cap T^\to \neq \varnothing} \sum_{T^\leftarrow: B_{p,q}\cap T^\leftarrow = \varnothing} \left|\kappa^{-1}\left(|\widehat V_{p,q}| * |\psi_\kappa|\right)(-(\alpha(T^\to)-\alpha(T^\leftarrow))\eta, (\alpha^2(T^\to)-\alpha^2(T^\leftarrow))\eta)\right|^2.$$

We substitute (3.18) into the sum above. Notice that the derivative of the map

$$(u,v) \mapsto \left(-(\alpha(T^\to)-u)v, (\alpha^2(T^\to)-u^2)v\right) \tag{4.7}$$

is nondegenerate if $C \geqslant |\alpha(T^{\to}) - u| \geqslant \delta$ and $C \geqslant |v| \geqslant \delta$ with positive δ. Then, averaging in η, we get for a given p and q:

$$\int_{I^{-1}} \sum_{T^{\leftarrow}: B_{p,q} \cap T^{\leftarrow} = \varnothing} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2 d\eta \lesssim \tag{4.8}$$

$$\int_{I^{-1}} \sum_{T^{\leftarrow}} \left|\kappa^{-1}\Big(|\widehat{V}_{p,q}| * |\psi_\kappa|\Big)(-(\alpha(T^{\to}) - \alpha(T^{\leftarrow}))\eta, (\alpha^2(T^{\to}) - \alpha^2(T^{\leftarrow}))\eta)\right|^2 d\eta \lesssim$$

$$\kappa^{-1} \int_{\mathbb{R}^2} (|\widehat{V}_{p,q}| * |\psi_\kappa|)^2(\xi_1, \xi_2) d\xi_1 d\xi_2 \overset{(3.20)}{\sim} \kappa^{-1} T^{1-2\gamma}\,.$$

Finally, we obtain $\int_{I^{-1}} |A| d\eta \lessapprox T^{1-2\gamma}$ and $\int_{I^{-1}} \|Qf\|^2 d\eta \lessapprox T^{1-2\gamma}$.

Example 2: evolution of a single k-independent wave packet. Define $\widetilde{\omega}_{n,\ell}(x) := \omega_{n,\ell}(x,1)$ (check (2.2)). Take $f = \widetilde{\omega}_{0,\ell^*}$ where $|\ell^*| \leqslant \delta\kappa$ where δ is a small positive parameter. Then, the "principal tubes" corresponding to such initial value will differ for different η. Specifically,

$$\widetilde{\omega}_{0,\ell^*}(x) = \sum_{j,n} r_{n,j,\ell^*}(\eta) \omega_{n,j}(x,\eta) \tag{4.9}$$

and (check (2.3))

$$|r_{0,j,\ell^*}(\eta)| \sim |\widehat{h^2}(\ell^* - j\eta)|\,, \tag{4.10}$$

so the principal direction j^* is

$$j^*(\ell^*, \eta) = \ell^*/\eta + O(1) \tag{4.11}$$

and the principal value of the base n is $n^* = O(1)$. Other j and n will amount to negligible contributions since $h \in C_c^\infty(\mathbb{R})$. The bound for A becomes $|A| \lesssim \sum_{T^{\to}} \sum_{T^{\leftarrow}} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2$, where the sum in $T^{\to}$ is extended to all possible directions which are at most $C(1 + |\ell^*|)$ due to (4.11). Notice that for every $\eta \in I^{-1}$, each backward $T^{\leftarrow}$ will intersect only at most C principal forward tubes $T^{\to}$. Averaging in η and recycling (4.8), we get $\int_{I^{-1}} |A| d\eta \lesssim (1 + |\ell^*|) T^{1-2\gamma}$. Since the same estimate holds for any n, we get

$$\int_{I^{-1}} \|Q\widetilde{\omega}_{n,\ell^*}\|^2 d\eta \lesssim \kappa T^{1-2\gamma} \tag{4.12}$$

for each n and ℓ^*.

4.2. **The rough bound for the general f.** As we already mentioned, our main goal is to handle f of "high complexity" which means we have no restrictions beyond the bounds $|n| \leqslant 2\kappa$ and $|\ell| \leqslant C\delta\kappa$ in (2.9). We start by obtaining a rough estimate via the wave packet decomposition.

We can bound $\mathcal{I}$ from (4.4) by Cauchy-Schwarz as follows

$$\mathcal{I} \lesssim \left(\sum_{T^{\to}} \sum_{T^{\leftarrow}} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2 \right)^2 ,$$

where both resonance and non-resonance interactions are taken into account. Consider the sum

$$\sum_{T^{\to}} \sum_{T^{\leftarrow}} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2 = \sum_{B_{p,q}} \sum_{\substack{T^{\to},T^{\leftarrow} \\ T^{\to} \cap T^{\leftarrow} \ni B_{p,q}}} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2 \tag{4.13}$$

and recall that $T^{\to} \in \mathcal{T}^{\to}$ and $T^{\leftarrow} \in \mathcal{T}^{\leftarrow}$. We now focus on the expression

$$\left|\kappa^{-1}\Big(|\widehat{V}_{p,q}| * |\psi_\kappa|\Big)(-(\alpha(T^{\to}) - \alpha(T^{\leftarrow}))\eta, (\alpha^2(T^{\to}) - \alpha^2(T^{\leftarrow}))\eta)\right|^2 \tag{4.14}$$

in (3.16). Notice that the derivative of a map $(\alpha, \beta) \mapsto (-\alpha + \beta, \alpha^2 - \beta^2)$ is $\begin{pmatrix} -1 & 1 \\ 2\alpha & -2\beta \end{pmatrix}$. It is continuous and nondegenerate on the compact set $\{(\alpha, \beta) : |\alpha| + |\beta| \leqslant C_1, |\beta| - |\alpha| \geqslant C_2\}$ for positive C_1 and C_2. Hence, for given $B_{p,q}$ and each $\eta \in I^{-1}$, the interior double sum in the r.h.s of (4.13) is bounded by the following quantity (use an estimate (3.18) to verify that):

$$\int_{\mathbb{R}^2} (|\widehat{V}_{p,q}| * |\psi_\kappa|)^2(\xi_1, \xi_2) d\xi_1 d\xi_2 \overset{(3.20)}{\lesssim} T^{1-2\gamma} \Rightarrow \sum_{T^{\to}} \sum_{T^{\leftarrow}} |F^{(0,0)}_{T^{\to},T^{\leftarrow}}(\eta)|^2 \lesssim T^{2-2\gamma} \tag{4.15}$$

and

$$\|Qf_o\| \lesssim T^{1-\gamma} \tag{4.16}$$

for *each* $\eta \in I^{-1}$. Our goal in the next three sections is to show that the bound (4.16) can be improved for generic η.

4.3. **The decomposition of $V_{p,q}$.** Let $\widehat{\phi}$ be a smooth nonnegative bump function supported inside $[-1,1]\times[-1,1]$ and chosen such that

$$1 = \sum_{\vec{n}\in\mathbb{Z}^2} \widehat{\varphi}(\xi - \vec{n}) \tag{4.17}$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Take $\widehat{\varphi}_{\vec{n}}(\xi) = \widehat{\varphi}(\kappa\xi_1 - n_1, \kappa\xi_2 - n_2)$ and rewrite (4.17) as

$$1 = \sum_{\vec{n}\in\mathbb{Z}^2} \widehat{\varphi}_{\vec{n}}(\xi)\,. \tag{4.18}$$

Here, each $\widehat{\varphi}_{\vec{n}}$ is nonnegative and it is supported on $B_{C\kappa^{-1}}(\kappa^{-1}\vec{n}), C > 0$. Then, define $V_{p,q,\vec{n}}$ by

$$\widehat{V}_{p,q,\vec{n}} = \widehat{V}_{p,q}\cdot\widehat{\varphi}_{\vec{n}}\,. \tag{4.19}$$

From the condition (3.5), we get

$$V_{p,q} = \sum_{C_1\kappa<|\vec{n}|<C_2\kappa} V_{p,q,\vec{n}} + Err, \quad \|Err\|_{(L^\infty\cap L^1)(\mathbb{R}^2)} \leqslant_m T^{-m}, m\in\mathbb{N}\,. \tag{4.20}$$

Using (3.6), (3.7), and the properties of convolution, one gets

$$\sum_{\vec{n}} \|\widehat{V}_{p,q,\vec{n}}\|^2_{L^\infty(\mathbb{R}^2)} \lesssim \kappa^2\|V_{p,q}\|^2_{L^2(\mathbb{R}^2)}\,. \tag{4.21}$$

Therefore,

$$\sum_{\vec{n}} \|\widehat{V}_{p,q,\vec{n}}\|^2_{L^\infty(\mathbb{R}^2)} \lesssim \kappa^2\|V_{p,q}\|^2_{L^2(\mathbb{R}^2)} \lesssim T^{2-2\gamma}\,. \tag{4.22}$$

Take $\lambda \in (0, \frac{1}{2})$ and define the set $\Omega(p,q,\lambda) = \{\vec{n} : \|\widehat{V}_{p,q,\vec{n}}\|_{L^\infty(\mathbb{R}^2)} > T^{1-\gamma-\lambda}\}$, $J_{p,q} := |\Omega(p,q,\lambda)|$ is its cardinality. Then, (4.22) yields $J_{p,q} \lesssim T^{2\lambda}$. For each (p,q), arbitrarily enumerate the frequencies in $\Omega(p,q,\lambda)$ as $\{\vec{n}^{(s)}_{p,q}\}, 1 \leqslant s \leqslant J_{p,q}$. Notice that $0 \leqslant J_{p,q} \lesssim T^{2\lambda}$, where taking $J_{p,q} = 0$ indicates that $\Omega(p,q,\lambda) = \varnothing$. Let $K := \max_{p,q} J_{p,q}$, then $K \lesssim T^{2\lambda}$. Take ρ as a smooth bump function which is equal to 1 on $B_1(0)$ (see (3.6)) and recall that $V_{p,q}$ is supported on $B_{C\kappa}(0)$. Hence $V_{p,q}(x,t) = V_{p,q}(x,t)\cdot\rho(x/(C_1\kappa), t/(C_1\kappa))$ if C_1 is large enough. Define functions $V^{(low)}_{p,q}, V^{(1)}_{p,q}, \dots, V^{(K)}_{p,q}$ by

$$\begin{aligned} V^{(s)}_{p,q}(x,t) &= \mathcal{F}^{-1}(\widehat{V}_{p,q}\cdot\widehat{\varphi}_{\vec{n}^{(s)}_{p,q}})(x,t)\cdot\rho(x/(C_1\kappa), t/(C_1\kappa)),\ s\leqslant K\,,\\ V^{(low)}_{p,q}(x,t) &= (V_{p,q}(x,t) - \sum_{s\leqslant K} V^{(s)}_{p,q}(x,t))\cdot\rho(x/(C_1\kappa)), t/(C_1\kappa))\,, \end{aligned} \tag{4.23}$$

where $\mathcal{F}^{-1}$ is inverse two-dimensional Fourier transform. If $s > J_{p,q}$, we set $V^{(s)}_{p,q} = 0$. The multiplication by ρ localizes the functions involved to $B_{C'\kappa}(0), C' > 0$ and, on the Fourier side, it is represented by the convolution with $(C_1\kappa)^2\widehat{\rho}(C_1\xi_1\kappa, C_1\xi_2\kappa)$. This way, the partition of unity gives

$$V_{p,q} = V^{(low)}_{p,q} + \sum_{s\leqslant K} V^{(s)}_{p,q} + Err, \quad \|Err\|_{(L^\infty\cap L^1)(\mathbb{R}^2)} \leqslant_m T^{-m},\ m\in\mathbb{N}\,. \tag{4.24}$$

By construction, $V^{(low)}_{p,q}$ satisfies

$$|\widehat{V}^{(low)}_{p,q}(\xi_1,\xi_2)| \lesssim T^{1-\gamma-\lambda}\,. \tag{4.25}$$

Recall the formulas (3.12) and (3.24). Substitute (4.24) into (3.12) and use triangle's inequality to get

$$\left(\sum_{T^{\leftarrow}} |D_{T^{\leftarrow}}|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{T^{\leftarrow}} |D^{(low)}_{T^{\leftarrow}}|^2\right)^{\frac{1}{2}} + \sum_{s\leqslant K}\left(\sum_{T^{\leftarrow}} |D^{(s)}_{T^{\leftarrow}}|^2\right)^{\frac{1}{2}} + Err,\ |Err| \leqslant_m T^{-m},\ m\in\mathbb{N}\,, \tag{4.26}$$

where $D^{(low)}_{T^{\leftarrow}}$ corresponds to changing $V_{p,q}$ by $V^{(low)}_{p,q}$ in (3.12) for every p and q. Similarly, $D^{(s)}_{T^{\leftarrow}}$ corresponds to changing $V_{p,q}$ by $V^{(s)}_{p,q}$ in (3.12) for every p and q.

5. Analysis of the λ-low-frequency part of V.

In that section, we control the first term in the right-hand side of (4.26) which corresponds to the λ-low-frequency part of V. This will give an improvement of (4.16).

Lemma 5.1. *Suppose V satisfies conditions of the Theorem 2.2. Then,*

$$\left\| \left(\sum_{T^{\leftarrow}} |D^{(low)}_{T^{\leftarrow}}(\eta)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_{I^{-1},d\eta}} \lessapprox T^{(1-\gamma)-\frac{\lambda}{6}}\,. \tag{5.1}$$

Proof. Recall that (4.25) holds for each p,q and we will use it when bounding $F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}$ below. Estimate $\mathcal{I}$ in (4.4) by applying Cauchy-Schwarz

$$\mathcal{I} \lessapprox \sum_{T^{\rightarrow}} \sum_{T'^{\rightarrow}} \left(\sum_{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}|^2 \right) \left(\sum_{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}|^2 \right), \tag{5.2}$$

and we recall that the backward tubes $T^{\leftarrow}$ in $\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})$ are those that create global resonance with $T^{\rightarrow}$ and $T'^{\rightarrow}$. In what follows, we always assume that $T^{\rightarrow}\in\mathcal{T}^{\rightarrow}$ and $T^{\leftarrow}\in\mathcal{T}^{\leftarrow}$, just like in the formula above. Let us start with considering a special case.

1. The case when all $T^{\rightarrow}$ and $T'^{\rightarrow}$ involved are at a distance $\sim T$ (or $\sim\kappa$ units) from each other. If B and B' are at a distance C_1T from each other, they define at most C_2 choices for $T^{\leftarrow}$ that intersect both of them (see Figure 5).

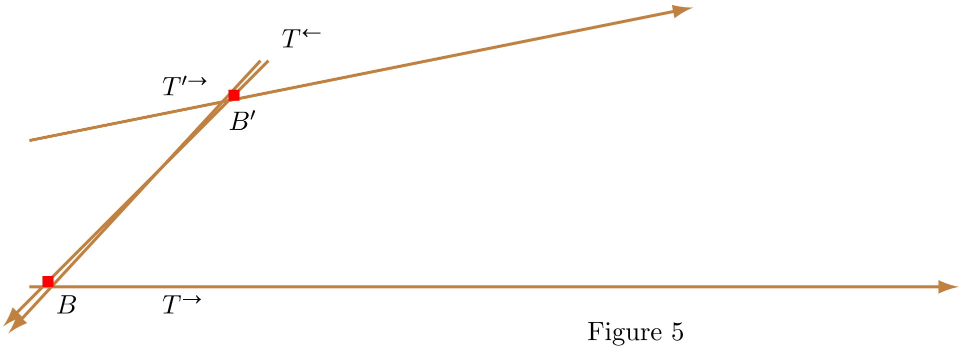

Figure 5

Hence, if we consider the sum that corresponds to only those $T^{\rightarrow}$ and $T'^{\rightarrow}$ that are T-separated, then the Lemma 3.4 gives the following estimate for the second factor in the right-hand side of (5.2):

$$\sum_{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}|^2 \overset{(3.16)+(4.25)}{\lessapprox} \kappa\cdot\frac{1}{\kappa^2}\cdot T^{2(1-\gamma-\lambda)} \tag{5.3}$$

for every η, where the first factor $C\kappa$ bounds the total number of cubes B' on $T'^{\rightarrow}$. Averaging in η, we have

$$\int_{I^{-1}} \mathcal{I}d\eta \lessapprox \kappa\cdot\frac{1}{\kappa^2}\cdot T^{2(1-\gamma-\lambda)}\cdot\sum_{T^{\rightarrow}}\sum_{B\cap T^{\rightarrow}\neq\varnothing}\left(\int_{I^{-1}}\sum_{T'^{\rightarrow}}\sum_{\substack{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow}):\\ B\cap T^{\leftarrow}\neq\varnothing}} |F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)|^2d\eta\right).$$

Now, notice that for every B, each backward tube $T^{\leftarrow}$ that intersects B contains $\sim\kappa$ cubes B' each of which defines $\#(s)$ tubes $T'^{\rightarrow}$ which are globally resonant with $T^{\rightarrow}$ and $T^{\leftarrow}$, see the calculations in (3.32) and (3.33). If $T^{\rightarrow}$ and $T'^{\rightarrow}$ are separated by $\sim T$, then $\#(s)\lessapprox 1$. Hence, for fixed $T^{\rightarrow}$ and B that intersects it, one has

$$\int_{I^{-1}}\sum_{T'^{\rightarrow}}\sum_{\substack{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow}):\\ B\cap T^{\leftarrow}\neq\varnothing}} |F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)|^2d\eta \lessapprox \kappa\int_{I^{-1}}\sum_{T^{\leftarrow}:B\cap T^{\leftarrow}=\varnothing}|F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)|^2d\eta \overset{(4.8)}{\lessapprox} \kappa\cdot\frac{T^{1-2\gamma}}{\kappa}\,. \tag{5.4}$$

Finally, $\int_{I^{-1}}\mathcal{I}d\eta \lesssim T^{4(1-\gamma)-2\lambda}$.

2. Another rough bound in the general case. The following estimate does not employ our additional assumption that $V_{p,q}$ is replaced by $V_{p,q}^{(low)}$ which is λ-low-frequency. However, rather than the argument in (4.16), it does require averaging in η. Consider (5.2). Fix $B' : B' \cap T'^{\rightarrow} \neq \varnothing$ and extend the sum $\sum_{T^{\leftarrow}}$ in $\sum_{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}|^2$ from only resonant directions $T^{\leftarrow}$ to all possible directions. Using (3.17) and (3.18), we get

$$\sum_{\substack{j,\ell:|j-\ell|>\delta\kappa\\ |j|,|\ell|<C\kappa}} \left|\kappa^{-1}\Big(|\widehat{V}_{B'}| * |\psi_\kappa|\Big)(-(j-\ell)\eta\kappa^{-1},(j^2-\ell^2)\eta\kappa^{-2})\right|^2 \lesssim T^{1-2\gamma}\,. \tag{5.5}$$

That gives the following bound for the second factor in (5.2):

$$\begin{aligned}\sum_{T^{\leftarrow}} |F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}|^2 &\lesssim \sum_{B':B'\cap T'^{\rightarrow}\neq\varnothing}\ \sum_{\substack{j,\ell:|j-\ell|>\delta\kappa\\ |j|,|\ell|<C\kappa}} \left|\kappa^{-1}\Big(|\widehat{V}_{B'}| * |\psi_\kappa|\Big)(-(j-\ell)\eta\kappa^{-1},(j^2-\ell^2)\eta\kappa^{-2})\right|^2\\ &\lesssim \kappa\cdot T^{1-2\gamma}\end{aligned} \tag{5.6}$$

for every η. Then, integrating in η, we get (check (5.4))

$$\int_{I^{-1}} \mathcal{I} d\eta \lesssim (T^{1-2\gamma}\kappa)\cdot \sum_{T^{\rightarrow}}\ \sum_{B\cap T^{\rightarrow}\neq\varnothing} \left(\int_{I^{-1}} \sum_{T'^{\rightarrow}}\ \sum_{\substack{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow}):\\ B\cap T^{\leftarrow}\neq\varnothing}} |F^{(0,0)}_{T^{\rightarrow},T^{\leftarrow}}(\eta)|^2 d\eta\right) \lesssim T^{4(1-\gamma)}\,. \tag{5.7}$$

3. Putting two arguments together. Now, we combine two previous arguments to treat the general case. We fix $\sigma := T^{-\nu}$ where a positive parameter ν will be selected later. Then, for every fixed $T^{\rightarrow}$ and B that intersects it, we split all $T'^{\rightarrow}$ into two groups: group $G_I(T^{\rightarrow},B)$, i.e., those that are "close" (at a distance $\leqslant \sigma T$) to B, and group $G_{II}(T^{\rightarrow},B)$, i.e., those that are "far" (at a distance more than σT).

We apply rough bound to group G_I. In contrast to (5.7), when we count the tubes $T'^{\rightarrow}$ that create a resonance with $T^{\rightarrow}$ and $T^{\leftarrow}$, we now take into account only those that are at a distance $\leqslant \sigma T$ from B and their total number is bounded by

$$\sum_{|s|<\sigma\kappa} \#(s) \overset{(3.32)}{\lessapprox} \sum_{|s|<\sigma\kappa} \kappa^{\frac12}\langle s-s_0\rangle^{-\frac12} \lessapprox \sigma^{\frac12}\kappa\,.$$

Hence, the contributions from all groups G_I in (5.7) add up to at most

$$\lessapprox \sigma^{\frac12} T^{4(1-\gamma)}\,. \tag{5.8}$$

Next, consider the contribution from the groups G_{II}. Take $T^{\rightarrow}$, B intersecting $T^{\rightarrow}$, and a tube $T'^{\rightarrow}$ at a distance $d\kappa$ to B, where the parameter d: $d\in\{\sigma\kappa,\ldots,\kappa\}$ is measuring the distance in units. We want to estimate $\sum_{T^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow},T^{\leftarrow}}|^2$ by modifying (5.3). Consider the case (see Figure 6) that provides the worst bound (the distance from B to L, the intersection point of $T^{\rightarrow}$ and $T'^{\rightarrow}$, is $\sim T$).

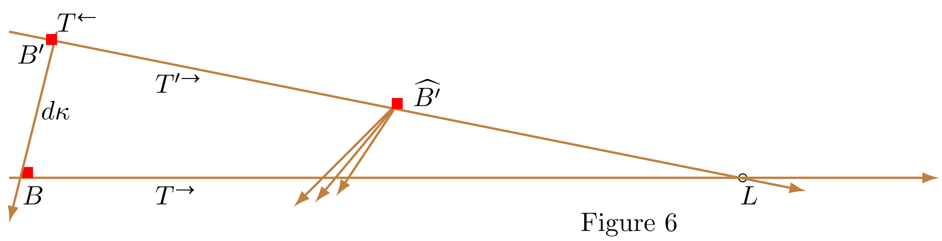

Figure 6

Take $\widehat{B'}$ intersecting $T'^{\rightarrow}$, which is at a distance $\alpha\kappa, \alpha\in\{1,\ldots,C\kappa\}$ from L. If X is a point on $T'^{\rightarrow}$ at a distance $\alpha\kappa$ from L, then $\mathrm{dist}(X,T^{\rightarrow})\sim\alpha d$. Hence, by Lemma 3.4, $\widehat{B'}$ defines $\lessapprox \kappa^2/(d\alpha)$ choices for backward tubes $\widehat{T}^{\leftarrow}$ that belong to $\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})$ and intersect $\widehat{B}'$. That gives

$$\sum_{\widehat{T}^{\leftarrow}\in\mathcal{G}(T^{\rightarrow},T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow},\widehat{T}^{\leftarrow}}|^2 \lessapprox \sum_{\alpha\lesssim\kappa} \frac{\kappa^2}{\alpha d}\cdot\frac{1}{\kappa^2} T^{2(1-\gamma-\lambda)} \lessapprox \frac{T^{2-2\gamma-2\lambda}}{d}\,.$$

Let n_d be the number of resonant tubes $T'^{\rightarrow}$ that intersect $T^{\leftarrow}$ at B' which is at a distance d units from B. The total number of $T^{\rightarrow}$'s times the total number of B's on $T^{\rightarrow}$ is bounded by $C\kappa^3$. So, the contribution from groups G_{II} is bounded by (the second factor in the formula below represents the sum of contributions over all $B' \in T^{\leftarrow}$ that are at least $\sigma\kappa$ far from B) the following quantity:

$$C\kappa^3 \cdot \left(\sum_{d=\sigma\kappa}^{\kappa} \frac{n_d T^{2-2\gamma-2\lambda}}{d}\right) \cdot \left(\kappa^{-1}T^{1-2\gamma}\right) \lesssim \frac{\kappa^3}{\sigma\kappa} \cdot \left(\sum_{s>s_0+O(R)} \#(s)\right) \cdot \left(T^{2-2\gamma-2\lambda}\right) \cdot \left(\kappa^{-1}T^{1-2\gamma}\right)$$

$$\overset{(3.33)}{\lessapprox} \frac{T^{4-4\gamma-2\lambda}}{\sigma}. \tag{5.9}$$

Combining the contributions from the first and the second groups (see (5.8) and (5.9)) and choosing $\sigma \sim T^{-\frac{4\lambda}{3}}$, we have $\int_{I^{-1}} \mathcal{I} d\eta \lessapprox T^{4(1-\gamma)-\frac{2\lambda}{3}}$, which gives

$$\int_{I^{-1}} \left| \left(\sum_{T^{\leftarrow}} |D^{(low)}_{T^{\leftarrow}}(\eta)|^2\right)^{\frac{1}{2}} \right|^4 d\eta \overset{(4.1)+(4.3)+(4.4)}{\lesssim} \int_{I^{-1}} \mathcal{I} d\eta \lessapprox T^{4(1-\gamma)-\frac{2\lambda}{3}}$$

and (5.1) holds. □

6. Bounds for the low-complexity part of V.

In that section, we control the entries in the second term in the right-hand side of (4.26). They correspond to the low-complexity part of V.

Lemma 6.1. *If V satisfies conditions of Theorem 2.2, then*

$$\left\| \left(\sum_{T^{\leftarrow}} |D^{(s)}_{T^{\leftarrow}}(\eta)|^2\right)^{\frac{1}{2}} \right\|_{L^2_{I^{-1},d\eta}} \lessapprox T^{1-\gamma-\frac{1}{64}} \tag{6.1}$$

for every $s \leqslant K$.

Proof. Fix s in (6.1) and consider a characteristic cube B with coordinates (p,q). If the frequency of $V^{(s)}_B$ is denoted as (see the representation (4.23)) $\xi^*(B) := (\xi^*_1(B), \xi^*_2(B)) = \kappa^{-1}\vec{n}^{(s)}_{p,q}$, then the formula (3.16), when written for $V^{(s)}_{p,q}$, implies that the contribution from tubes that satisfy (here, again, $|R| \lessapprox 1$)

$$|(\alpha(T^{\rightarrow}) - \alpha(T^{\leftarrow}))\eta + \xi^*_1(B)| \gtrapprox R/\kappa$$

or

$$|(\alpha(T^{\rightarrow}) + \alpha(T^{\leftarrow}))\xi^*_1(B) + \xi^*_2(B)| \gtrapprox R/\kappa$$

are negligible and we can assume that *local resonance conditions*

$$|(\alpha(T^{\rightarrow}) - \alpha(T^{\leftarrow}))\eta + \xi^*_1(B)| \lessapprox R/\kappa, \quad |(\alpha(T^{\rightarrow}) + \alpha(T^{\leftarrow}))\xi^*_1(B) + \xi^*_2(B)| \lessapprox R/\kappa \tag{6.2}$$

are satisfied. We recall that $|\alpha(T^{\rightarrow})| \leqslant C\delta$ with a small positive constant δ and $\alpha(T^{\leftarrow}) \sim 1$ so $|\alpha(T^{\leftarrow}) - \alpha(T^{\rightarrow})| \gtrsim 1$. Hence, we get $|\xi^*_1(B)| \sim 1$ and $|\xi^*_2(B)| \sim 1$. Now,

$$\alpha(T^{\rightarrow}) = -\frac{1}{2}\left(\frac{\xi^*_1(B)}{\eta} + \frac{\xi^*_2(B)}{\xi^*_1(B)}\right) + R\kappa^{-1}, \quad \alpha(T^{\leftarrow}) = \frac{1}{2}\left(\frac{\xi^*_1(B)}{\eta} - \frac{\xi^*_2(B)}{\xi^*_1(B)}\right) + R\kappa^{-1}. \tag{6.3}$$

We will say that B satisfies the *local resonance conditions* for given η if (6.3) holds. Here, we recall that $T^{\rightarrow} \cap T^{\leftarrow}$ can intersect at most one characteristic cube, check here (3.23) and the discussion after that formula. Our immediate goal is to study the geometry of such resonances. Given our assumptions, we can make (notice that the bounds below are coming from counting the local conditions only) the following observations:

I. *If B, η are known, then we have at most R choices for $T^{\leftarrow}$ and for $T^{\rightarrow}$ to create a local resonance at B.*

   That follows directly from (6.3).

II. *For fixed $T'^{\rightarrow}, T^{\rightarrow}$, each B that intersects $T^{\rightarrow}$ can define at most R resonance configurations that are both local and global simultaneously for each given η.*

   This is immediate from I.

III. *For fixed $T'^{\to}, T^{\to}$ and η, we can have at most κR tubes $T^{\leftarrow}$ that satisfy both local and global resonance conditions with these $T^{\to}$ and $T'^{\to}$.*

That is immediate from the previous claim.

IV. *Given $\eta, T^{\to}, T^{\leftarrow}$ and $B' : B' \cap T^{\leftarrow} = \varnothing$, there are at most R choices for $T'^{\to}$ that intersect B' and form global and local resonance with $T^{\to}$ and $T^{\leftarrow}$.*

That again is due to the formula

$$\alpha(T'^{\to}) = -\frac{1}{2}\left(\frac{\xi_1^*(B')}{\eta} + \frac{\xi_2^*(B')}{\xi_1^*(B')}\right) + R\kappa^{-1}\,.$$

Recall that we need to estimate

$$\mathcal{I} = \sum_{T^{\to}, T'^{\to}} \left(\sum_{T^{\leftarrow} \in \mathcal{G}(T^{\to}, T'^{\to})} |F^{(0,0)}_{T^{\to}, T^{\leftarrow}} \overline{F^{(0,0)}_{T'^{\to}, T^{\leftarrow}}}| \right)^2 \tag{6.4}$$

for generic η. In the definition of $\mathcal{I}$, the sum is symmetric in $T^{\to}$ and $T'^{\to}$ and one can write

$$\mathcal{I} \lesssim \sum_{T^{\to}, T'^{\to} : |\alpha(T^{\to})| \leqslant |\alpha(T'^{\to})|} \left(\sum_{T^{\leftarrow} \in \mathcal{G}(T^{\to}, T'^{\to})} |F^{(0,0)}_{T^{\to}, T^{\leftarrow}} \overline{F^{(0,0)}_{T'^{\to}, T^{\leftarrow}}}| \right)^2 . \tag{6.5}$$

We will need some notation. Fix $\epsilon \in (0,1)$. Given $T^{\to}$, define the set

$$R_\epsilon(T^{\to}) = \{T'^{\to} : \max_{z \in T'^{\to}} \mathrm{dist}(z, T^{\to})\} < \kappa^{2-\epsilon}\}, \quad z = (x,t)\,.$$

The tubes in $R_\epsilon(T^{\to})$ will be called $\kappa^{2-\epsilon}$-close to $T^{\to}$. We can write $\mathcal{I} = \mathcal{I}' + \mathcal{I}''$, where $\mathcal{I}''$ comes from $T'^{\to} \in R_\epsilon(T^{\to})$ and $\mathcal{I}'$ accounts for the rest.

We first consider the contribution from $\mathcal{I}''$ and recycle the argument in (5.7). Notice that the local resonance condition can be satisfied by at most R many $T'^{\to}$ that intersect $T^{\leftarrow}$ at given B' (see item IV above). So, if we denote by $\#\#(s)$ the number of $T'^{\to}$ intersecting $T^{\leftarrow}$ at B', which is characterized by a parameter $s = q(B') - q(B)$, and forming both global and local resonances, then $\#\#(s) \leqslant R$. That gives us (compare with (5.7)):

$$\int_{I^{-1}} \mathcal{I}'' d\eta \lesssim C\kappa^3 \cdot \left(\sum_{|s| \lesssim \kappa^{1-\epsilon}} \#\#(s) \right) \cdot \left(\kappa^{-1} T^{1-2\gamma}\right) \cdot \left(\kappa T^{1-2\gamma}\right) \overset{\#\#(s) \leqslant R}{\lessapprox} T^{4(1-\gamma)} \kappa^{-\epsilon}\,. \tag{6.6}$$

We only need to study $\mathcal{I}'$ and now we deal with tubes $T^{\to}$ and $T'^{\to}$ that satisfy the restrictions below:

(A) $\max_{z \in T'^{\to}} \mathrm{dist}(z, T^{\to}) > \kappa^{2-\epsilon}$,
(B) $|\alpha(T^{\to})| \leqslant |\alpha(T'^{\to})|$.

We need the following simple lemma.

Lemma 6.2. *Let $\epsilon \in (0, \frac{1}{2})$. If $T^{\to}$ and $T'^{\to}$ are not $\kappa^{2-\epsilon}$-close, $|\alpha(T'^{\to})| \geqslant |\alpha(T^{\to})|$, and they form a global resonance through some $T^{\leftarrow}$, then $|\alpha(T'^{\to}) - \alpha(T^{\to})| \geqslant C_{\epsilon_1} \kappa^{-\epsilon_1}$, $\forall \epsilon_1 \in (\epsilon, \frac{1}{2})$.*

Proof. Suppose $|\alpha(T^{\to}) - \alpha(T'^{\to})| \leqslant C\kappa^{-\epsilon_1}$ with some $\epsilon_1 > \epsilon$. Let $T^{\leftarrow}$ and $T'^{\to}$ "intersect" at B'. If $\mathrm{dist}(B', T^{\to}) \leqslant C\kappa^{2-\epsilon_1}$, then $\max_{z \in T'^{\to}} \mathrm{dist}(z, T^{\to}) \lesssim \kappa^{2-\epsilon_1}$ by a simple geometric reasoning. Since $T^{\to}$ and $T'^{\to}$ are not $\kappa^{2-\epsilon}$-close, this is not possible. So, $\mathrm{dist}(B', T^{\to}) > C_{\epsilon_1}\kappa^{2-\epsilon_1}$ with arbitrary $\epsilon_1 > \epsilon$ and $\mathrm{dist}(B, B') > C_{\epsilon_1}\kappa^{2-\epsilon_1}$ where $T^{\to}$ and $T^{\leftarrow}$ "intersect" at B. Recall that $|\alpha(T'^{\to})| \geqslant |\alpha(T^{\to})|$. Then, the analysis of quantity (3.34) shows that $s = q(B') - q(B)$ is positive and is at least $C_{\epsilon_1}\kappa^{1-\epsilon_1}$. Application of (3.36) and the trivial bound $|\alpha(T'^{\to}) - \alpha(T^{\to})| \geqslant |\alpha(T'^{\to})| - |\alpha(T^{\to})|$ finishes the proof. □

In the next argument, we will use the following observation whose proof is based on an elementary counting argument.

Lemma 6.3. *Suppose $N, M \in \mathbb{N}, N \leqslant M$ are given and the finite sets $\{E_j\}, j = 1, \ldots, M$ satisfy $\cap_{j=1}^N E_{k_j} = \varnothing$, $k_1 < \ldots < k_N$. Then, $\sum_{j=1}^M |E_j| \leqslant |\cup_{j=1}^M E_j| N$.*

For a given $T^{\to}$ and $T'^{\to}$ that satisfy (A) and (B), we define $\mathcal{P}(T^{\to}, T'^{\to}, \eta)$ as the set of all tubes $T^{\leftarrow}$ such that $T^{\to}, T'^{\to}, T^{\leftarrow}$ create both local and global resonances for a given η. We denote by

$\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ its cardinality and write $\mathcal{P}(T^{\rightarrow}, T'^{\rightarrow}, \eta) = \{T_1^{\leftarrow}, \ldots, T^{\leftarrow}_{\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta)}\}$. If B is the cube intersecting $T^{\rightarrow}$, then there are at most R elements in $\mathcal{P}(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ that intersect that B, as follows from the analysis of local resonance condition. Order such cubes $\{B_j\}$ by the coordinate $q : q(B_1) < \ldots < q(B_S)$ and hence

$$\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta)R^{-1} \lesssim S \leqslant \#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta). \tag{6.7}$$

In the case $S \geqslant 3$, we can write $\mathcal{P}(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ as a disjoint union $\mathcal{P}(T^{\rightarrow}, T'^{\rightarrow}, \eta) = \cup_{j=1}^3 \mathcal{P}_j(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ where $\mathcal{P}_1(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ corresponds to $B_1, \ldots, B_{S/3}$, i.e., the "left third" of the group of all cubes. Similarly, $\mathcal{P}_2(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ corresponds to the "middle third", and $\mathcal{P}_3(T^{\rightarrow}, T'^{\rightarrow}, \eta)$ to the "right third", e.g., to cubes $\{B_j\}$ with $j > 2S/3$ (check the Figure 7). Recall that we consider only those $T^{\rightarrow}$ and $T'^{\rightarrow}$ that satisfy (A) and (B). For a given $\upsilon \in (0, \frac{1}{2})$, we define $Long(\upsilon, \eta)$ as the set of pairs $(T^{\rightarrow}, T'^{\rightarrow})$ for which $\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta) > \kappa^{1-\upsilon}$ and $Short(\upsilon, \eta)$ denotes all other pairs. Now, $\mathcal{I}' \leqslant \mathcal{I}'_{(1)} + \mathcal{I}'_{(2)}$, where $\mathcal{I}'_{(1)}$ stands for the sum in (6.5) over $Short(\upsilon, \eta)$. Then, like in (5.2), we apply Cauchy-Schwarz:

$$\mathcal{I}'_{(1)}(\upsilon, \eta) \lesssim \sum_{Short(\upsilon,\kappa)} \left(\sum_{T^{\leftarrow} \in \mathcal{G}(T^{\rightarrow}, T'^{\rightarrow})} |F^{(0,0)}_{T^{\rightarrow}, T^{\leftarrow}}|^2 \right) \left(\sum_{T^{\leftarrow} \in \mathcal{G}(T^{\rightarrow}, T'^{\rightarrow})} |F^{(0,0)}_{T'^{\rightarrow}, T^{\leftarrow}}|^2 \right).$$

Given our assumptions, we can use (5.5) (compare with (5.6)) to get,

$$\sum_{T^{\leftarrow}} |F^{(0,0)}_{T'^{\rightarrow}, T^{\leftarrow}}(\eta)|^2 \lesssim T^{1-2\gamma}\kappa^{1-\upsilon}$$

for every η. So, applying our previous rough bound (5.7), we obtain

$$\int_{I^{-1}} \mathcal{I}'_{(1)}(\upsilon, \eta) d\eta \leqslant T^{4-4\gamma}\kappa^{-\upsilon}. \tag{6.8}$$

Consider $\mathcal{I}'_{(2)}$. Two estimates $|F^{(0,0)}_{T^{\rightarrow}, T^{\leftarrow}}(\eta)| \lesssim T^{\frac{1}{2}-\gamma}$, $|F^{(0,0)}_{T'^{\rightarrow}, T^{\leftarrow}}(\eta)| \lesssim T^{\frac{1}{2}-\gamma}$ follow from (5.5) and they hold for each η. We plug them into (6.4) to get

$$\mathcal{I}'_{(2)} \lesssim T^{2-4\gamma} \sum_{Long(\upsilon,\eta)} (\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta))^2 \overset{\text{(III)}}{\lessapprox} T^{2-4\gamma} \cdot \kappa^2 |Long(\upsilon, \eta)|, \tag{6.9}$$

where $|Long(\upsilon, \eta)|$ is the cardinality of the set $Long(\upsilon, \eta)$ which we are going to estimate now. First, fix η and denote by $L_\upsilon(\eta)$ the set of tubes $T^{\rightarrow}$ that have at least one $T'^{\rightarrow}$ so that $\#Res(T^{\rightarrow}, T'^{\rightarrow}, \eta) > \kappa^{1-\upsilon}$, i.e., a pair $(T^{\rightarrow}, T'^{\rightarrow}) \in Long(\upsilon, \eta)$. For each such tube $T^{\rightarrow}$, define $LB(T^{\rightarrow}, \eta)$ as a set of cubes intersecting $T^{\rightarrow}$ that satisfy local resonance conditions with some $T^{\leftarrow}$. By (6.7), we get $|LB(T^{\rightarrow}, \eta)| \geqslant \kappa^{1-\upsilon}R^{-1}$. Notice that every cube $B \in LB(T^{\rightarrow}, \eta)$ cannot belong to more than R other such forward tubes in $L_\upsilon(\eta)$ (that follows from (6.3), the first formula). Hence, we can apply Lemma 6.3 to write $\sum_{T^{\rightarrow} \in L_\upsilon} |LB(T^{\rightarrow}, \eta)| \lessapprox |\cup_{T^{\rightarrow} \in L_\upsilon} LB(T^{\rightarrow}, \eta)|$. Since the total number of cubes is $\sim \kappa^2$, we get a bound

$$|L_\upsilon(\eta)| \lessapprox \kappa^{1+\upsilon}. \tag{6.10}$$

Now, fix $T^{\rightarrow} \in L_\upsilon(\eta)$ and let $T_1^{\rightarrow}, \ldots, T_M^{\rightarrow}$ be all tubes such that (A) and (B) hold for each pair $T^{\rightarrow}, T_j^{\rightarrow}$ and that $\#Res(T^{\rightarrow}, T_j^{\rightarrow}, \eta) > \kappa^{1-\upsilon}$. For each j, we define Ω_j as the set of ordered pairs $(\widehat{T}^{\leftarrow}, \widetilde{T}^{\leftarrow})$ such that $\widehat{T}^{\leftarrow} \in \mathcal{P}_1(T^{\rightarrow}, T_j^{\rightarrow}, \eta), \widetilde{T}^{\leftarrow} \in \mathcal{P}_3(T^{\rightarrow}, T_j^{\rightarrow}, \eta)$ (such a pair $(\widehat{T}^{\leftarrow}, \widetilde{T}^{\leftarrow})$ is pictured in blue on Figure 7).

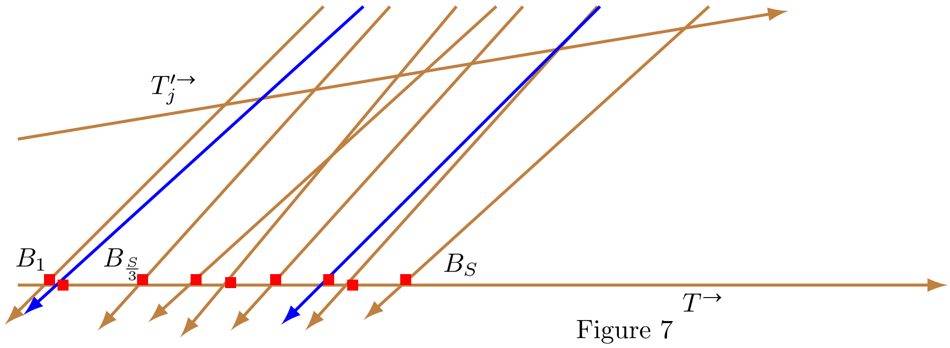

Figure 7

Notice that $|\Omega_j| \gtrsim \kappa^{2(1-\upsilon)}R^{-2}$. We assume now that $2(\upsilon+\epsilon) < 1$. Then, Lemma 3.5 and Lemma 6.2 give us $\cap_{j=1}^N \Omega_{k_j} = \varnothing$, $N = C_{\epsilon_1}\kappa^{2(\upsilon+\epsilon_1)}$, $k_1 < \ldots < k_N$ for every $\epsilon_1 : \epsilon_1 > \epsilon, 2(\upsilon+\epsilon_1) < 1$. The Lemma 6.3 provides a bound

$$\sum_{j=1}^M |\Omega_j| \leqslant |\cup_{j=1}^M \Omega_j| \cdot N$$

and, since $|\cup_{j=1}^M \Omega_j| \overset{\text{(III)}}{\lesssim} \kappa^2 R^2$, one has $M \lessapprox \kappa^{4\upsilon+2\epsilon}$. Given (6.10), we get

$$|Long(\upsilon,k)| \leqslant |L_\upsilon(\eta)| \max_{T^{\to}\in L^\upsilon} M \lessapprox \kappa^{1+5\upsilon+2\epsilon} \quad \Rightarrow \quad \mathcal{I}'_{(2)} \overset{(6.9)}{\lessapprox} T^{4(1-\gamma)}\kappa^{5\upsilon+2\epsilon-1}$$

for every $\eta \in I^{-1}$. Take $\upsilon = \epsilon = \frac{1}{8}$ and combine (6.6) with (6.8) to get

$$\int_{I^{-1}} \mathcal{I} d\eta \lessapprox T^{4(1-\gamma)-\frac{1}{16}} \quad \Rightarrow \quad (6.1)\,, \tag{6.11}$$

finishing the proof of Lemma 6.1. □

7. Proofs of the main theorems.

Now, we are ready to finish the proof of Theorem 2.2.

Proof of Theorem 2.2. Combining (4.26), (5.1), and (6.1), we get

$$\|\|Qf_o\|\|_{L^2_{I^{-1},d\eta}} \lessapprox T^{2\lambda}T^{1-\gamma-\frac{1}{64}} + T^{1-\gamma-\frac{\lambda}{6}}\,.$$

Choose $\lambda = \frac{3}{416}$ to obtain $\|\|Qf_o\|\|_{L^2_{I^{-1},d\eta}} \lessapprox T^{1-\gamma-\frac{1}{832}}$. □

Proof of Theorem 1.3. Given our assumptions, $U(0,t,k)f = U(0,t,k)f_o + o(1)$ uniformly in t and k. Take $t_j = 2\pi jT/N$ where $j \in \{0,\ldots,N\}$ and N is to be chosen later. Write $u := U(0,t,k)f_o$ and define ε_j through the formula $U(0,t_j,k)f_o = e^{ik\Delta t_j}f_o + \varepsilon_j(k)$. By the group property, $U(0,t_{j+1},k) = U(t_j,t_{j+1},k)U(0,t_j,k)$. We can use the Duhamel expansion (1.8) for the first factor to get

$$U(0,t_{j+1},k)f_o = e^{ik\Delta t_{j+1}}f_o - i\int_{t_j}^{t_{j+1}} e^{ik\Delta(t_{j+1}-\tau)}Ve^{ik\Delta\tau}f_o d\tau + U(t_j,t_{j+1},k)\varepsilon_j(k) + \Delta_j\,,$$

where

$$\Delta_j := -\int_{t_j}^{t_{j+1}} e^{ik\Delta(t_{j+1}-\tau_1)}V(\cdot,\tau_1)\int_{t_j}^{\tau_1} e^{ik\Delta(\tau_1-\tau_2)}V(\cdot,\tau_2)u(\cdot,\tau_2,k)d\tau_2 d\tau_1$$

and $\|\Delta_j\| \lesssim (T/N)^2 T^{-2\gamma}$ because $\|u(\cdot,t)\| \lesssim 1$ for all t. Hence,

$$\epsilon_{j+1}(k) = -i\int_{t_j}^{t_{j+1}} e^{ik\Delta(t_{j+1}-\tau)}Ve^{ik\Delta\tau}f_o d\tau + U(t_j,t_{j+1},k)\varepsilon_j(k) + \Delta_j\,.$$

Consider the first term in the formula above. Following (3.2), we can write

$$\chi_{t_j<t<t_{j+1}} \cdot V(x,t) = V^{(j)}(x,t) + V_{err}(x,t)\,,$$

where

$$V^{(j)} := \sum_{(p,q):2B_{p,q}\subset\mathbb{R}\times[t_j,t_{j+1}]} V_{p,q}(x-2\pi\kappa p, t-2\pi\kappa q)\,.$$

The term $V_{err}(x,t)$ is supported in t on $[t_j, t_j+C\kappa]\cup[t_{j+1}-C\kappa, t_{j+1}]$ and $\|V_{err}\|_{L^\infty(\mathbb{R}^2)}\lesssim T^{-\gamma}$. We write

$$\int_{t_j}^{t_{j+1}} e^{ik\Delta(t_{j+1}-\tau)}V^{(j)}e^{ik\Delta\tau}f_o d\tau =$$

$$\int_0^{t_{j+1}} e^{ik\Delta(t_{j+1}-\tau)}V^{(j)}e^{ik\Delta\tau}f_o d\tau = e^{-ik\Delta(T-t_{j+1})}\int_0^T e^{ik\Delta(T-\tau)}V^{(j)}e^{ik\Delta\tau}f_o d\tau$$

and apply Theorem 2.2 to the

$$\int_0^T e^{ik\Delta(T-\tau)}V^{(j)}e^{ik\Delta\tau}f_o d\tau$$

recalling that $V(x,t)=0$ for $t<C_1T$. That gives

$$\|\|\epsilon_{j+1}\|\|_{L^2_{I,dk}} \lessapprox CT^{1-\gamma-\frac{1}{832}} + \|\|\epsilon_j\|\|_{L^2_{I,dk}} + CT^{2-2\gamma}N^{-2} + CT^{\frac12-\gamma}\,.$$

Iterating N times and plugging in $\varepsilon_1=0$, we get $\|\|\epsilon_N\|\|_{L^2_{I,dk}}\lessapprox NT^{1-\gamma-\frac{1}{832}}+T^{2-2\gamma}N^{-1}+NT^{\frac12-\gamma}$. Choosing $N=T^{0.5(1-\gamma)+\frac{1}{1664}}$, one has $\|\|\epsilon_N\|\|_{L^2_{I,dk}}\lessapprox T^{1.5(1-\gamma)-\frac{1}{1664}}+T^{1-\frac32\gamma+\frac{1}{1664}}$ and the proof is finished by letting $\gamma_0=1-\frac{1}{2496}$. □

8. Creation of the resonances.

In that section, we show how the free evolution can be distorted by the potential V of a small uniform norm. In particular, we will see that the Corollary 1.2 does not hold for $\gamma<1$. That explains that the set of resonant parameters in Theorem 1.3 can indeed be nonempty.

Definition. In our perturbation analysis, we will say that the solution u in (1.1) experiences the *anomalous dynamics* for a given T-dependent initial data f and k if $\lim_{T\to\infty}\|u(x,T,k)-e^{ik\Delta T}f\|_2$ either does not exist or is not equal to zero.

Lemma 8.1 (Approximation Lemma). *Given real-valued $V(x,t)$ that satisfies $\|V(\cdot,t)\|_{L^\infty(\mathbb{R})}\lesssim T^{-\gamma}, t\in[0,T]$, we suppose N is chosen such that $N^{-1}T^{2-2\gamma}\lesssim 1$. Then,*

$$\|U(0,t_j,k)-(e^{ik\Delta d}+Q_j)\cdot\ldots\cdot(e^{ik\Delta d}+Q_1)\|\lesssim N^{-1}T^{2-2\gamma},\ j=\{1,\ldots,N\}, \tag{8.1}$$

where $t_j:=jT/N, d:=T/N$, and $Q_j:=-i\int_{t_{j-1}}^{t_j}e^{ik\Delta(t_j-\tau)}Ve^{ik\Delta(\tau-t_{j-1})}d\tau$.

Proof. If $\Delta_j:=U(0,t_j,k)-(e^{ik\Delta d}+Q_j)\cdot\ldots\cdot(e^{ik\Delta d}+Q_1)$, $R_j:=U(t_{j-1},t_j,k)-(e^{ik\Delta d}+Q_j)$, then

$$\begin{aligned}U(0,t_j,k) &= (e^{ik\Delta d}+Q_j+R_j)U(0,t_{j-1},k)\\ &= (e^{ik\Delta d}+Q_j+R_j)((e^{ik\Delta d}+Q_{j-1})\cdot\ldots\cdot(e^{ik\Delta d}+Q_1)+\Delta_{j-1})\\ &= (e^{ik\Delta d}+Q_j)\cdot\ldots\cdot(e^{ik\Delta d}+Q_1)+R_jU(0,t_{j-1},k)+(e^{ik\Delta d}+Q_j)\Delta_{j-1}\,.\end{aligned}$$

Hence, $\Delta_j=R_jU(0,t_{j-1},k)+(U(t_{j-1},t_j,k)-R_j)\Delta_{j-1}$ and we use Lemma 1.1 to get a bound $\|\Delta_j\|\leqslant(1+\alpha)\|\Delta_{j-1}\|+\alpha$, $\|\Delta_1\|\leqslant\alpha$, where $\alpha:=CT^{-2\gamma}d^2$. Then, by induction, $\|\Delta_j\|\leqslant(1+\alpha)^j-1\leqslant e^{\alpha j}-1\lesssim\alpha j$ and the last estimate holds provided that $\alpha j\lesssim 1$. Taking $j=N$, we get the statement of our lemma. □

Remark. Taking $N=T^{2-2\gamma}\mu(T), \lim_{T\to\infty}\mu(T)=+\infty$ in this lemma, we get a good approximation for the dynamics by a product of N relatively simple factors. Clearly, the same result holds if we replace $x\in\mathbb{R}$ by $x\in\mathbb{R}\backslash(CT)\mathbb{Z}$.

(A) *Creation of anomalous dynamics using a bound state.* Consider any smooth non-positive function $q(s)$ supported on $[-1,1]$ which is not equal to zero identically. The standard variational principle yields the existence of a positive bound state $\varphi(s)$ that solves $-\varphi''+\lambda q\varphi=E\varphi, E<0$ when a positive coupling constant λ is large enough. The number E is the smallest eigenvalue and φ is an exponentially decaying smooth function. We can normalize it as $\|\varphi\|_{L^2(\mathbb{R})}=1$. Hence, one gets

$$iy_\tau=-\Delta y+\lambda qy,\quad y(s,\tau)=\varphi(s)e^{-iE\tau},\quad y(s,0)=\varphi(s)\,.$$

Given T and α_T, we can rescale the variables $u(x,t):=\alpha_T^{\frac12}y(\alpha_Tx,\alpha_T^2t)$. Then,

$$iu_t=-\Delta u+\lambda\alpha_T^2q(\alpha_Tx)u,\quad \|u(x,0)\|_{L^2(\mathbb{R})}=1.$$

The initial value $u(x,0)$ is now supported around the origin at scale α_T^{-1}, and its Fourier transform is supported around the origin on scale α_T. The potential $V_2 = \lambda\alpha_T^2 q(\alpha_T x)$ satisfies $|V_2| \lesssim \lambda\alpha_T^2$. Hence, taking $\alpha_T = T^{-\frac{\gamma}{2}}$, we satisfy $|V| \lesssim \lambda T^{-\gamma}$. Nonetheless, the function $u(x,0)$ of scale $T^{\gamma/2} \ll T^{1/2}$ evolves into $u(x,T)$ which is supported around zero and has the same scale as $u(x,0)$. On the other hand, $e^{i\Delta T}u(x,0)$ is supported around the origin and has the scale $T^{1-\frac{\gamma}{2}} \gg T^{\frac{\gamma}{2}}$ if $\gamma < 1$. This is one example of anomalous transfer when the potential, small in the uniform norm, traps the wave and prevents its propagation.

(B) *Anomalous dynamics: construction of a resonance.* We now focus on another problem with time-independent potential:

$$iu_t = -\Delta u + T^{-\gamma}\cos(2x)u, \quad u(x,0) = T^{-\frac{1}{2}}(a_0e^{ix} + b_0e^{-ix}), \quad x \in \mathbb{R}/(T\mathbb{Z}), \quad T \in 2\pi\mathbb{N}, \tag{8.2}$$

$\gamma \in (\frac{1}{2}, 1)$ and $|a_0|^2 + |b_0|^2 = 1$. Consider operators Q_j from Approximation Lemma. They do not depend on j and, if we denote $Q := Q_j$, then

$$T^{-\frac{1}{2}}Q(a_0e^{ix} + b_0e^{-ix}) = -iT^{-\gamma-\frac{1}{2}}\int_0^d e^{i\Delta(d-\tau)}\cos(2x)\cdot e^{i\Delta\tau}(a_0e^{ix} + b_0e^{-ix})d\tau\,.$$

For arbitrary a and b, one can write

$$-T^{-\frac{1}{2}}Q(ae^{ix} + be^{-ix}) = iT^{-\gamma-\frac{1}{2}}\int_0^d e^{i\Delta(d-\tau)}\cos(2x)(ae^{i(x-\tau)} + be^{-i(x+\tau)})d\tau =$$
$$i\frac{a}{2}T^{-\gamma-\frac{1}{2}}\int_0^d e^{-i\tau}e^{i\Delta(d-\tau)}e^{3ix}d\tau + i\frac{b}{2}T^{-\gamma-\frac{1}{2}}\int_0^d e^{-i\tau}e^{i\Delta(d-\tau)}e^{-3ix}d\tau +$$
$$i\frac{a}{2}T^{-\gamma-\frac{1}{2}}\int_0^d e^{-i\tau}e^{i\Delta(d-\tau)}e^{-ix}d\tau + i\frac{b}{2}T^{-\gamma-\frac{1}{2}}\int_0^d e^{-i\tau}e^{i\Delta(d-\tau)}e^{ix}d\tau\,.$$

We can arrange d to make sure that $d = T/N \in 2\pi\mathbb{N}$ and $N \sim T^{2-2\gamma}$. Then,

$$T^{-\frac{1}{2}}Q(ae^{ix} + be^{-ix}) = \left(-\frac{ib}{2}T^{-\gamma-\frac{1}{2}}d\right)e^{ix} + \left(-\frac{ia}{2}T^{-\gamma-\frac{1}{2}}d\right)e^{-ix}$$

and the product in (8.1) takes the form

$$\left((e^{i\Delta d} + Q_j)\cdot\ldots\cdot(e^{i\Delta d} + Q_1)\right)(a_0e^{ix} + b_0e^{-ix}) = a_je^{ix} + b_je^{-ix}\,,$$

where

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}^j \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad \lambda = -\frac{idT^{-\gamma}}{2}\,.$$

From the Approximation Lemma, we get

$$\|u(x,dj) - T^{-\frac{1}{2}}(a_je^{ix} + b_je^{-ix})\|_2 = o(1), \quad T \to \infty$$

for $j = o(N)$. The eigenvalues of the matrix $\left(\begin{smallmatrix} 1 & \lambda \\ \lambda & 1 \end{smallmatrix}\right)$ are $z_\pm = 1 \mp i|\lambda|$ so

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = z_+^j\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{a_0 + b_0}{2} + z_-^j\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{a_0 - b_0}{2}\,.$$

Since $N \sim T^{2(1-\gamma)}$, $z_+^j = e^{\log(1-i|\lambda|)j} = e^{-i(|\lambda|j + O(|\lambda|^2j))} = e^{-i|\lambda|j}(1 + O(|\lambda|^2j))$ if $j = o(N)$. A similar calculation can be done for z_-^j. Hence, already for $j \gg NT^{\gamma-1}$, the solution u with initial data $e^{ix}T^{-\frac{1}{2}}$ will carry nontrivial L^2 norm on frequency band $\{e^{i\alpha x}, |\alpha - 1| > 0.1\}$ which indicates the anomalous dynamics. That is achieved by the creation of a resonance that changes the direction of the wave.

Remark. We notice that introduction of the parameter $k > 0$ in the form

$$iu_t = -k\Delta u + T^{-\gamma}\cos(2x)u, \quad u(x,0,k) = T^{-\frac{1}{2}}e^{ix}$$

only rescales the time and potential, and the resonance occurs for all positive k. That also indicates that δ in (1.17) must be taken sufficiently small for (1.18) to hold.

Remark. In the evolution $iu_t = -\Delta u + V(x,t)u$, the energy

$$E(t) = \int\left(|u_x|^2 + V|u|^2\right)dx$$

satisfies $E'(t) = \int V_t(x,t)|u(x,t)|^2dx$ provided that V is smooth in t. In particular, E is a conserved quantity for time-independent V. For initial data $f = e^{ix}T^{-\frac{1}{2}}$ and $V = T^{-\gamma}\cos(2x)$, we get $E = 1$. That, however, does not contradict the existence of a resonance since other functions also give rise to the same energy, e.g., $\widetilde{f} = e^{-ix}T^{-\frac{1}{2}}$. Hence, in our example above, we realize the transfer of L^2–norm along the equi-energetic set.

The resonance phenomenon described above in (8.2) where $x \in \mathbb{R}/(T\mathbb{Z})$ also takes place if we consider the same equation on $\mathbb{R}$ and the initial data is replaced by $e^{ix}T^{-\frac{1}{2}}\mu(x/T)$ where μ is a compactly supported smooth bump satisfying $\mu(x) = 1$ for $|x| < 1$. Next, we introduce parameter k and consider the problem

$$iy_t = -k\Delta y + T^{-\gamma}\cos(2x)y, \quad y(x,0,k) = T^{-\frac{1}{2}}e^{i\frac{1}{2k}x}\mu(x/T), \quad x \in \mathbb{R}\,, \tag{8.3}$$

which exhibits the resonance for $k = \frac{1}{2}$ as we just established. We recast it using the modulation scaling described in the Appendix. In particular, $\psi(x,t,k) := e^{-i(\frac{t}{4k}+\frac{x}{2k})}y(x+t,t,k)$ solves

$$i\psi_t = -k\Delta\psi + T^{-\gamma}\cos(2x+2t)\psi, \quad \psi(x,0,k) = T^{-\frac{1}{2}}\mu(x/T), \quad x \in \mathbb{R}\,. \tag{8.4}$$

The solution to problem (8.4) satisfies (1.18) and so it is non-resonant for generic k. Nevertheless, for $k = \frac{1}{2}$, it is resonant: it has no local oscillation when $t = 0$ but starts to oscillate like e^{-2ix} locally when $t \gg T^\gamma$ giving a boost to the Sobolev norms. In particular, the Corollary 1.2 does not hold for $\gamma < 1$ and the set Res of resonant parameters k in Theorem 1.3 can indeed be nonempty

9. Appendix.

Basic properties of 1d Schrödinger evolution. The following two scaling properties of Schrödinger evolution can be checked by direct inspection.

(a) Modulation. If $u(x,t,k)$ solves

$$iu_t = -k\Delta u + q(x,t)u, \quad u(x,0,k) = f(x)\,,$$

then $\psi(x,t,k) := e^{-i\frac{\beta^2}{4k}t - i\frac{\beta}{2k}x}u(x+\beta t,t,k)$ solves

$$i\psi_t = -k\Delta\psi + q(x+\beta t,t)\psi, \quad \psi(x,0,k) = e^{-i\frac{\beta}{2k}x}f(x). \tag{9.1}$$

(b) Scaling of time and space variables. If $u(x,t)$ solves

$$iu_t = -\Delta u + q(x,t)u, \quad u(x,0) = f(x)\,,$$

then $\phi(x,t) := u(\beta x, \sigma t)$ solves

$$i\phi_t = -\sigma\beta^{-2}\Delta\phi + \sigma q(\beta x,\sigma t)\phi, \quad \phi(x,0) = f(\beta x)$$

for all $\beta \neq 0$ and $\sigma > 0$.

(c) Evolution of a bump function. Suppose $\phi \in \mathbb{S}(\mathbb{R})$ and let

$$\mathcal{U}(x,t) = e^{it\Delta}\phi\,. \tag{9.2}$$

Then, $\langle x\rangle^\beta|(\partial_t^\nu\partial_x^\alpha\mathcal{U})(x,t)| < C_{\alpha,\beta,\nu}$, $\alpha,\beta,\nu \in \mathbb{Z}^+$ uniformly in $t \in [-t_0,t_0]$ with an arbitrary fixed positive t_0. That is immediate from the Fourier representation of Schrödinger evolution.

(d) Evolution of a scaled bump. Suppose $\phi \in \mathbb{S}(\mathbb{R})$ and $\kappa = \sqrt{T} \geqslant 1$. Then,

$$e^{it\Delta}\phi(x/\kappa) = \mathcal{U}(x/\kappa, t/\kappa^2) \tag{9.3}$$

and hence

$$\partial_t^\nu\partial_x^\alpha\left(e^{it\Delta}\phi(x/\kappa)\right) = \kappa^{-\alpha-2\nu}(\partial_t^\nu\partial_x^\alpha\mathcal{U})(x/\kappa,t/\kappa^2), \quad \alpha,\nu \in \mathbb{Z}^+\,.$$

That follows directly from the previous observation. For $\alpha = \nu = 0$ and $\kappa \to \infty$, the function in the right-hand side is essentially supported in the neighborhood of the tube $x \in [-\kappa,\kappa] \times [-\kappa^2,\kappa^2]$, when t is restricted to $[-T,T]$ but that localization is not exact as function $\mathcal{U}$ is not compactly supported in x. To have a sharper form of localization, we apply the following decomposition. Suppose $\phi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\phi \in [-2\pi,2\pi]$ and $1 = \sum_{\lambda\in\mathbb{Z}}\phi(x-2\pi\lambda)$. Then, we can write

$$\mathcal{U}(x,t) = \sum_{\lambda\in\mathbb{Z}}\mathcal{U}_\lambda(x,t), \quad \mathcal{U}_\lambda(x,t) := \mathcal{U}(x,t)\phi(x-2\pi\lambda) \tag{9.4}$$

and

$$\langle\lambda\rangle^{\beta}|\partial_t^{\nu}\partial_x^{\alpha}\mathcal{U}_\lambda(x,t)| \leqslant C_{\alpha,\nu,\beta}, \quad \alpha,\beta,\nu \in \mathbb{Z}^+ \tag{9.5}$$

if $t \in [-1,1]$. Hence, we can rewrite

$$e^{it\Delta}\phi(x/\kappa) = \sum_{\lambda\in\mathbb{Z}} \mathcal{U}_\lambda(x/\kappa, t/\kappa^2)\,, \tag{9.6}$$

where each $\mathcal{U}_\lambda(x/\kappa, t/\kappa^2)$ is supported on the tube $[-2\pi\kappa + 2\pi\lambda\kappa, 2\pi\kappa + 2\pi\lambda\kappa] \times [-T,T]$ when $t \in [-T,T]$. Moreover, (9.5) indicates that the contribution from $\mathcal{U}_\lambda(x/\kappa, t/\kappa^2)$ is negligible for large λ.

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