

ASYMPTOTICS OF POLYNOMIALS ORTHOGONAL ON AN INTERVAL WITH VARYING WEIGHTS

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ABSTRACT. We study strong asymptotics of orthogonal polynomials on an interval with varying weights, extending the framework of Totik's theorem on weighted orthogonal polynomials. Our results generalize this theorem in several directions. In particular, we allow the supports of the associated equilibrium measures to converge to a proper subinterval of the original interval of orthogonality or even collapse to a point. These extensions are motivated by applications to multiple orthogonality where such varying measures arise naturally.

Keywords: orthogonal polynomials, varying weights, strong asymptotics

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1. INTRODUCTION

In this paper we study strong (Szegő) asymptotics of orthogonal polynomials on an interval with respect to varying weights. Such problems arise naturally in several settings, in particular, in the theory of multiple orthogonal polynomials. Our primary motivation comes from this connection.

Recall that measures $\mu_1, \dots, \mu_d$ on the real line form an Angelesco system if $\Delta(\mu_i) \cap \Delta(\mu_k) = \emptyset$ for each $i \neq k$, where $\Delta(\mu)$ denotes the convex hull of the support of a measure μ . Given a multi-index $\vec{n} = (n_1, \dots, n_d)$, $n_i \geq 0$, the corresponding type II multiple

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orthogonal polynomial is a monic polynomial $P_{\vec{n}}$ of degree $n_1 + \dots + n_d$ characterized by orthogonality conditions

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k \in \{0, \dots, n_i - 1\}, \quad i \in \{1, \dots, d\}.$$

A classical theorem of Angelesco (see [1]) states that $P_{\vec{n}}$ has exactly n_i zeros on Δ_i . Hence, it can be written in the form $P_{\vec{n}} = P_{\vec{n},1} \cdots P_{\vec{n},d}$, where $P_{\vec{n},i}$ is a monic polynomial of degree n_i with all of its zeros contained in the interval $\Delta(\mu_i)$. This factorization transforms multiple orthogonality into an ordinary orthogonality problem with a varying weight. Indeed, it holds that

$$(1.1) \quad \int x^k P_n(x) w_n^{2n}(x) d\mu(x) = 0, \quad k \in \{0, \dots, n - 1\},$$

with $n = n_i$, $\mu = \mu_i$, $P_n = P_{\vec{n},i}$, and $w_n^{2n} = \prod_{k \neq i} P_{\vec{n},k}$ for any $i \in \{1, \dots, d\}$. Denote by V^ω the logarithmic potential of a measure ω :

$$V^\omega(z) := \int \log |z - x|^{-1} d\omega(x).$$

If ν_n is the normalized zero counting measure of $\prod_{k \neq i} P_{\vec{n},k}$, then

$$w_n(x) = \exp \left(-\frac{N}{n} V^{\nu_n}(x) \right), \quad N = \frac{1}{2} \sum_{k \neq i} n_k.$$

Hence, polynomials $P_{\vec{n},i}$ can be viewed as ordinary orthogonal polynomials with non-trivial and varying weights. In this paper, we investigate strong asymptotics of such polynomials. Then, in a follow-up paper we intend to use these results to derive strong asymptotics of the polynomials $P_{\vec{n},i}$ themselves. Throughout the text, we denote by $T_n(w_n^{2n}\mu)$ the monic polynomials satisfying (1.1), where, with a slight abuse of notation, we write $f\mu$ for the measure $f d\mu$.

The asymptotic theory of orthogonal polynomials with fixed weights is classical; see the monographs [17, 8, 16, 7, 13, 14]. Orthogonal polynomials with varying weights on a finite interval were studied in [18], see also [3] for the special case of reciprocal polynomial weights. The starting point for our work is [18, Theorem 14.4]. Before we state it, let us describe in detail all the necessary ingredients.

Given a closed interval $\Delta = [\alpha, \beta]$, we denote by $L^p(\omega_\Delta)$ the space of real-valued functions whose moduli are p -summable with respect to

$$(1.2) \quad d\omega_\Delta(x) := \frac{dx}{\pi \sqrt{(x - \alpha)(\beta - x)}},$$

the arcsine distribution on Δ . We further denote by $H^2(D_\Delta)$ the Hardy space of functions holomorphic in $D_\Delta := \mathbb{C} \setminus \Delta$ whose squared moduli admit harmonic majorants in D_Δ , see [4, Chapter 10, p.168]. This definition is conformally invariant, meaning that $f \in H^2(\mathbb{D})$, the standard Hardy space on the unit disk, if and only if $f \circ \phi_\Delta \in H^2(D_\Delta)$, where

$$(1.3) \quad \phi_\Delta(z) := \frac{2}{\beta - \alpha} \left(z - \frac{\beta + \alpha}{2} - w_\Delta(z) \right) \quad \text{and} \quad w_\Delta(z) := \sqrt{(z - \alpha)(z - \beta)}.$$

The branch in the definition of w_Δ is taken to be holomorphic off Δ and normalized so that $w_\Delta(z) = z + \mathcal{O}(1)$ as $z \rightarrow \infty$ (ϕ_Δ is simply the conformal map of D_Δ onto $\mathbb{D}$ such that $\phi_\Delta(\infty) = 0$ and $\phi_\Delta(\beta) = 1$). Every function $g \in H^2(D_\Delta)$ possesses non-tangential limits $g_\pm$ from above and below Δ that satisfy

$$g_\pm \in L^2(\omega_\Delta) \quad \text{and} \quad \log |g_\pm| \in L^1(\omega_\Delta).$$

We shall say that G is an *outer function* in $H^2(D_\Delta)$ if $G \circ \phi_\Delta^{-1}$ is an outer function in $H^2(\mathbb{D})$, see [4, Section 2.4]. For instance, given a nonnegative function $f \in L^2(\omega_\Delta)$ such

that $\log f \in L^1(\omega_\Delta)$, the function

$$(1.4) \quad \Omega_\Delta(f; z) := \exp \left(w_\Delta(z) \int_\Delta \log f(x) \frac{d\omega_\Delta(x)}{z-x} \right)$$

is an outer function in $H^2(D_\Delta)$. It also satisfies $\Omega_\Delta(f; \infty) > 0$ and is conjugate-symmetric. For such functions it holds that

$$(1.5) \quad |\Omega_{\Delta\pm}(f; x)| = f(x) \quad \text{for almost every } x \in \Delta.$$

In particular, any conjugate-symmetric outer function in $H^2(D_\Delta)$ that is positive at infinity can be recovered through the modulus of its boundary values via (1.4)–(1.5). Notice also that

$$(1.6) \quad \log \Omega_\Delta(f; \infty) = \int_\Delta \log f d\omega_\Delta.$$

Given a measure μ on the real line, we write

$$d\mu(x) = \mu'(x)dx + d\mu^s(x),$$

where $\mu'(x)$ is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on the real line and μ^s is singular to the Lebesgue measure. We denote by $|\mu|$ the total mass of μ .

We say that μ is a *Szegő measure* on $\Delta \subseteq \Delta(\mu)$, written $\mu \in \text{Sz}(\Delta)$, if $\log \mu' \in L^1(\omega_\Delta)$. In this case, $\log v_\Delta \in L^1(\omega_\Delta)$, where

$$(1.7) \quad d\mu_{|\Delta}(x) = v_\Delta(x)d\omega_\Delta(x) + d\mu_\Delta^s(x).$$

Equivalently, $v_\Delta(x) = \pi\mu'(x)\sqrt{(x-\alpha)(\beta-x)}$, $x \in \Delta$. In the classical theory of strong asymptotics, one usually assumes that $\Delta = \Delta(\mu)$ in (1.7). Our primary motivation, however, comes from multiple orthogonality, where it is essential to work with restrictions of μ to proper subintervals of its support. This is the reason for writing $\mu_{|\Delta}$ in (1.7). When $\mu \in \text{Sz}(\Delta)$, one can define the associated *Szegő function* by

$$(1.8) \quad G(\mu_{|\Delta}; z) := \Omega_\Delta(\sqrt{v_\Delta}; z), \quad z \in D_\Delta.$$

By construction, $G(\mu_{|\Delta}; \cdot)$ is an outer function in $H^2(D_\Delta)$ and its boundary values on Δ satisfy $|G_\pm(\mu_{|\Delta}; x)|^2 = v_\Delta(x)$ for almost every $x \in \Delta$. In what follows, we remove the subscript $|\Delta$ from μ in (1.8) if $\Delta = \Delta(\mu)$.

Let w be a continuous nonnegative function on Δ that is non-zero quasi-everywhere on Δ (that is, outside a set of zero logarithmic capacity). Among all probability measures ω on Δ , there exists a unique one, denoted ω_w , that minimizes the weighted energy functional

$$I_w[\omega] := \iint \log [|x-y|w(x)w(y)]^{-1} d\omega(x)d\omega(y).$$

This measure is known as *extremal measure* (or *weighted equilibrium measure/distribution*), see [18, Theorem A]. Let

$$F_w := I_w[\omega_w] + \int \log w d\omega_w.$$

It is known that

$$(1.9) \quad V^\omega(x) \begin{cases} \geq \log w(x) + F_w, & x \in \Delta, \\ \leq \log w(x) + F_w, & x \in \text{supp}(\omega_w). \end{cases}$$

We are now ready to state the theorem that serves as the starting point for our work (we provide its reduced version for the asymptotics of monic polynomials).

Theorem (Totik, [18, Theorem 14.4]). *Let $\{w_n\}$ be a sequence of continuous, nonnegative functions on $[-1, 1]$ that are non-vanishing quasi-everywhere. Assume that the corresponding extremal measures ω_{w_n} are supported on the entire interval $[-1, 1]$, are absolutely continuous, and satisfy¹*

$$(1 - x^2)^{\varkappa_L} \lesssim \omega'_{w_n}(x) \lesssim (1 - x^2)^{\varkappa_U}, \quad x \in [-1, 1],$$

for some constants $-1 < \varkappa_U \leq \varkappa_L$. Furthermore, let $\mu \in \text{Sz}([-1, 1])$ be an absolutely continuous measure. Then, the asymptotics

$$T_n(w_n^{2n}\mu)(z) = (1 + o(1)) \exp\left(n \int \log(z - x) d\omega_{w_n}(x)\right) \frac{G(\mu; \infty)}{G(\mu; z)}$$

holds locally uniformly in $D_{[-1, 1]}$.

Observe that Totik's theorem requires $\text{supp}(\omega_n) = [-1, 1]$. In this setting, relation (1.9) shows that the weight of orthogonality may equivalently be written as $e^{2nV^{\omega_{w_n}}}\mu$ since multiplying the weight by a positive constant does not change the corresponding monic orthogonal polynomial. For the applications that motivate our work, however, this result is not sufficiently general. One objective of the present paper is therefore to extend Totik's theorem in several ways.

In Theorem 3.1, we replace a single absolutely continuous measure μ with a sequence of not necessarily absolutely continuous measures $e^{h_n}\mu_n$, where functions h_n belong to a compact subset of continuous functions on $[-1, 1]$ (in fact, the theorem is formulated for a general interval Δ). Totik's proof of the above theorem relies on [18, Theorem 10.2] together with results from [3] concerning orthogonal polynomials with reciprocal polynomial weights. We follow the same general strategy. The main difference is that we replace the key results from [3] by our Theorem 2.2, which is itself derived from its analogue on the unit circle, namely Theorem 2.1.

As discussed earlier, our applications also require treating situations in which the supports of the extremal measures ω_{w_n} are proper subsets of the original interval of orthogonality. This leads to two additional extensions. In Theorem 4.3, we address the situation where $\text{supp}(\omega_{w_n})$ converge to a non-degenerate interval and in Theorem 5.1 we deal with the case where $\text{supp}(\omega_{w_n})$ collapse into a point. Theorems 4.3 and 5.1 require us to strengthen the notion of a Szegő measure. Accordingly, in Section 4 we introduce the notion of a *uniformly Szegő measure* and in Section 5 we define the class of *strongly Szegő measures with index γ* .

2. OPs WITH RECIPROCAL POLYNOMIAL WEIGHTS

Let σ be a measure on the unit circle $\mathbb{T}$. A sequence of polynomials ϕ_n , $n \geq 0$, is said to be orthonormal with respect to σ if $\deg(\phi_n) = n$ and

$$\int \phi_n(\xi) \overline{\phi_m(\xi)} d\sigma(\xi) = \delta_{nm},$$

where δ_{nm} denotes the usual Kronecker symbol. As standard, we assume that the leading coefficient of ϕ_n is positive. Given a polynomial p of degree at most n , denote by p^* its reciprocal polynomial (w.r.t. index n). That is,

$$p^*(z) = z^n \overline{p_n(1/\bar{z})}.$$

We are interested in the asymptotic behavior of the polynomials ϕ_n^* . It is known [17, Theorem 11.4.1] that the zeros of ϕ_n belong to the unit disk $\mathbb{D}$ and consequently the zeros of ϕ_n^* belong to the exterior of the closed unit disk. A celebrated result of Szegő, see [17,

¹By $f(x) \lesssim g(x)$, $x \in S$, we mean that $f(x) \leq cg(x)$, $x \in S$, for some constant $c > 0$.

Theorem 12.1.1], states that if $d\sigma = v dm$, where $dm = (2\pi)^{-1}|d\xi|$ is the normalized arclength measure on $\mathbb{T}$, and $\log v \in L^1(\mathbb{T})$, then

$$(2.1) \quad \phi_n^*(z)D(\sigma; z) = 1 + o(1) \quad \text{as } n \rightarrow \infty$$

locally uniformly in the unit disk $\mathbb{D}$, where $D(\sigma; z)$ is the Szegő function of the measure σ defined by

$$(2.2) \quad D(\sigma; z) := \exp \left(\frac{1}{2} \int \frac{\xi + z}{\xi - z} \log v(\xi) dm(\xi) \right), \quad z \notin \mathbb{T}.$$

The function $D(\sigma; z)$ is analytic in $\overline{\mathbb{C}} \setminus \mathbb{T}$. In fact, it is an outer function in both $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \mathbb{D}$, its values inside and outside of the unit disk are related via the identity

$$D^{-1}(\sigma; z) = \overline{D(\sigma; 1/\bar{z})}, \quad z \notin \mathbb{T}.$$

It has a non-tangential limit on $\mathbb{T}$ (taken within $\mathbb{D}$) that satisfies $|D(\sigma, \xi)|^2 = v(\xi)$ for almost every ξ on $\mathbb{T}$.

Verblunsky and Kolmogorov observed that (2.1) does not change if σ is of the more general form $d\sigma = v dm + d\sigma^s$, where $d\sigma^s$ is singular with respect to dm . In this case, the Szegő function of such measures is still defined by (2.2). A further extension of (2.1) was obtained in [3] (see also [15]), where σ was modified by an n -dependent reciprocal polynomial weight. Such weights arise naturally in the theory of multi-point Padé approximants, while we need them as a technical tool in the proof of Theorem 3.1. Our applications, however, require a more general formulation which we discuss below.

Accordingly, let (σ_n, g_n, W_n) be a sequence of triples consisting of a finite measure on $\mathbb{T}$, a real-valued continuous function on $\mathbb{T}$, and a monic polynomial W_n of degree n whose zeros all lie in $\mathbb{D}$. For each n , define the inner product:

$$\langle f, k \rangle_{\mathbb{T}, n} := \int f(\xi) \overline{k(\xi)} \frac{e^{g_n(\xi)} d\sigma_n(\xi)}{|W_n(\xi)|^2}.$$

We are interested in the orthonormal polynomials ϕ_n , $\deg \phi_n = n$, satisfying

$$(2.3) \quad \begin{cases} \langle \phi_n, \xi^k \rangle_{\mathbb{T}, n} = 0, & k \in \{0, 1, \dots, n-1\}, \\ \langle \phi_n, \phi_n \rangle_{\mathbb{T}, n} = 1, \end{cases}$$

and normalized to have a positive leading coefficient, i.e., $\phi_n(z) = \alpha_n z^n + \dots$, $\alpha_n > 0$ (the special case $g_n = 0$ and $\sigma_n = \sigma$ is exactly the setting considered in [15]). If we denote the zeros of W_n by $b_{n,i}$, $i \in \{1, 2, \dots, n\}$, then

$$W_n(z) = \prod_{i=1}^n (z - b_{n,i}) \quad \text{and} \quad W_n^*(z) = \prod_{i=1}^n (1 - \overline{b_{n,i}} z).$$

Recall that all $b_{n,i}$ belong to the unit disk. Write $d\sigma_n = v_n dm + d\sigma_n^s$, where the measures σ_n^s are singular to m . Assume that

(A $_{\mathbb{T}}$) *there exists a finite measure $d\sigma = v dm + d\sigma^s$, where σ^s is singular to m , such that*

$$\limsup_{n \rightarrow \infty} \int f d\sigma_n \leq \int f d\sigma;$$

for every nonnegative continuous function f on $\mathbb{T}$;

(B $_{\mathbb{T}}$) *the functions $\log v_n$ and $\log v$ belong to $L^1(\mathbb{T})$ and $\|\log v_n - \log v\|_{L^1(\mathbb{T})} \rightarrow 0$ as $n \rightarrow \infty$;*

(C $_{\mathbb{T}}$) *the real-valued functions g_n belong to $\mathcal{E}$, a fixed compact subset of $C(\mathbb{T})$, the space of continuous functions on the unit circle;*

(D $_{\mathbb{T}}$) *the zeros of W_n satisfy $\sum_{i=1}^n (1 - |b_{n,i}|) \rightarrow \infty$ as $n \rightarrow \infty$.*

It is known, see [5, Section II.2], that condition $(D_{\mathbb{T}})$ is equivalent to

$$(2.4) \quad \frac{W_n(z)}{W_n^*(z)} = o(1) \quad \text{as } n \rightarrow \infty$$

locally uniformly in $\mathbb{D}$.

Theorem 2.1. *Let ϕ_n be the orthonormal polynomials defined by (2.3), where the triples (σ_n, g_n, W_n) satisfy conditions $(A_{\mathbb{T}})$ – $(D_{\mathbb{T}})$. Then,*

$$(2.5) \quad \frac{\phi_n^*(z)}{W_n^*(z)} D_n(z) = 1 + o_{\mathcal{E}}(1) \quad \text{and} \quad \frac{\phi_n(z)}{\phi_n^*(z)} = o_{\mathcal{E}}(1) \quad \text{as } n \rightarrow \infty$$

locally uniformly in $\mathbb{D}$, where $D_n(z) := D(e^{g_n} \sigma_n; z)$. In particular,

$$(2.6) \quad \alpha_n D_n(0) = 1 + o_{\mathcal{E}}(1) \quad \text{as } n \rightarrow \infty.$$

It is well-known, see [17, Theorem 11.5], that orthogonal polynomials on $[-1, 1]$ can be expressed through associated orthogonal polynomials on the unit circle. In particular, one can carry over the Szegő asymptotics obtained for polynomials orthogonal on the unit circle to polynomials orthogonal on the interval, see [17, Theorem 12.1.2] and [15]. This is exactly what we do next. That is, we translate Theorem 2.1 into an analogous theorem about polynomials orthogonal on $[-1, 1]$. To this end, we look at triples $(\tilde{\mu}_n, h_n, \tau_n)$, where $\tilde{\mu}_n$ is a finite measure, h_n is a continuous function, and τ_n is a polynomial of degree at most $2n$ with real coefficients that does not vanish on $[-1, 1]$ and is normalized by $\tau_n(0) = 1$. For each n we define an inner product

$$\langle f, g \rangle_{[-1,1],n} := \int f(x) \overline{g(x)} \frac{e^{h_n(x)} d\tilde{\mu}_n(x)}{\tau_n(x)}.$$

We consider the corresponding orthonormal polynomials p_n characterized by $\deg p_n = n$ and

$$(2.7) \quad \begin{cases} \langle p_n, x^k \rangle_{[-1,1],n} = 0, & k \in \{0, 1, \dots, n-1\}, \\ \langle p_n, p_n \rangle_{[-1,1],n} = 1, \end{cases}$$

with the normalization that the leading coefficient γ_n is positive, i.e., $\gamma_n > 0$ where $p_n(z) = \gamma_n z^n + \dots$. The assumptions we made about the polynomials τ_n can be equivalently stated in the following way. Let $\{a_{n,1}, a_{n,2}, \dots, a_{n,2n}\}$ be a conjugate-symmetric multi-set (points $a_{n,i}$ can coincide and be either real or come in complex-conjugate pairs) such that $a_{n,i} \in \overline{\mathbb{C}} \setminus [-1, 1]$. Then,

$$\tau_n(x) = \prod_{i=1}^{2n} \left(1 - \frac{x}{a_{n,i}} \right),$$

where we interpret $x/a_{n,i}$ as 0 when $a_{n,i} = \infty$. Below, we assume the following hypothesis:

$(A_{[-1,1]})$ *there exists a finite measure μ on $[-1, 1]$ such that*

$$\limsup_{n \rightarrow \infty} \int f d\tilde{\mu}_n \leq \int f d\mu$$

for every nonnegative continuous function f on $[-1, 1]$;

$(B_{[-1,1]})$ *it holds that $\|\log \tilde{v}_n - \log v\|_{L^1(\omega_{[-1,1]})} \rightarrow 0$ as $n \rightarrow \infty$, where $\tilde{v}_n$ and v denote the Radon-Nikodym derivatives of $\tilde{\mu}_n$ and μ with respect to $\omega_{[-1,1]}$, see (1.2) and (1.7);*

$(C_{[-1,1]})$ *the functions h_n belong to $\mathcal{K}$, a fixed compact subset of $C[-1, 1]$, the space of continuous functions on $[-1, 1]$;*

$(D_{[-1,1]})$ *the zeros $\{a_{n,i}\}$ of the polynomials τ_n satisfy $\sum_{i=1}^{2n} (1 - |\phi(a_{n,i})|) \rightarrow \infty$ as $n \rightarrow \infty$, where $\phi = \phi_{[-1,1]}$ is the conformal map defined in (1.3).*

Theorem 2.2. Let p_n be the orthonormal polynomials defined in (2.7), where the triples $(\tilde{\mu}_n, h_n, \tau_n)$ satisfy conditions $(A_{[-1,1]})-(D_{[-1,1]})$. Then,

$$(2.8) \quad 2\tilde{G}_n^2(z) \frac{p_n^2(z)}{\tau_n(z)} \prod_{j=1}^{2n} \frac{\phi(z) - \phi(a_{n,j})}{1 - \overline{\phi(a_{n,j})}\phi(z)} = 1 + o_K(1) \quad \text{as } n \rightarrow \infty$$

locally uniformly in $D_{[-1,1]}$, where $\tilde{G}_n(z) = G(e^{h_n}\tilde{\mu}_n; z)$. In particular,

$$(2.9) \quad \gamma_n^2 \tilde{G}_n^2(\infty) 2^{1-2n} \prod_{i: a_{n,i} \neq \infty} (2a_{n,i} \phi(a_{n,i})) = 1 + o_K(1) \quad \text{as } n \rightarrow \infty.$$

2.1. Proof of Theorem 2.1. The proof closely follows the argument in [15], although several modifications are required to accommodate the greater generality of our assumptions. For clarity, we divide the proof into a sequence of lemmas.

We define the Caratheodory function of a measure σ by

$$F(\sigma; z) := \int \frac{\xi + z}{\xi - z} d\sigma(\xi).$$

Clearly, $\text{Re}(F(\sigma; \cdot))$ is the Poisson integral of σ and therefore is a positive harmonic function in $\mathbb{D}$. From [9, Theorem 1.2.4] we know that every function λ , which is harmonic in $\mathbb{D}$ and extends continuously to $\overline{\mathbb{D}}$, admits representation $\lambda = \text{Re}(F(\lambda m; \cdot))$ and $\lambda(0) = |\lambda m|$. For each orthonormal polynomial ϕ_n , we further introduce its companion polynomial ψ_n . This is the polynomial of degree at most n such that ψ_n^* interpolates $\phi_n^*(z)F(e^{g_n}\sigma_n; z)$ at the zeros of $zW_n(z)$. This polynomial has an explicit integral representation, see [15, Equation (3.6)], which fixes its normalization.

Lemma 2.3. Under the conditions of Theorem 2.1, we have

$$F(e^{g_n}\sigma_n; z) - \frac{\psi_n^*(z)}{\phi_n^*(z)} = o_{\mathcal{E}}(1) \quad \text{as } n \rightarrow \infty$$

locally uniformly in $\mathbb{D}$. Moreover,

$$F(e^{g_n}\sigma_n; 0) = \frac{\psi_n^*(0)}{\phi_n^*(0)}.$$

Proof. Combining formulas (3.23), (3.24), and (3.26) from [15], we get

$$\left| F(e^{g_n}\sigma_n; z) - \frac{\psi_n^*(z)}{\phi_n^*(z)} \right| \leq 2\sqrt{2} \left| z \frac{W_n(z)}{W_n^*(z)} \right| \frac{|e^{g_n}\sigma_n|}{(1-|z|)^{3/2}}, \quad z \in \mathbb{D},$$

where $|\mu|$ denotes the total mass of the measure μ . Taking $f = 1$ in $(A_{\mathbb{T}})$, we get

$$\limsup_{n \rightarrow \infty} |\sigma_n| \leq |\sigma| \quad \text{and} \quad \limsup_{n \rightarrow \infty} |e^{g_n}\sigma_n| \lesssim_{\mathcal{E}} |\sigma|,$$

where the last bound follows from $(C_{\mathbb{T}})$. Applying (2.4) finishes the proof of the lemma. $\square$

Notice that Jensen's inequality and $(B_{\mathbb{T}})$ yield that

$$\liminf_{n \rightarrow \infty} |\sigma_n| \geq \liminf_{n \rightarrow \infty} \exp \left(\int \log v_n dm \right) = \exp \left(\int \log v dm \right) > -\infty$$

and $(C_{\mathbb{T}})$ gives a lower bound

$$\liminf_{n \rightarrow \infty} |e^{g_n}\sigma_n| \gtrsim_{\mathcal{E}} \exp \left(\int \log v dm \right).$$

Let λ_n be a harmonic function in some neighborhood of $\overline{\mathbb{D}}$ defined by

$$(2.10) \quad \lambda_n(z) := \text{Re} \left(\frac{\psi_n^*(z)}{\phi_n^*(z)} \right)$$

(recall that ϕ_n^* has no zeros in $\overline{\mathbb{D}}$). It has been shown in [15, Equation (3.9)] that

$$(2.11) \quad \frac{\psi_n^*(z)}{\phi_n^*(z)} = \int \frac{\xi + z}{\xi - z} \left| \frac{W_n(\xi)}{\phi_n(\xi)} \right|^2 dm(\xi).$$

Hence, λ_n is a Poisson integral of an absolutely continuous measure with a strictly positive density and therefore is a strictly positive harmonic function on the closed unit disk. It readily follows from the second claim of Lemma 2.3 that

$$(2.12) \quad |\lambda_n m| = \lambda_n(0) = F(e^{g_n} \sigma_n; 0) = |e^{g_n} \sigma_n|.$$

Lemma 2.4. *Let $\mathcal{Z}$ be a compact set in $C(\mathbb{T})$. Under the conditions of Theorem 2.1, it holds that for any $\epsilon > 0$ there exists $N_{\mathcal{E}, \mathcal{Z}}(\epsilon)$ such that*

$$\left| \int h e^{g_n} d\sigma_n - \int h \lambda_n dm \right| \leq \epsilon$$

for all $n \geq N_{\mathcal{E}, \mathcal{Z}}(\epsilon)$ and each $h \in \mathcal{Z}$.

Proof. We use a standard approximation argument. Recall that

$$F(\sigma; z) = |\sigma| + 2 \sum_{i=1}^{\infty} z^i \int \xi^{-i} d\sigma(\xi) = |\sigma| + 2 \sum_{i=1}^{\infty} z^i \int \overline{\xi^i d\sigma(\xi)}.$$

Since locally uniform convergence of analytic functions implies convergence of their Taylor coefficients, we can use Lemma 2.3 and (2.11) to conclude that

$$\int h(e^{g_n} d\sigma_n - \lambda_n dm) = o_{\mathcal{E}, h}(1) \quad \text{as } n \rightarrow \infty$$

for every monomial $h(\xi) = \xi^i$, $i \in \mathbb{Z}$. By linearity, the same conclusion holds for every trigonometric polynomial h . Now, given $\epsilon > 0$, we can use compactness and denseness of trigonometric polynomials in $C(\mathbb{T})$ to find a finite collection of trigonometric polynomials $\{h_1, h_2, \dots, h_{K(\epsilon)}\}$ such that for each $h \in \mathcal{Z}$ there is $k \in \{1, 2, \dots, K(\epsilon)\}$ for which

$$\|h - h_k\|_{\infty} < \frac{\epsilon}{3M_{\mathcal{E}}},$$

where $M_{\mathcal{E}}$ is chosen so that $|e^{g_n} \sigma_n| \leq M_{\mathcal{E}}$ for all n . Then,

$$\left| \int (h - h_k) e^{g_n} d\sigma_n \right| \leq \frac{\epsilon}{3} \quad \text{and} \quad \left| \int (h - h_k) \lambda_n dm \right| \leq \frac{\epsilon}{3}$$

by (2.12). As just observed, for each h_k one can find a natural number $N_{\mathcal{E}, h_k}(\epsilon)$ such that

$$\left| \int h_k (e^{g_n} d\sigma_n - \lambda_n dm) \right| \leq \frac{\epsilon}{3}$$

for $n \geq N_{\mathcal{E}, h_k}(\epsilon)$. Taking $N_{\mathcal{E}, \mathcal{Z}}(\epsilon) = \max_{1 \leq k \leq K(\epsilon)} N_{\mathcal{E}, h_k}(\epsilon)$ yields the desired claim. $\square$

For the next step, we shall need the mutual entropy of two measures. Let μ and ν be two measures on $\mathbb{T}$ such that μ is absolutely continuous with respect to ν . The entropy $S(\mu|\nu)$ is defined as

$$S(\mu|\nu) = - \int \log \left(\frac{d\mu}{d\nu} \right) d\mu.$$

It is known that $S(\mu|\nu) \leq \log |\nu|$ and, see [13, Lemma 2.3.3], it holds that

$$(2.13) \quad S(\mu|\nu) = \inf_f \left(\int f d\nu - \int (1 + \log f) d\mu \right),$$

where the infimum is taken over all positive continuous functions on $\mathbb{T}$. Moreover, see [13, Example 2.3.2], if $\mu = m$ and $d\nu = \nu' dm + d\nu^s$, then

$$(2.14) \quad S(m|\nu) = - \int \log \left(\frac{dm}{d\nu} \right) dm = \int \log \nu' dm.$$

Lemma 2.5. *Under the assumptions of Theorem 2.1, for every $\epsilon > 0$, there exists $N_{\mathcal{E}}(\epsilon)$, depending only on the compact set $\mathcal{E}$, such that*

$$\int \log \lambda_n dm \leq \int (g_n + \log v_n) dm + \epsilon, \quad n \geq N_{\mathcal{E}}(\epsilon).$$

Proof. By (2.13), (2.14) and condition $(B_{\mathbb{T}})$, there exists a positive continuous function f_{ϵ} and a natural number $N_1(\epsilon)$ such that

$$\int f_{\epsilon} d\sigma - \int (1 + \log f_{\epsilon}) dm \leq S(m|\sigma) + \frac{\epsilon}{4} \leq \int \log v dm + \frac{\epsilon}{4} \leq \int \log v_n dm + \frac{\epsilon}{2}$$

for all $n \geq N_1(\epsilon)$. Applying (2.13) and (2.14) one more time, we use the inequality above to get

$$\begin{aligned} \int \log \lambda_n dm &= S(m|\lambda_n m) \leq \int (e^{-g_n} f_{\epsilon}) \lambda_n dm - \int (1 + \log(e^{-g_n} f_{\epsilon})) dm \\ (2.15) \quad &\leq \int (g_n + \log v_n) dm + \int (e^{-g_n} f_{\epsilon}) \lambda_n dm - \int f_{\epsilon} d\sigma + \frac{\epsilon}{2} \end{aligned}$$

for all $n \geq N_1(\epsilon)$. Furthermore, condition $(A_{\mathbb{T}})$ implies that there exists a natural number $N_2(\epsilon)$ such that

$$\int f_{\epsilon} d\sigma_n \leq \int f_{\epsilon} d\sigma + \frac{\epsilon}{4}$$

for all $n \geq N_2(\epsilon)$. Applying Lemma 2.4 with $\mathcal{Z} = \{e^{-g} f_{\epsilon} : g \in \mathcal{E}\}$, we conclude that there exists $N_{\mathcal{E}}(\epsilon) \geq \max\{N_1(\epsilon), N_2(\epsilon)\}$ such that

$$(2.16) \quad \int (e^{-g_n} f_{\epsilon}) \lambda_n dm \leq \int (e^{-g_n} f_{\epsilon}) e^{g_n} d\sigma_n + \frac{\epsilon}{4} \leq \int f_{\epsilon} d\sigma + \frac{\epsilon}{2}$$

for all $n \geq N_{\mathcal{E}}(\epsilon)$. Clearly, inequalities (2.15) and (2.16) yield the desired claim. $\square$

Lemma 2.6. *Under the assumptions of Theorem 2.1, asymptotic relation (2.6) holds.*

Proof. It follows directly from (2.10) and (2.11) combined with [9, Theorem 1.2.4], see also [15, Equation (3.8)], that

$$\left| \frac{W_n^*(\xi)}{\phi_n^*(\xi)} \right|^2 = \left| \frac{W_n(\xi)}{\phi_n(\xi)} \right|^2 = \lambda_n(\xi), \quad |\xi| = 1.$$

As W_n^*/ϕ_n^* is analytic and non-vanishing on the closed unit disk, the logarithm of its absolute value is harmonic there. Hence, the mean-value property gives

$$\log \alpha_n = -\log \left| \frac{W_n^*(0)}{\phi_n^*(0)} \right| = -\frac{1}{2} \int \log \lambda_n dm,$$

where we also used $\phi_n^*(0) = \alpha_n > 0$ and $W_n^*(0) = 1$. Applying Lemma 2.5 together with the definition of D_n , we obtain

$$\log \alpha_n \geq -\frac{1}{2} \int (g_n + \log v_n) dm + o_{\mathcal{E}}(1) = -\log D_n(0) + o_{\mathcal{E}}(1).$$

On the other hand, (2.3) provides

$$0 = \log \int \left| \frac{\phi_n}{W_n} \right|^2 e^{g_n} d\sigma_n \geq \log \int \left| \frac{\phi_n}{W_n} \right|^2 e^{g_n} v_n dm.$$

Applying Jensen's inequality gives

$$0 \geq \int \log \frac{e^{g_n} v_n}{\lambda_n} dm = 2 \log (\alpha_n D_n(0)).$$

Combining the two estimates finishes the proof of the lemma. $\square$

Lemma 2.7. *Under the assumptions of Theorem 2.1, it holds that*

$$\int \left| \frac{\phi_n^*}{W_n^*} D_n - 1 \right|^2 dm = o_{\mathcal{E}}(1) \quad \text{as } n \rightarrow \infty,$$

where $D_n(\xi)$ denotes the non-tangential boundary values of $D_n(z)$ on $\mathbb{T}$ taken from within $\mathbb{D}$. Consequently, the first asymptotic formula in (2.5) takes place.

Proof. Denote the integral in the statement of the lemma by I . Expanding the square gives

$$I = \int \left| \frac{\phi_n^*}{W_n^*} D_n \right|^2 dm + 1 - 2 \int \operatorname{Re} \left(\frac{\phi_n^*}{W_n^*} D_n \right) dm.$$

Recall that $|D_n(\xi)|^2 = e^{g_n(\xi)} v_n(\xi)$ for almost every $|\xi| = 1$. By the mean-value property for harmonic functions, one has

$$I = \int \left| \frac{\phi_n}{W_n} \right|^2 e^{g_n} v_n dm + 1 - 2 \operatorname{Re} \left(\frac{\phi_n^*(0)}{W_n^*(0)} D_n(0) \right) \leq 2 - 2\alpha_n D_n(0),$$

where we used that $W_n^*(0) = 1$. The first claim of the lemma now follows from (2.6). In particular, we have that the functions $1 - \phi_n^* D_n / W_n^*$ belong to the Hardy space $H^2(\mathbb{D})$. Thus, the second claim of the lemma now follows from the first one and the Cauchy integral formula for functions in $H^2(\mathbb{D})$. $\square$

Lemma 2.8. *Under the assumptions of Theorem 2.1, the second asymptotic formula in (2.5) holds.*

Proof. It follows from the first asymptotic formula in (2.5) that

$$\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{W_n^*(z)}{\phi_n^*(z)} \frac{\phi_n(z)}{W_n^*(z)} = (1 + o_{\mathcal{E}}(1)) \frac{\phi_n(z) D_n(z)}{W_n^*(z)}.$$

Thus, it is sufficient to study the behavior of $\phi_n D_n / W_n^*$ in the unit disk. Since these functions belong to $H^2(\mathbb{D})$ and $dm(\xi) = d\xi / (2\pi i \xi)$, the Cauchy integral formula gives

$$\frac{\phi_n(z) D_n(z)}{W_n^*(z)} = \int \frac{\phi_n(\xi) D_n(\xi)}{W_n^*(\xi)} \frac{dm(\xi)}{1 - z\bar{\xi}} = \int B_n(\xi) U_n(\xi) \frac{\phi_n(\xi) \overline{D_n(\xi)}}{W_n(\xi)} \frac{dm(\xi)}{1 - z\bar{\xi}},$$

where $B_n = W_n / W_n^*$ and $U_n(\xi) = D_n(\xi) / \overline{D_n(\xi)}$, $\xi \in \mathbb{T}$. Since $|B_n U_n| \equiv 1$ on $\mathbb{T}$, it follows from the Cauchy-Schwarz inequality that

$$\left| \int \left(\frac{\phi_n(\xi) \overline{D_n(\xi)}}{W_n(\xi)} - 1 \right) B_n(\xi) U_n(\xi) \frac{dm(\xi)}{1 - z\bar{\xi}} \right| \leq \frac{\|\phi_n \overline{D_n} / W_n - 1\|_{L^2(\mathbb{T})}}{\sqrt{1 - |z|^2}}.$$

Thus, we deduce from the first claim of Lemma 2.7 that

$$\frac{\phi_n(z) D_n(z)}{W_n^*(z)} = o_{\mathcal{E}}(1) + \int B_n(\xi) U_n(\xi) \frac{dm(\xi)}{1 - z\bar{\xi}} = o_{\mathcal{E}}(1) + \langle B_n, u(\cdot; g_n, v_n, z) \rangle,$$

where $o_{\mathcal{E}}(1)$ holds locally uniformly in the unit disk and

$$u(\xi; g_n, v_n, z) := \frac{1}{1 - \bar{z}\xi} \frac{\overline{D(e^{g_n}; \xi)}}{D(e^{g_n}; \xi)} \frac{\overline{D(v_n; \xi)}}{D(v_n; \xi)}.$$

It follows from (2.4) that $\langle B_n, u \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any fixed $u \in L^2(\mathbb{T})$. Indeed, this property clearly holds for each $u(\xi) = \xi^i$, $i \in \{0, 1, \dots\}$, and therefore for the whole space by the density of the monomials. As Blaschke products B_n have unit norms in $L^2(\mathbb{T})$, it also holds that $\langle B_n, u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ whenever $u_n \rightarrow u$ in $L^2(\mathbb{T})$. Thus, we would get the desired bound

$$\langle B_n, u_n(\cdot; g_n, v_n, z) \rangle = o_{\mathcal{E}}(1)$$

for $z \in K \subset \mathbb{D}$, K is an arbitrary compact in $\mathbb{D}$, if we can only show that the functions $u(\cdot; g, v_n, z)$ form a precompact family in $L^2(\mathbb{T})$ when $z \in K$ and $g \in \mathcal{E}$. Denote by $\mathcal{H}$ the Hilbert transform on $\mathbb{T}$. The functions $u(\xi; g_n, v_n, z)$ can be written as

$$(1 - \bar{z}\xi)^{-1} e^{-2i(\mathcal{H}g)(\xi)} e^{-2i(\mathcal{H} \log v_n)(\xi)}.$$

Consider the following subsets of $L^2(\mathbb{T})$:

$$\begin{cases} S_1 &= \{(1 - z\bar{\xi})^{-1} : z \in K\}, \\ S_2 &= \{e^{2i(\mathcal{H}g)(\xi)} : g \in \mathcal{E}\}, \\ S_3 &= \{e^{2i(\mathcal{H} \log v_n)(\xi)} : n \in \mathbb{N}\}. \end{cases}$$

The compactness of S_1 in $L^2(\mathbb{T})$ is trivial. Since $\mathcal{E}$ is compact in $C(\mathbb{T})$, it is also compact in $L^2(\mathbb{T})$. The operator $\mathcal{H}$ is a bounded on $L^2(\mathbb{T})$. Hence, $\mathcal{H}\mathcal{E}$ is a compact subset of real-valued functions in $L^2(\mathbb{T})$. Given two real numbers f and g , we have

$$(2.17) \quad |e^{if} - e^{ih}|^2 = 4 \sin^2((f - h)/2) \leq 4|f - h|^p, \quad p \in (0, 2].$$

Therefore, S_2 is compact in $L^2(\mathbb{T})$. Finally, since $\log v_n \rightarrow \log v$ in $L^1(\mathbb{T})$ due to our condition $(B_{\mathbb{T}})$, it holds that $\mathcal{H} \log v_n \rightarrow \mathcal{H} \log v$ in $L^p(\mathbb{T})$ for any $p \in (0, 1)$ by Kolmogorov's theorem. Using (2.17) with any such p yields that the functions $\exp(2i\mathcal{H} \log v_n)$ converge to $\exp(2i\mathcal{H} \log v)$ in $L^2(\mathbb{T})$, which shows that S_3 is precompact in $L^2(\mathbb{T})$. Since S_1, S_2 , and S_3 are bounded in $L^\infty(\mathbb{T})$, the product set $S_1 S_2 S_3$ is precompact in $L^2(\mathbb{T})$ as claimed. That completes the proof of the lemma. $\square$

2.2. Proof of Theorem 2.2. To translate conditions $(A_{[-1,1]})-(D_{[-1,1]})$ into conditions $(A_{\mathbb{T}})-(D_{\mathbb{T}})$, set $b_{2n,i} = \phi(a_{n,i})$, $i \in \{1, 2, \dots, 2n\}$. That is, $J(b_{2n,i}) = a_{n,i}$, where $J(z) = (z + z^{-1})/2$ is the Joukowski map. Then,

$$\tau_n(z) = \prod_{a_{n,i} \neq \infty} \frac{(\zeta - b_{2n,i})(1 - \zeta b_{2n,i})}{2a_{n,i} b_{2n,i} \zeta} = \frac{W_{2n}(\zeta) W_{2n}^*(\zeta)}{c_n \zeta^{2n}},$$

$z = J(\zeta)$, where $c_n = \prod_{a_{n,i} \neq \infty} 2a_{n,i} b_{2n,i}$, $W_{2n}(\zeta) = \prod_{i=1}^{2n} (\zeta - b_{2n,i})$, and we used the conjugate symmetry of the multi-set $\{b_{2n,1}, b_{2n,2}, \dots, b_{2n,2n}\}$. Clearly, conditions $(D_{[-1,1]})$ and $(D_{\mathbb{T}})$ are equivalent to each other. Define $\mathcal{E} = \{h \circ J : h \in \mathcal{K}\}$ and let $g_{2n} = h_n \circ J$. It trivially holds that condition $(C_{[-1,1]})$ implies condition $(C_{\mathbb{T}})$. Every measure μ defined on $[-1, 1]$ can be mapped to a measure σ on $\mathbb{T}$ by the formula

$$2\sigma(A) = \mu(J(A_+)) + \mu(J(A_-)),$$

where A is any subset in $\mathbb{T}$, $A_+ = A \cap \{e^{it} : t \in [0, \pi)\}$, and $A_- = A \setminus A_+$. For example, the mapping of the arcsine law $\omega_{[-1,1]}$ results in the normalized Lebesgue measure m on the circle. More generally, if one has a function v that is integrable with respect to $\omega_{[-1,1]}$, the measure $v d\omega_{[-1,1]}$ is mapped to $v dm$, where $v = v \circ J$. We use this map to define measures σ_{2n} on $\mathbb{T}$ that correspond to the measures $\tilde{\mu}_n$ on $[-1, 1]$. This gives

$$d\sigma_{2n} = v_{2n} dm + d\sigma_{2n}^s \quad \text{and} \quad d\sigma = v dm + d\sigma^s,$$

where $v_{2n} = \tilde{v}_n \circ J$ and $v = v \circ J$. Notice that condition $(B_{[-1,1]})$ implies $(B_{\mathbb{T}})$. Similarly, condition $(A_{[-1,1]})$ implies condition $(A_{\mathbb{T}})$.

Let now ϕ_{2n} be the polynomials satisfying orthogonality relations (2.3) with the above defined $(\sigma_{2n}, g_{2n}, W_{2n})$. By [17, Theorem 11.5] (see also [15, Lemma 4.13]), it holds that

$$p_n^2(z) = \frac{\phi_{2n}^*(\zeta)^2}{2c_n \zeta^{2n}} \frac{(1 + \phi_{2n}(\zeta)/\phi_{2n}^*(\zeta))^2}{1 + \phi_{2n}(0)/\phi_{2n}^*(0)} \quad \text{and} \quad \gamma_n^2 = 2^{2n-1} \frac{\alpha_{2n}^2}{c_n} \left(1 + \frac{\phi_{2n}(0)}{\phi_{2n}^*(0)}\right),$$

where $z = J(\zeta)$. Moreover, as observed in [15, Lemma 4.3], $\tilde{G}_n(z) = D_{2n}(\zeta)$. Thus, asymptotic formulae (2.8) and (2.9) follow from (2.5) and (2.6), where one needs to use the

identity

$$\frac{\tilde{G}_n^2(z)}{\tau_n(z)} \cdot \prod_{i=1}^{2n} \frac{\phi(z) - \phi(a_{n,i})}{1 - \overline{\phi(a_{n,i})}\phi(z)} = c_n \left(\frac{\zeta^n D_{2n}(\zeta)}{W_{2n}^*(\zeta)} \right)^2.$$

3. OPs WITH VARYING WEIGHTS AND SZEGŐ MEASURES

In this section we present the first generalization of Totik's theorem (Totik). To define the weight of orthogonality, we consider triples (μ_n, h_n, ω_n) , where μ_n, ω_n are measures on a closed interval $\Delta = [\alpha, \beta]$ and h_n is a continuous function on it. Assume further that

(A) *there exists a finite measure μ on Δ such that*

$$\limsup_{n \rightarrow \infty} \int f d\mu_n \leq \int f d\mu$$

for every nonnegative function $f \in C(\Delta)$;

(B) *if v_n and v denote the Radon-Nikodym derivatives of μ_n and μ with respect to the arcsine distribution ω_Δ , see (1.7), then $\|\log v_n - \log v\|_{L^1(\omega_\Delta)} \rightarrow 0$ as $n \rightarrow \infty$;*

(C) *the functions h_n belong to $\mathcal{K}$, a fixed compact subset of $C(\Delta)$;*

(D) *$d\omega_n(x) = \omega'_n(x)dx$ are probability measures whose densities ω'_n form a uniformly equicontinuous family on every compact subset of (α, β) and satisfy*

$$\begin{cases} \omega'_n(x) & \gtrsim |w_\Delta(x)|^{\varkappa_L}, & x \in [\alpha + n^{-\tau}, \beta - n^{-\tau}], \\ \omega'_n(x) & \lesssim |w_\Delta(x)|^{\varkappa_U}, & x \in (\alpha, \beta), \end{cases}$$

for some $\tau > 0$ and $\varkappa_L, \varkappa_U > -2$, see (1.3).

We recall our notation: given a nonnegative measure μ on an interval Δ and a continuous nonnegative function f on Δ , $T_n(f\mu)$ denotes the n -th monic orthogonal polynomial with respect to the measure $f\mu$.

Theorem 3.1. *Assume that the triples (μ_n, h_n, ω_n) satisfy conditions (A)–(D) and define $\theta_n := 2nV^{\omega_n} + h_n$. Then,*

$$(3.1) \quad T_n(e^{\theta_n} \mu_n)(z) = (1 + o_{\mathcal{K}}(1)) \exp \left(n \int \log(z - x) d\omega_n(x) \right) \frac{G_n(e^{h_n} \mu_n; \infty)}{G_n(e^{h_n} \mu_n; z)}$$

as $n \rightarrow \infty$ locally uniformly in $D_\Delta = \overline{\mathbb{C}} \setminus \Delta$. Moreover,

$$\int_{\Delta} T_n^2(e^{\theta_n} \mu_n)(x) e^{\theta_n(x)} d\mu_n(x) = 2(1 + o_{\mathcal{K}}(1)) G_n^2(e^{h_n} \mu_n; \infty).$$

We prove Theorem 3.1 in three steps that we organize as separate lemmas.

Lemma 3.2. *It is enough to prove Theorem 3.1 for $\Delta = [-1, 1]$ only.*

Proof. Let $l(z) = az + b$ be any linear transformation with $a > 0$ and b real. Set $\Delta^{(l)} := l^{-1}(\Delta)$ and, given a measure μ on Δ , let $\mu^{(l)}$ denote a measure on $\Delta^{(l)}$ such that $\mu^{(l)}(B) = \mu(l(B))$ for any set $B \subseteq \Delta^{(l)}$. Notice that $\omega_\Delta^{(l)} = \omega_{\Delta^{(l)}}$ and that the Radon-Nikodym derivative of $\mu^{(l)}$ with respect to the Lebesgue measure (resp. $\omega_{\Delta^{(l)}}$) is equal to $a\mu'(l(x))$ (resp. $v(l(x))$), where $\mu'(x)$ (resp. $v(x)$) is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure (resp. ω_Δ). Observe also that

$$V^\omega(l(x)) = - \int \log |l(x) - l(y)| d\omega^{(l)}(y) = -\log a + V^{\omega^{(l)}}(x)$$

for any measure ω . Set $\theta_n^{(l)} := 2nV^{\omega_n^{(l)}} + h_n^{(l)}$, $h_n^{(l)} = h \circ l$. Hence, it holds that

$$T_n(e^{\theta_n} \mu_n)(l(z)) = a^n T_n(e^{\theta_n^{(l)}} \mu_n^{(l)})(z).$$

Since $w_\Delta(l(z)) = aw_{\Delta^{(l)}}(z)$, the above considerations show that triples $(\mu_n^{(l)}, h_n^{(l)}, \omega_n^{(l)})$ satisfy conditions (A)–(D) on $\Delta^{(l)}$ if the triples (μ_n, h_n, ω_n) satisfy (A)–(D) on Δ . Finally, we get from (1.4) and (1.8) that

$$e^{n \int \log(l(z)-x) d\omega_n(x)} \frac{G(e^{h_n} \mu_n; \infty)}{G(e^{h_n} \mu_n; l(z))} = a^n e^{n \int \log(z-x) d\omega_n^{(l)}(x)} \frac{G(e^{h_n^{(l)}} \mu_n^{(l)}; \infty)}{G(e^{h_n^{(l)}} \mu_n^{(l)}; z)},$$

which finishes the proof of the lemma. $\square$

Condition (D), placed on the measures ω_n in Theorem 3.1, comes from [18, Theorem 10.2]. Under this assumption, it was shown there that there exist polynomials H_n , $\deg H_n \leq n$, that do not vanish on $[-1, 1]$ and such that the functions $\iota_n = e^{2nV^{\omega_n}} |H_n|^2$ satisfy

$$(3.2) \quad \begin{cases} 0 < \iota_n(x) \leq 1, & x \in [-1, 1], \\ \lim_{n \rightarrow \infty} \int \log \iota_n d\omega_{[-1, 1]} = 0. \end{cases}$$

We remark that [18, Theorem 10.2] was formulated on $[0, 1]$, but its results can be easily brought to $[-1, 1]$ by a linear transformation. In that theorem, we put $\gamma = 1/2$ and $u \equiv 1$, and the degree satisfies $\deg H_n = n - i_n$, where $i_n \rightarrow \infty$. The non-vanishing of H_n was claimed only on $(-1, 1)$, but it is clear from the construction, see [18, pages 58 and 75], that these polynomials also do not vanish at the endpoints. Set

$$(3.3) \quad \tau_n(z) := \frac{H_n(z) \overline{H_n(\bar{z})}}{|H_n(0)|^2}.$$

The polynomial τ_n has an even degree not exceeding $2n$ and has the following properties:

- τ_n has real coefficients, does not vanish on $[-1, 1]$, and $\tau_n(0) = 1$;
- if we denote the zeros of τ_n by $a_{n,1}, \dots, a_{n, \deg \tau_n}$, then at least a half of them, see [18, page 94], are located in $\{|\operatorname{Re} z| < 0.9, |\operatorname{Im} z| > L_n/n\}$ with $\lim_{n \rightarrow \infty} L_n = +\infty$ (this guarantees that condition $(D_{[-1, 1]})$ of Theorem 2.2 is satisfied by τ_n).

Lemma 3.3. *Under the conditions of Theorem 3.1 with $\Delta = [-1, 1]$, it holds that*

$$(3.4) \quad T_n^2(e^{\theta_n} \mu_n)(z) = (1 + o_K(1)) \frac{G^2(e^{h_n} \iota_n \mu_n; \infty)}{G^2(e^{h_n} \iota_n \mu_n; z)} \times \frac{1}{2^{2n}} \prod_{j=1}^{\deg \tau_n} (2a_{n,j} \phi(a_{n,j})) \frac{\tau_n(z)}{\phi^{2n - \deg \tau_n}(z)} \prod_{j=1}^{\deg \tau_n} \frac{1 - \overline{\phi(a_{n,j})} \phi(z)}{\phi(z) - \phi(a_{n,j})}$$

locally uniformly in $D = \mathbb{C} \setminus [-1, 1]$, where, as before, $\phi = \phi_{[-1, 1]}$, see (1.3). Moreover,

$$\int_{-1}^1 T_n^2(e^{\theta_n} \mu_n) e^{\theta_n} d\mu_n = (1 + o_K(1)) \frac{G^2(e^{h_n} \iota_n \mu_n; \infty)}{2^{2n-1}} \prod_{j=1}^{\deg \tau_n} (2\phi(a_{n,j})).$$

Proof. Since monic orthogonal polynomials do not depend on the normalization of the measure of orthogonality, we have that

$$T_n(e^{\theta_n} \mu_n) = T_n(e^{h_n} \tau_n^{-1} \tilde{\mu}_n), \quad \tilde{\mu}_n := \iota_n \mu_n.$$

Let us show that conditions $(A_{[-1, 1]})$ – $(D_{[-1, 1]})$ of Theorem 2.2 are satisfied by the triples $(\tilde{\mu}_n, h_n, \tau_n)$. We have already mentioned that the polynomials τ_n fulfill $(D_{[-1, 1]})$. Moreover, condition (C) of Theorem 3.1 is identical to condition $(C_{[-1, 1]})$ of Theorem 2.2. Since the Radon-Nikodym derivative of $\tilde{\mu}_n$ with respect to $\omega_{[-1, 1]}$ is $\iota_n v_n$, the functions ι_n obey the first line of (3.2), and

$$|\log(\iota_n v_n)(x) - \log v(x)| \leq |\log v_n(x) - \log v(x)| - \log \iota_n(x),$$

condition $(B_{[-1,1]})$ of Theorem 2.2 follows from condition (B) of Theorem 3.1 and the second line of (3.2). Finally, condition (A) of Theorem 3.1 implies condition $(A_{[-1,1]})$ of Theorem 2.2 for the same measure μ due to the upper bound in the first line of (3.2).

If $p_n(\sigma)(z) = \gamma_n z^n + \dots$ denotes the n -th orthonormal polynomial with respect to some measure σ , then we can write

$$T_n(\sigma)(z) = \gamma_n^{-1} p_n(\sigma)(z), \quad \text{where} \quad \gamma_n^{-2} = \int T_n^2(\sigma) d\sigma.$$

So, the first claim of the lemma is deduced from Theorem 2.2. To get the second one, we first observe that (3.3) implies $|H_n(0)|^2 = \prod_i a_{n,i}$. Then,

$$\int_{-1}^1 T_n^2(e^{\theta_n} \mu_n) e^{\theta_n} d\mu_n = \frac{1}{|H_n(0)|^2} \int_{-1}^1 T_n^2(e^{h_n} \tau_n^{-1} \tilde{\mu}_n) e^{h_n} \frac{d\tilde{\mu}_n}{\tau_n}$$

and we only need to apply (2.9) to the last integral. $\square$

Lemma 3.4. *Theorem 3.1 holds on $\Delta = [-1, 1]$.*

Proof. We will show that the right-hand side of (3.4) can be written in a form consistent with (3.1). We readily get from (1.4) and (1.8) that

$$G^2(e^{h_n} \iota_n \mu_n; z) G_n^{-2}(e^{h_n} \mu_n; z) = \Omega(\iota_n; z) = \Omega(e^{2nV^{\omega_n}}; z) \Omega(|H_n|^2; z),$$

where $\Omega(\cdot; \cdot)$ is used as a shorthand for $\Omega_{[-1,1]}(\cdot; \cdot)$. Let us show that

$$\begin{cases} \Omega(|H_n|^2; z) &= \tau_n(z) \phi^{\deg \tau_n}(z) \left(\prod_{j=1}^{\deg \tau_n} a_{n,j} \right) \prod_{j=1}^{\deg \tau_n} \frac{1 - \overline{\phi(a_{n,j})} \phi(z)}{\phi(z) - \phi(a_{n,j})}, \\ \Omega(e^{V^{\omega_n}}; z) &= \phi^{-1}(z) \exp \left(\int \log(z - x) d\omega_n(x) \right). \end{cases}$$

Both equalities follow from the same general principle: if f is a continuous function on $[-1, 1]$ and Ω is a holomorphic non-vanishing function in D such that $|\Omega|$ is continuous in the entire extended complex plane and $|\Omega| = f$ on $[-1, 1]$, then $\Omega = \Omega(f; \cdot)$. Continuity of $|H_n|^2$ is obvious while continuity of V^{ω_n} follows from condition (D) and properties of the logarithmic potentials. Recall that $\deg \tau_n$ is an even integer and that $2z\phi(z) \rightarrow 1$ as $z \rightarrow \infty$. Now, to complete the proof of the lemma, it remains to notice that the above explicit representations yield

$$\Omega(|H_n|^2; \infty) = \prod_{i=1}^{\deg \tau_n} (2\phi(a_{n,i}))^{-1} \quad \text{and} \quad \Omega(e^{V^{\omega_n}}; \infty) = 2. \quad \square$$

4. OPS WITH VARYING WEIGHTS AND UNIFORMLY SZEGŐ MEASURES

In this section, the weights of orthogonality come from a single measure μ (rather than a sequence of measures μ_n as in Theorem 3.1) and triples $(h_n, \kappa_n, \omega_n)$, where each h_n, κ_n are continuous functions on $\Delta(\mu)$, the convex hull of the support of μ , and ω_n is a measure whose support now can be a proper subset of $\Delta(\mu)$. The latter possibility necessitates strengthening of the notion of a Szegő measure. We shall say that a measure μ is *uniformly Szegő* on a closed interval $\Delta \subseteq \Delta(\mu)$, and denote this by $\mu \in \text{USz}(\Delta)$, if $\mu \in \text{Sz}(\Delta)$ and for any sequence of closed intervals $\{\Delta_n\}$ such that $\Delta_n \subseteq \Delta(\mu)$ and $\Delta_n \rightarrow \Delta$ as $n \rightarrow \infty$, there is n_0 such that $\mu \in \text{Sz}(\Delta_n)$ for $n \geq n_0$ and

$$(4.1) \quad \lim_{n \rightarrow \infty} \int |\log \mu'(x) - \log \mu'(l_{\Delta \rightarrow \Delta_n}(x))| d\omega_{\Delta}(x) \rightarrow 0,$$

where $l_{\Delta \rightarrow \Delta_n}$ is a linear function with a positive leading coefficient that maps Δ onto Δ_n . We adopt the term “uniformly Szegő” to emphasize that small perturbations of the endpoints of Δ result in small changes of the value of the Szegő function at infinity. In fact, this is true for the whole Szegő function locally uniformly on $\overline{\mathbb{C}} \setminus \Delta$.

Proposition 4.1. *Let μ be a compactly supported measure such that $\mu \in \text{USz}(\Delta)$ for some $\Delta \subseteq \Delta(\mu)$ and $\{h_n\}$ be a sequence of continuous functions that converges uniformly on some closed interval that contains Δ in its interior to a continuous function h . Further, let $\{\Delta_n\}$ be collections of closed intervals such that $\Delta_n \subseteq \Delta(\mu)$ and $\Delta_n \rightarrow \Delta$ as $n \rightarrow \infty$. Then, the asymptotics*

$$G(e^{h_n} \mu|_{\Delta_n}; z) = (1 + o(1))G(e^h \mu|_{\Delta}; z).$$

holds locally uniformly in D_Δ

Since the concept of uniformly Szegő measures is important to our analysis, we provide a different characterization of this class. To this end, given an integrable function θ on an interval Δ , we let

$$(I_\gamma \theta)(x) := \frac{1}{\sqrt{\pi}} \int_\gamma^x \frac{\theta(t) dt}{\sqrt{|x-t|}}, \quad x \in \Delta,$$

where $\gamma \in \Delta$ is fixed. That is a version of the so-called Riemann-Liouville fractional integral and it corresponds to the exponent $1/2$. As an integral transform, I_γ is a continuous operator from $L^1(\Delta)$ into weak- $L^2(\Delta)$, see [2, Lemma 2.13] (these are Lebesgue spaces with respect to the Lebesgue measure on Δ).

Proposition 4.2. *Let $[\alpha, \beta] = \Delta \subseteq \Delta(\mu)$. The following are equivalent:*

- (i) $\mu \in \text{USz}(\Delta)$;
- (ii) $(I_\gamma \log \mu')(x)$, $x \in \Delta(\mu)$, is continuous at β and α , where $\gamma \in (\alpha, \beta)$ is any;
- (iii) for every $\epsilon > 0$ there exists $d_\epsilon > 0$ such that

$$\left| \int_{a_\alpha}^{b_\alpha} \frac{\log \mu'(t) dt}{\sqrt{t-a_\alpha}} \right|, \left| \int_{a_\beta}^{b_\beta} \frac{\log \mu'(t) dt}{\sqrt{b_\beta-t}} \right| < \epsilon$$

when $\text{dist}(\alpha, \{a_\alpha, b_\alpha\}), \text{dist}(\beta, \{a_\beta, b_\beta\}) < d_\epsilon$.

If $\theta \in L^p(\Delta)$ for some $p > 2$, then $I_\gamma \theta$ is Hölder continuous on Δ with exponent at least $2 - 1/p$, see [6, Theorem 12]. Since μ' is an integrable function, $\log^+ \mu'$ is in $L^p(\Delta)$ for any $p > 2$. Hence, $I_\gamma \log^+ \mu'$ is necessarily Hölder continuous and therefore Proposition 4.2 could be equivalently stated with $\log \mu'$ replaced by either $|\log \mu'|$ or $\log^- \mu'$.

Below, we assume that the triples $(h_n, \kappa_n, \omega_n)$ are such that

- (1) ω_n is a probability measure supported on some interval $[\alpha_n, \beta_n] = \Delta_n \subseteq \Delta(\mu)$ and these intervals converge to an interval $\Delta = [\alpha, \beta]$ as $n \rightarrow \infty$;
- (2) We have that $d\omega_n(x) = \omega'_n(x)dx$ and the densities ω'_n form a uniformly equicontinuous family on any compact subset of (α, β) . Moreover, they satisfy the bounds

$$(4.2) \quad |w_{\Delta_n}(x)|^{\varkappa_L} \lesssim \omega'_n(x) \lesssim |w_{\Delta_n}(x)|^{\varkappa_U}, \quad x \in (\alpha_n, \beta_n),$$

for some $\varkappa_L, \varkappa_U > -2$. If $\alpha_n > \alpha(\mu)$, we additionally assume that the inequalities

$$(4.3) \quad \omega'_n(x) \lesssim |x - \alpha_n|^{\hat{\varkappa}_U}, \quad x \in (\alpha_n, \alpha_n + \delta),$$

hold for $\hat{\varkappa}_U > 0$ and n -independent positive δ . The similar assumption is made for the case when $\beta_n < \beta(\mu)$;

- (3) The functions h_n belong to $\mathcal{K}$, a fixed compact subset of $C(\Delta(\mu))$;
- (4) The functions κ_n are such that

$$\kappa_n(x) \begin{cases} \leq 0, & x \in \Delta(\mu), \\ = 0, & x \in \Delta_n, \end{cases}$$

and the bound

$$(4.4) \quad |w_{\Delta_n}(x)|^\varkappa \lesssim |\kappa_n(x)| \lesssim |w_{\Delta_n}(x)|^\varkappa, \quad x \in \Delta(\mu) \setminus \Delta_n,$$

holds for some $\varkappa > 0$.

The functions κ_n are introduced to model the behavior of the external field away from the support of the extremal measure. Recall (1.9). It shows that

$$w_n^{2n} = e^{2nV^{\omega_n}} e^{2n(\log w_n - V^{\omega_n})},$$

where ω_n is the extremal measure corresponding to the weight function w_n . The difference $\log w_n - V^{\omega_n} + \int \log w_n d\omega_n$ is non-positive on $\Delta(\mu)$ and is identically zero on $\text{supp}(\omega_n)$. The functions κ_n account for those differences.

Theorem 4.3. *Let μ be a compactly supported measure. Assume that the triples $(h_n, \kappa_n, \omega_n)$ possess the above properties (1)–(4) and that $\mu \in \text{USz}(\Delta)$. Set $\theta_n := 2n(V^{\omega_n} + \kappa_n) + h_n$. Then, we have that*

$$T_n(e^{\theta_n} \mu)(z) = (1 + o_K(1)) \exp\left(n \int \log(z - x) d\omega_n(x)\right) \frac{G(e^{h_n} \mu|_{\Delta}; \infty)}{G(e^{h_n} \mu|_{\Delta}; z)}$$

as $n \rightarrow \infty$ locally uniformly in D_{Δ} . Moreover,

$$\int_{\Delta(\mu)} T_n^2(e^{\theta_n} \mu)(x) e^{\theta_n(x)} d\mu(x) = 2(1 + o_K(1)) G^2(e^{h_n} \mu|_{\Delta}; \infty).$$

In the above two formulae, the functions $G(e^{h_n} \mu|_{\Delta}; z)$ can be replaced by $G(e^{h_n} \mu|_{\Delta_n}; z)$.

4.1. Proof of Proposition 4.1. Given a (real-valued) function u in $L^1(\omega_{\Delta})$, we set

$$(H_{\Delta} u)(z) := \log |\Omega_{\Delta}(e^u; z)|, \quad z \in D_{\Delta}.$$

Then, $H_{\Delta} u$ is a harmonic function in D_{Δ} whose non-tangential boundary values from above and below Δ exist almost everywhere and are equal to u , see (1.5). That is, $H_{\Delta} u$ is a solution of the Dirichlet problem in D_{Δ} with boundary data u . One can readily see from (1.4) and (1.8) that to prove the proposition it is enough to show that

$$(H_{\Delta_n} u_n)(z) \rightarrow (H_{\Delta} u)(z) \quad \text{as } n \rightarrow \infty$$

locally uniformly in D_{Δ} , where $u_n = \log v_n + h_n$ and $u = \log v + h$ while v_n and v are the Radon-Nikodym derivatives of $\mu|_{\Delta_n}$ and $\mu|_{\Delta}$ with respect to ω_{Δ_n} and ω_{Δ} , respectively.

Let $l_n = l_{\Delta \rightarrow \Delta_n}$ be as in (4.1). Observe that $l_n(x)$ converges to x uniformly on Δ . Then,

$$(H_{\Delta_n} u_n)(z) - (H_{\Delta} u)(z) = \int_{\Delta} \text{Re} \left(w_{\Delta_n}(z) \frac{u_n(l_n(x))}{z - l_n(x)} - w_{\Delta}(z) \frac{u(x)}{z - x} \right) d\omega_{\Delta}(x).$$

The function in parenthesis above can be rewritten as

$$\begin{aligned} (w_{\Delta_n}(z) - w_{\Delta}(z)) \frac{u(x)}{z - x} + w_{\Delta_n}(z) \frac{u_n(l_n(x)) - u(x)}{z - l_n(x)} + \\ w_{\Delta_n}(z) \frac{(l_n(x) - x)u(x)}{(z - l_n(x))(z - x)} =: (J_{n,1} + J_{n,2} + J_{n,3})(x, z). \end{aligned}$$

Observe that the functions $w_{\Delta_n} - w_{\Delta}$ converge to zero uniformly in the whole extended complex plane. Hence,

$$(4.5) \quad \int_{\Delta} J_{n,1}(x, z) d\omega_{\Delta}(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

locally uniformly in D_{Δ} . Furthermore, since

$$v_n(x) = \pi \mu'(x) \sqrt{(x - \alpha_n)(\beta_n - x)}, \quad x \in \Delta_n = [\alpha_n, \beta_n],$$

it follows that $u_n(l_n(x)) - u(x)$ is equal to

$$[h_n(l_n(x)) - h(x)] + \frac{1}{2} \log \frac{\beta_n - \alpha_n}{\beta - \alpha} + [\log \mu'(l_n(x)) - \log \mu'(x)]$$

on Δ , where $\Delta = [\alpha, \beta]$. Due to uniform convergence of h_n to h , uniform continuity of h on Δ , and (4.1), these functions converge to zero in $L^1(\omega_{\Delta})$. As functions $|w_{\Delta_n}(z)/(z -$

$l_n(x))$ are uniformly bounded for $x \in \Delta$ and z on closed subsets of D_Δ , this necessarily yields that (4.5) holds with $J_{n,1}(x, z)$ is replaced by $J_{n,2}(x, z)$. Uniform convergence to zero of $l_n(x) - x$ on Δ now guarantees that (4.5) remains valid if $J_{n,1}(x, z)$ is replaced by $J_{n,3}(x, z)$ as well. This, of course, finishes the proof of the proposition.

4.2. Proof of Proposition 4.2. Set $\theta(t) := |\log \mu'(t)|$. As mentioned right after the statement of Proposition 4.2, we can consider $(I_\gamma \theta)(x)$ instead of $(I_\gamma \log \mu')(x)$.

Lemma 4.4. *Proposition 4.2(i) implies Proposition 4.2(ii).*

Proof. We shall prove continuity at β ; the continuity at α can be proven analogously. We need to show that

$$\lim_{n \rightarrow \infty} (I_\gamma \theta)(\beta_n) = (I_\gamma \theta)(\beta)$$

for any sequence $\{\beta_n\} \subset \Delta(\mu)$ such that $\beta_n \rightarrow \beta$ as $n \rightarrow \infty$. Clearly, in the limit above we can replace γ by α . Let $l_n(t) = \alpha + \frac{\beta_n - \alpha}{\beta - \alpha}(t - \alpha)$. Observe also that

$$(I_\alpha \theta)(\beta_n) = \sqrt{\frac{\beta_n - \alpha}{\beta - \alpha}} \int_\alpha^\beta \frac{\theta(l_n(t))}{\sqrt{\beta - t}} dt.$$

The claim of the lemma now follows from (4.1) and the estimate

$$\begin{aligned} \left| \int_\alpha^\beta \frac{\theta(t)}{\sqrt{\beta - t}} dt - \int_\alpha^\beta \frac{\theta(l_n(t))}{\sqrt{\beta - t}} dt \right| &\leq \pi \sqrt{\beta - \alpha} \int_\alpha^\beta |\theta(t) - \theta(l_n(t))| d\omega_\Delta(t) \\ &\leq \pi \sqrt{\beta - \alpha} \int_\alpha^\beta |\log \mu'(t) - \log \mu'(l_n(t))| d\omega_\Delta(t). \quad \square \end{aligned}$$

Lemma 4.5. *Proposition 4.2(iii) implies Proposition 4.2(i).*

Proof. Let $\{\Delta_n\}$ be a sequence of closed subintervals of $\Delta(\mu)$ that converges to Δ . Pick $\epsilon > 0$ and let δ be some positive number to be specified later. Then,

$$\begin{aligned} \int_\alpha^\beta |\log \mu'(t) - \log \mu'(l_n(t))| d\omega_\Delta(t) &\leq \int_{\alpha+\delta}^{\beta-\delta} |\log \mu'(t) - \log \mu'(l_n(t))| d\omega_\Delta(t) + \\ &\quad \int_{\beta-\delta}^\beta (\theta(t) + \theta(l_n(t))) d\omega_\Delta(t) + \int_\alpha^{\alpha+\delta} (\theta(t) + \theta(l_n(t))) d\omega_\Delta(t), \end{aligned}$$

where $l_n(t)$ is the linear transformation with the positive leading coefficient that maps Δ onto Δ_n . If α_n and β_n are the endpoints of Δ_n , i.e., $\Delta_n = [\alpha_n, \beta_n]$, then

$$\int_{\beta-\delta}^\beta \theta(l_n(t)) d\omega_\Delta(t) = \int_{\beta_n-\delta_n}^{\beta_n} \theta(t) d\omega_{\Delta_n}(t),$$

where $\delta_n = \frac{\beta_n - \alpha_n}{\beta - \alpha} \delta$, and a similar equality holds for the integral of $\theta(l_n(t))$ on $[\alpha, \alpha + \delta]$. Notice that $\lim_{n \rightarrow \infty} \delta_n = \delta$. Now, it becomes clear that the assumption (iii) of Proposition 4.2 implies that there exists $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned} \int_{\beta-\delta}^\beta \theta(t) d\omega_\Delta(t) &< \frac{\epsilon}{5}, \quad \int_\alpha^{\alpha+\delta} \theta(t) d\omega_\Delta(t) < \frac{\epsilon}{5}, \\ \int_{\beta-\delta}^\beta \theta(l_n(t)) d\omega_\Delta(t) &< \frac{\epsilon}{5}, \quad \int_\alpha^{\alpha+\delta} \theta(l_n(t)) d\omega_\Delta(t) < \frac{\epsilon}{5} \end{aligned}$$

for all $n \geq N$. To see that the first integral in (4.6) also can be made smaller than $\epsilon/5$ for all large enough n , observe that $d\omega_\Delta(t) \leq (\pi\delta)^{-1} dt$ on the interval of integration and that

$$\lim_{n \rightarrow \infty} \int_{\alpha+\delta}^{\beta-\delta} |\log \mu'(t) - \log \mu'(l_n(t))| dt = 0.$$

Indeed, this limit is obviously zero if we replace $\log \mu'$ by a continuous function. The general case follows by approximating $\log \mu'$ in $L^1(\Delta(\mu))$ -norm with continuous functions. $\square$

Lemma 4.6. *Proposition 4.2(ii) implies Proposition 4.2(iii).*

Proof. Since $(I_\gamma \theta)(x)$ is continuous at β , there exists an interval, say $[a, b]$, that contains β in its interior (unless β is the right endpoint of $\Delta(\mu)$, in which case $b = \beta$), on which $(I_\gamma \theta)(x)$ is bounded. It is known [12, Theorem 2.1] that

$$(4.7) \quad \theta(x) = \frac{d}{dx} \left(\frac{1}{\sqrt{\pi}} \int_\gamma^x \frac{(I_\gamma \theta)(t)}{\sqrt{x-t}} dt \right) = \frac{d}{dx} (I_\gamma (I_\gamma \theta))(x), \quad x \in (\gamma, b).$$

Let us write $(I_\gamma^2 \theta)(x)$ for $(I_\gamma (I_\gamma \theta))(x)$, which is an absolutely continuous function on $[\gamma, b]$ that vanishes at γ , see again [12, Theorem 2.1]. Fix some $\delta \in (0, (b-a)/2)$. Notice that

$$\begin{aligned} \frac{d}{dx} \left(\int_\gamma^{x-\delta} \frac{(I_\gamma^2 \theta)(t)}{\sqrt{x-t}} dt \right) &= \frac{d}{dx} \left(\int_\delta^{x-\gamma} \frac{(I_\gamma^2 \theta)(x-s)}{\sqrt{s}} ds \right) = \\ &= \lim_{h \rightarrow 0} \int_\delta^{x-\gamma} \frac{(I_\gamma^2 \theta)(x+h-s) - (I_\gamma^2 \theta)(x-s)}{h} \frac{ds}{\sqrt{s}} + \lim_{h \rightarrow 0} \frac{1}{h} \int_\gamma^{\gamma+h} \frac{(I_\gamma^2 \theta)(t)}{\sqrt{x+h-t}} dt. \end{aligned}$$

The second limit is equal to zero due to continuity of the integrand and vanishing of $(I_\gamma^2 \theta)(t)$ at γ . It is known, see [10, Theorem 6.9], that absolute continuity of a function is equivalent to uniform integrability of its divided differences. As $(I_\gamma^2 \theta)(x)$ is absolutely continuous and $1/\sqrt{s}$ is continuous on $[\delta, b-\gamma]$, we get from Vitali's convergence theorem and (4.7) that the first limit is equal to

$$\int_\delta^{x-\gamma} \frac{\theta(x-s)}{\sqrt{s}} ds$$

and hence

$$(4.8) \quad \frac{d}{dx} \left(\int_\gamma^{x-\delta} \frac{(I_\gamma^2 \theta)(t)}{\sqrt{x-t}} dt \right) = \int_\delta^{x-\gamma} \frac{\theta(x-s)}{\sqrt{s}} ds = \int_\gamma^{x-\delta} \frac{\theta(t)}{\sqrt{x-t}} dt.$$

Writing the outer I_γ transform explicitly and changing the order of integration gives us

$$(4.9) \quad \begin{aligned} \int_\gamma^{x-\delta} \frac{(I_\gamma^2 \theta)(t)}{\sqrt{x-t}} dt &= \frac{1}{\sqrt{\pi}} \int_\gamma^{x-\delta} \left(\int_s^{x-\delta} \frac{dt}{\sqrt{(t-s)(x-t)}} \right) (I_\gamma \theta)(s) ds = \\ &= \int_\gamma^{x-\delta} F \left(1 - \frac{\delta}{x-s} \right) (I_\gamma \theta)(s) ds, \quad F(s) := \frac{1}{\sqrt{\pi}} \int_0^s \frac{dt}{\sqrt{t(1-t)}}. \end{aligned}$$

Again, we need to justify changing the order of differentiation and integration. To this end, assume now that $x \in (a+2\delta, b)$. By the mean-value theorem and its very definition, the derivative of the last integral in (4.9) is equal to the limit as $h \rightarrow 0$ of the following sum of three terms

$$(4.10) \quad \begin{aligned} &\delta \int_\gamma^a F' \left(1 - \frac{\delta}{x+\xi_h-s} \right) \frac{(I_\gamma \theta)(s)}{(x-s)^2} ds + \\ &\quad \frac{1}{h} \int_a^{x-\delta} \left(F \left(1 - \frac{\delta}{x+h-s} \right) - F \left(1 - \frac{\delta}{x-s} \right) \right) (I_\gamma \theta)(s) ds + \\ &\quad \quad \quad \frac{1}{h} \int_{x-\delta}^{x+h-\delta} F \left(1 - \frac{\delta}{x+h-s} \right) (I_\gamma \theta)(s) ds, \end{aligned}$$

where $\xi_h = \xi_h(x, s)$ is such that $|\xi_h| \leq |h|$. The first term in the sum above converges to

$$\sqrt{\frac{\delta}{\pi}} \int_\gamma^a \frac{(I_\gamma \theta)(s) ds}{(x-s)\sqrt{x-\delta-s}},$$

when $h \rightarrow 0$. This follows from the dominated convergence theorem because $(I_\gamma \theta)(s)$ is a fixed integrable function and the other factor in the integrand is a function continuous in s that converges uniformly when $h \rightarrow 0$. The second term in (4.10) has the following limit

$$\delta \int_a^{x-\delta} F' \left(1 - \frac{\delta}{x-s} \right) \frac{(I_\gamma \theta)(s)}{(x-s)^2} ds = \sqrt{\frac{\delta}{\pi}} \int_a^{x-\delta} \frac{(I_\gamma \theta)(s) ds}{(x-s)\sqrt{x-\delta-s}}$$

as $h \rightarrow 0$ according to Vitali's convergence theorem. Indeed, $(I_\gamma \theta)(s)$ is bounded on the interval of integration by assumptions of Proposition 4.2(ii) and the divided differences of $F(1 - \delta(x-s)^{-1})$ are uniformly integrable. The last term in (4.10) can be rewritten as

$$\frac{\delta}{h} \int_0^{h/(h+\delta)} (I_\gamma \theta) \left(x + h - \frac{\delta}{1-t} \right) \frac{F(t)}{(1-t)^2} dt.$$

Its limit as $h \rightarrow 0$ is equal to 0 due to the boundedness of $(I_\gamma \theta)(x)$ on (a, b) as well as the continuity of the function $F(t)/(1-t)^2$ around the origin and its vanishing at $t = 0$. Altogether, we get from (4.8), (4.9), and the reasoning above that

$$\int_\gamma^{x-\delta} \frac{\theta(t) dt}{\sqrt{x-t}} = \sqrt{\frac{\delta}{\pi}} \int_\gamma^{x-\delta} \frac{(I_\gamma \theta)(s) ds}{(x-s)\sqrt{x-\delta-s}} = \frac{2}{\sqrt{\pi}} \int_0^{L_{x,\delta}} (I_\gamma \theta)(x - \delta - \delta t^2) \frac{dt}{1+t^2},$$

where $L_{x,\delta} = \sqrt{(x-\delta-\gamma)/\delta}$. We can use the identity $\int_0^\infty (t^2+1)^{-1} dt = \pi/2$ to rewrite

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{x-\delta}^x \frac{\theta(t) dt}{\sqrt{x-t}} &= (I_\gamma \theta)(x) - \frac{1}{\sqrt{\pi}} \int_\gamma^{x-\delta} \frac{\theta(t) dt}{\sqrt{x-t}} = \frac{2}{\pi} \left(\int_{\ell_\delta/\sqrt{\delta}}^\infty \frac{(I_\gamma \theta)(x) dt}{1+t^2} \right. \\ &\quad \left. - \int_{\ell_\delta/\sqrt{\delta}}^{L_{x,\delta}} \frac{(I_\gamma \theta)(x - \delta - \delta t^2) dt}{1+t^2} + \int_0^{\ell_\delta/\sqrt{\delta}} \frac{(I_\gamma \theta)(x) - (I_\gamma \theta)(x - \delta - \delta t^2)}{1+t^2} dt \right) \end{aligned}$$

for any positive ℓ_δ . If we choose $\ell_\delta = \delta^{\frac{1}{8}}$, we get that

$$\left| \int_{\ell_\delta/\sqrt{\delta}}^\infty \frac{(I_\gamma \theta)(x) dt}{1+t^2} \right| \leq |(I_\gamma \theta)(x)| \frac{\sqrt{\delta}}{\ell_\delta} = |(I_\gamma \theta)(x)| \delta^{\frac{3}{8}}.$$

Similarly,

$$\begin{aligned} \left| \int_{\ell_\delta/\sqrt{\delta}}^{L_{x,\delta}} \frac{(I_\gamma \theta)(x - \delta - \delta t^2) dt}{1+t^2} \right| &\leq \frac{\sqrt{\delta}}{\ell_\delta^2} \int_\gamma^{x-\delta-\ell_\delta^2} \frac{|(I_\gamma \theta)(s)| ds}{\sqrt{x-\delta-s}} \\ &\leq \frac{\sqrt{\delta}}{\ell_\delta^3} \int_\gamma^b |(I_\gamma \theta)(s)| ds = \delta^{\frac{1}{8}} \int_\gamma^b |(I_\gamma \theta)(s)| ds \lesssim \delta^{\frac{1}{8}}. \end{aligned}$$

Next, we have that

$$\left| \int_0^{\ell_\delta/\sqrt{\delta}} \frac{(I_\gamma \theta)(x) - (I_\gamma \theta)(x - \delta - \delta t^2)}{1+t^2} dt \right| \leq \frac{\pi}{2} \max_{s \in [x-\delta-\delta^{\frac{1}{4}}, x-\delta]} |(I_\gamma \theta)(x) - (I_\gamma \theta)(s)|.$$

Now, using continuity of $(I_\gamma \theta)(x)$ at β , the above estimates show that given $\epsilon > 0$, we can always find d_ϵ so that $|x - \beta| < d_\epsilon$ and $|x - \delta - \beta| < d_\epsilon$ imply

$$\int_{x-\delta}^x \frac{\theta(t) dt}{\sqrt{x-t}} < \epsilon.$$

This is precisely the statement of the second part in Proposition 4.2(iii). The first one is proved similarly. $\square$

An example. The uniform Szegő condition is subtle and depends on the direction in which the point is approached, as shown by the following example. Given $\epsilon \in [0, 1]$, let μ_ϵ be an absolutely continuous measure on $[-1, 1]$ such that $\log \mu'_\epsilon(x) = -\theta_\epsilon(x)$, where

$$\theta_\epsilon(x) := \begin{cases} 0, & x \in [-1, 0], \\ x^{-1/2}(1 - \log x)^{-\epsilon}, & x \in (0, 1]. \end{cases}$$

When $\epsilon = 0$, it holds that $\mu_0 \in \text{Sz}([-1, a])$ for any $a \in (-1, 1]$. Indeed, the claim is obvious when $a \leq 0$. When $a > 0$, it holds that

$$\frac{1}{\pi} \int_{-1}^a \frac{\theta_0(x) dx}{\sqrt{(x+1)(a-x)}} = \frac{1}{\pi} \int_0^a \frac{dx}{\sqrt{(x+1)x(a-x)}} \leq \frac{1}{\pi} \int_0^a \frac{dx}{\sqrt{x(a-x)}} = 1.$$

This computation also shows that

$$\frac{1}{\pi} \int_{-1}^\delta \frac{\theta_0(x) dx}{\sqrt{\delta-x}} = \begin{cases} 0, & \delta \leq 0, \\ 1, & \delta > 0, \end{cases}$$

and therefore $\mu_0 \notin \text{USz}([-1, 0])$ as follows from Proposition 4.2(ii). On the other hand, when $\epsilon > 0$, it holds that

$$0 \leq \frac{1}{\pi} \int_{-1}^\delta \frac{\theta_\epsilon(x) dx}{\sqrt{\delta-x}} \leq \frac{1}{(1 - \log \delta)^\epsilon} \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+,$$

and therefore $\mu_\epsilon \in \text{USz}([-1, 0])$ again by Proposition 4.2(ii) (in fact, $\mu_\epsilon \in \text{USz}([-1, a])$ for any $a \in (-1, 1]$ in this case). However, when $\epsilon \leq 1$, we have that $\mu_\epsilon \notin \text{Sz}([0, 1])$ since

$$\frac{1}{\pi} \int_0^1 \frac{\theta_\epsilon(x) dx}{\sqrt{x(1-x)}} \geq \frac{1}{\pi} \int_0^1 \frac{dx}{x(1 - \log x)^\epsilon} = \frac{1}{\pi} \int_1^\infty \frac{du}{u^\epsilon} = \infty.$$

4.3. Proof of Theorem 4.3. We prove Theorem 4.3 in four steps organized as separate lemmas. We write $\Delta(\mu) = [\alpha(\mu), \beta(\mu)]$, $\Delta_n = [\alpha_n, \beta_n]$, and $\Delta = [\alpha, \beta]$.

Let $\varkappa$ be a parameter in (4.4) and $\{\delta_n\}$ be a sequence of positive numbers given by

$$(4.11) \quad \delta_n = (\xi_n/n)^{2/(\varkappa+1)},$$

where $\xi_n \rightarrow 0$ as $n \rightarrow \infty$ (this sequence will be specified later in the formula (4.25)). Set

$$\Delta_n^* := [\alpha_n^*, \beta_n^*], \quad \alpha_n^* := \max\{\alpha_n - \delta_n, \alpha(\mu)\}, \quad \beta_n^* := \min\{\beta_n + \delta_n, \beta(\mu)\}.$$

Our strategy consists in first applying Theorem 3.1 to obtain the asymptotics of polynomials P_n^* orthogonal with respect to the same measures as T_n from Theorem 4.3 but restricted to Δ_n^* . Then, we show that asymptotics of T_n coincides with that of P_n^* .

Lemma 4.7. *Recall that $\theta_n = 2n(V^{\omega_n} + \kappa_n) + h_n$. It holds that*

$$P_n^*(z) := T_n \left(e^{\theta_n} \mu_{|\Delta_n^*} \right) (z) = (1 + o_K(1)) F_n^*(z)$$

locally uniformly in D_Δ , where

$$(4.12) \quad F_n^*(z) := \exp \left(n \int \log(z-x) d\omega_n(x) \right) \frac{G(e^{h_n} \mu_{|\Delta_n^*}; \infty)}{G(e^{h_n} \mu_{|\Delta_n^*}; z)}.$$

Proof. To apply Theorem 3.1, we need to rescale intervals Δ_n^* to the fixed limit interval Δ . To this end, let $l_n(x) = a_n x + b_n$, $a_n > 0$, be the linear function that maps Δ onto Δ_n^* . Since $\Delta_n^* \rightarrow \Delta$, we have $a_n \rightarrow 1$ and $b_n \rightarrow 0$ as $n \rightarrow \infty$. In the notation of Lemma 3.2, we denote the rescaled equilibrium measure by $\tilde{\omega}_n := \omega_n^{(l_n)}$. Then, one has

$$\text{supp}(\tilde{\omega}_n) = l_n^{-1}(\Delta_n) \subseteq \Delta \quad \text{and} \quad d\tilde{\omega}_n(x) = \tilde{\omega}_n'(x) dx = a_n \omega_n'(l_n(x)) dx.$$

Write $l_n^{-1}(\Delta_n) = [\tilde{\alpha}_n, \tilde{\beta}_n]$. Since $a_n \rightarrow 1$ as $n \rightarrow \infty$, the upper bound in (4.2) gives

$$\tilde{\omega}_n'(x) \lesssim (x - \tilde{\alpha}_n)^{\varkappa_U/2} (\tilde{\beta}_n - x)^{\varkappa_V/2}, \quad x \in (\tilde{\alpha}_n, \tilde{\beta}_n).$$

If $\tilde{\alpha}_n > \alpha$ for some index n , then $\alpha_n > \alpha_n^* \geq \alpha(\mu)$ and $(x - \tilde{\alpha}_n)^{\varkappa_U/2}$ can be replaced by $(x - \tilde{\alpha}_n)^{\hat{\varkappa}_U}$ in the above estimate as required by (4.3), where $\hat{\varkappa}_U > 0$. Similarly, if $\tilde{\beta}_n < \beta$



FIGURE 1. Segments $\Delta \subset \Delta_n \subset \Delta_n^* \subset \Delta(\mu)$ and $l_n^{-1}(\Delta_n) \subset \Delta = l_n^{-1}(\Delta_n^*)$.

for some n , then $(\tilde{\beta}_n - x)^{\varkappa_U/2}$ can again be replaced by $(\tilde{\beta}_n - x)^{\hat{\varkappa}_U}$. Either way, the upper bound in assumption (D) of Theorem 3.1 is fulfilled. Similarly to the upper bound, we have that

$$\tilde{\omega}'_n(x) \gtrsim |w_{[\tilde{\alpha}_n, \tilde{\beta}_n]}(x)|^{\varkappa_L}, \quad x \in (\tilde{\alpha}_n, \tilde{\beta}_n),$$

by the lower bound in (4.2). If $l_n^{-1}(\Delta_n) = \Delta$ for all n , the above inequality gives the desired lower bound in assumption (D) of Theorem 3.1. If at least one of the intervals $l_n^{-1}(\Delta_n)$ is a proper subinterval of Δ , then the corresponding upper bound requires that $\varkappa_L > 0$. It can be readily checked that

$$|w_{[\tilde{\alpha}_n, \tilde{\beta}_n]}(x)| \gtrsim |w_\Delta(x)|, \quad x \in (\alpha + 2(\tilde{\alpha}_n - \alpha), \beta - 2(\beta - \tilde{\beta}_n)).$$

Notice that $a_n(\tilde{\alpha}_n - \alpha) = \alpha_n - \alpha_n^* \leq \delta_n$ and similarly that $a_n(\beta - \tilde{\beta}_n) \leq \delta_n$. Then, (4.11) guarantees existence of $\tau \geq \frac{2}{1+\varkappa}$ such that

$$\tilde{\omega}'_n(x) \gtrsim |w_\Delta(x)|^{\varkappa_L}, \quad x \in (\alpha + n^{-\tau}, \beta - n^{-\tau}).$$

As equicontinuity of the functions $\tilde{\omega}'_n(x)$ on compact subsets of (α, β) follows from the equicontinuity of the densities $\omega'_n(x)$, all the requirements of assumption (D) of Theorem 3.1 are satisfied.

Next, let $\tilde{h}_n = h_n \circ l_n$. Functions l_n converge to the identity map locally uniformly, hence assumption (C) of Theorem 3.1 follows from assumption (3) of Theorem 4.3 with the compact subset of $C(\Delta)$ being the closure of $\cup_n \{h \circ l_n : h \in \mathcal{K}\}$.

Now, we verify assumption (A) for the same measure μ as in the statement of Theorem 4.3 and a sequence of measures

$$d\tilde{\mu}_n(x) = e^{2n\tilde{\kappa}_n(x)} d(\mu|_{\Delta_n^*})^{(l_n)}(x), \quad \tilde{\kappa}_n = \kappa_n \circ l_n.$$

Given a continuous nonnegative function f on Δ , it holds that

$$\int_{\Delta} f d(\mu|_{\Delta_n^*}^{(l_n)} - \mu) = \int_{\Delta \cap \Delta_n^*} (f \circ l_n^{-1} - f) d\mu + \int_{\Delta_n^* \setminus \Delta} f \circ l_n^{-1} d\mu - \int_{\Delta \setminus \Delta_n^*} f d\mu.$$

The first integral on the right-hand side above converges to zero due to uniform continuity of f . The second one converges to zero because f is bounded and $\cap_n \Delta_n^* \setminus \Delta = \emptyset$ so that $\mu(\Delta_n^* \setminus \Delta) \rightarrow 0$ as $n \rightarrow \infty$. The third integral is always nonnegative. Hence, as $\tilde{\kappa}_n(x) \leq 0$, assumption (A) of Theorem 3.1 is clearly fulfilled.

To verify assumption (B), we write

$$\log \tilde{v}_n(x) = 2n\tilde{\kappa}_n(x) + \log v_{\Delta_n^*}(l_n(x)),$$

where $\tilde{v}_n$ is the Radon-Nikodym derivative of $\tilde{\mu}_n$ with respect to ω_Δ , see (1.7). Observe that

$$|\log v_\Delta(x) - \log v_{\Delta_n^*}(l_n(x))| = |\log \mu'(x) - \log \mu'(l_n(x)) - \log a_n|.$$

Since $a_n \rightarrow 1$ and $\mu \in \text{Usz}(\Delta)$, $\|\log v_\Delta - \log v_{\Delta_n^*}(l_n)\|_{L^1(\omega_\Delta)} \rightarrow 0$ as $n \rightarrow \infty$ due to assumption (4.1). Furthermore, we have

$$\int_{\Delta} |\tilde{\kappa}_n(x)| d\omega_\Delta(x) = \int_{\Delta_n^*} |\kappa_n(x)| d\omega_{\Delta_n^*}(x) \lesssim \int_{\Delta_n^* \setminus \Delta_n} |w_{\Delta_n}(x)|^{\varkappa} d\omega_{\Delta_n^*}(x),$$

where we used (4.4) and identity $\kappa_n \equiv 0$ on Δ_n . Hence, it readily follows from the definition of Δ_n^* that

$$(4.13) \quad 2n \int_{\Delta} |\tilde{\kappa}_n(x)| d\omega_{\Delta}(x) \lesssim_{\Delta} n \int_0^{\delta_n} x^{(\kappa+1)/2} d\omega_{[0,\delta_n]}(x) \lesssim_{\Delta} n \delta_n^{(\kappa+1)/2} = \xi_n.$$

Since $\lim_{n \rightarrow \infty} \xi_n = 0$ by our assumptions, we get $\|2n\tilde{\kappa}_n\|_{L^1(\omega_{\Delta})} \rightarrow 0$ as $n \rightarrow \infty$, which shows that condition (B) of Theorem 3.1 is also satisfied.

Altogether, we have that the triples $(\tilde{\mu}_n, \tilde{h}_n, \tilde{\omega}_n)$ satisfy all the conditions of Theorem 3.1 and therefore

$$T_n \left(e^{2nV^{\tilde{\omega}_n} + \tilde{h}_n} \tilde{\mu}_n \right) (z) = (1 + o_{\mathcal{K}}(1)) \exp \left(n \int \log(z-x) d\tilde{\omega}_n(x) \right) \frac{G(e^{\tilde{h}_n} \tilde{\mu}_n; \infty)}{G(e^{\tilde{h}_n} \tilde{\mu}_n; z)}$$

locally uniformly in D_{Δ} . Computations in Lemma 3.2 now show that the above formula is equivalent to the statement of the lemma. $\square$

The next lemma provides a simple uniform estimate on the boundary behavior for the sequence of outer functions. Recall the definition of the function G given in (1.8) and (1.4).

Lemma 4.8. *There exists a non-decreasing function $\epsilon_{\mu}(t)$ such that $\lim_{t \downarrow 0} \epsilon_{\mu}(t) = 0$ and*

$$\log G^{-2}(e^{h_n} \mu|_{\Delta_n}; \beta_n + t) \leq \frac{\epsilon_{\mu}(t)}{\sqrt{t}}$$

for every $t \in [0, \beta(\mu) - \beta_n]$ and all n . Moreover, an analogous estimate holds with $\beta_n + t$ replaced by $\alpha_n - t$ for $t \in [0, \alpha_n - \alpha(\mu)]$.

Proof. From (1.4) and (1.8), we get

$$\log G^{-2}(e^{h_n} \mu|_{\Delta_n}; \beta_n + t) \leq w_{\Delta_n}(\beta_n + t) \int_{\Delta_n} \frac{|h_n(x)| + |\log v_{\Delta_n}(x)|}{\beta_n + t - x} d\omega_{\Delta_n}(x).$$

The application of the Cauchy integral formula gives

$$\frac{1}{w_{\Delta_n}(z)} = \frac{1}{\pi i} \int_{\Delta_n} \frac{1}{x - z} \frac{dx}{w_{\Delta_n+}(x)} = \int_{\Delta_n} \frac{d\omega_{\Delta_n}(x)}{z - x},$$

where we used the fact that $w_{\Delta_n+} = i|w_{\Delta_n}|$ on Δ_n . Since $\mathcal{K}$ is a compact family, the functions $|h_n|$ are uniformly bounded. Therefore, we get that

$$w_{\Delta_n}(\beta_n + t) \int_{\Delta_n} \frac{|h_n(x)|}{\beta_n + t - x} d\omega_{\Delta_n}(x) \lesssim_{\mathcal{K}} 1.$$

Recall further that $v_{\Delta_n}(x) = \pi\mu'(x)|w_{\Delta_n}(x)|$. We have

$$\begin{aligned} w_{\Delta_n}(\beta_n + t) \int_{\Delta_n} \frac{|\log |w_{\Delta_n}(x)||}{\beta_n + t - x} d\omega_{\Delta_n}(x) &\lesssim_{\Delta} 1 + \sqrt{t} \int_{\beta_n-1}^{\beta_n} \frac{-\log(\beta_n - x)}{\beta_n + t - x} \frac{dx}{\sqrt{\beta_n - x}} \\ &\lesssim_{\Delta} 1 + |\log t| \end{aligned}$$

when $t \in [0, \beta(\mu) - \beta_n]$, where the first estimate is deduced exactly as the estimate for the integrals of $|h_n|$ above and the second one follows by making a substitution $\beta_n - x = ty$. Moreover,

$$\begin{aligned} \int_{\Delta_n} \frac{|\log \mu'(x)|}{\beta_n + t - x} d\omega_{\Delta_n}(x) &\lesssim_{\Delta} \frac{1}{\sqrt{t}} \int_{\alpha_n}^{\beta_n - \sqrt{t}} \frac{|\log \mu'(x)|}{\sqrt{\beta_n - x}} dx + \frac{1}{t} \int_{\beta_n - \sqrt{t}}^{\beta_n} \frac{|\log \mu'(x)|}{\sqrt{\beta_n - x}} dx \\ &\stackrel{(4.1)}{\lesssim_{\Delta}} \frac{\|\log \mu'\|_{L^1(\omega_{\Delta})}}{\sqrt{t}} + \frac{1}{t} \int_{\beta_n - \sqrt{t}}^{\beta_n} \frac{|\log \mu'(x)|}{\sqrt{\beta_n - x}} dx. \end{aligned}$$

We define $\epsilon(t)$ by the formula:

$$\epsilon(t) := \sup_{n > N_0} \int_{\beta_n - \sqrt{t}}^{\beta_n} \frac{|\log \mu'(x)|}{\sqrt{\beta_n - x}} dx,$$

where N_0 is sufficiently large to make sure that integrals converge. Clearly, this is a non-decreasing function of t . We claim that $\lim_{t \downarrow 0} \epsilon(t) = 0$. Indeed, assume to the contrary that there exist $\epsilon_0 > 0$, a sequence $\{t_m\}$ decreasing to 0, and a set $\{m_n\}$ such that

$$(4.14) \quad \int_{\beta_{m_n} - \sqrt{t_m}}^{\beta_{m_n}} \frac{|\log \mu'(x)|}{\sqrt{\beta_{m_n} - x}} dx \geq \epsilon_0.$$

If $\{m_n\}$ is bounded, it contains a constant sequence. However, (4.14) cannot hold along this sequence as the integrals of a fixed integrable function over sets of decreasing measure must vanish. On the other hand, if $\{m_n\}$ contains a strictly increasing sequence, (4.14) contradicts Proposition 4.2(iii) since $\mu \in \text{USz}(\Delta)$ (recall that we can replace $\log \mu'$ by $|\log \mu'|$ in that proposition). Hence, altogether,

$$w_{\Delta_n}(\beta_n + t) \int_{\Delta_n} \frac{|\log \mu'(x)|}{\beta_n + t - x} d\omega_{\Delta_n}(x) \lesssim_{\mu, \Delta} 1 + \frac{\epsilon(t)}{\sqrt{t}}.$$

Combining the previous estimates, we obtain

$$\log G^{-2}(e^{h_n} \mu|_{\Delta_n}, \beta_n + t) \lesssim_{\Delta, \mu, \mathcal{K}} \frac{\sqrt{t}(1 + |\log t|) + \epsilon(t)}{\sqrt{t}}.$$

Since the numerator on the right-hand side is estimated by a non-decreasing function that has zero limit at zero, the claim of the lemma follows. $\square$

In the proof of the next lemma, we will be using the following estimate. Suppose $f \in H^2(\mathbb{D})$, then the Cauchy integral formula and Cauchy-Schwarz inequality give

$$|f(z)| \leq \int_{\mathbb{T}} \frac{|f(\eta)|}{|\eta - z|} \frac{|d\eta|}{2\pi} \leq \frac{\|f\|_{L^2(\mathbb{T})}}{\sqrt{1 - |z|}}.$$

Thus, if $F \in H^2(\overline{\mathbb{C}} \setminus \Delta)$, then $F \circ \phi_{\Delta}$ is a function in $H^2(\mathbb{D})$, see (1.3), and therefore

$$(4.15) \quad |F(\beta + \delta)| \leq \frac{\|F\|_{L^2(\omega_{\Delta})}}{\sqrt{1 - \phi_{\Delta}(\beta + \delta)}} \lesssim_{\Delta} \frac{\|F\|_{L^2(\omega_{\Delta})}}{\sqrt[4]{\delta}}.$$

Lemma 4.9. *The sequence $\{\xi_n\}$ from (4.11) can be chosen so that*

$$\int_{\Delta(\mu) \setminus \Delta_n^*} P_n^*(x)^2 e^{\theta_n(x)} d\mu(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. As in the proof of Lemma 4.7, let $l_n(x) = a_n x + b_n$, $a_n > 0$, be the linear function that maps Δ onto Δ_n^* . Due to the compactness of $\mathcal{K}$, the functions $h_n - h_n \circ l_n$ converge to zero uniformly on Δ . Thus, according to Proposition 4.1,

$$(4.16) \quad \begin{cases} G(e^{h_n} \mu|_{\Delta_n^*}; z) &= (1 + o_{\mathcal{K}}(1)) G(e^{h_n} \mu|_{\Delta}; z), \\ G(e^{h_n} \mu|_{\Delta_n}; z) &= (1 + o_{\mathcal{K}}(1)) G(e^{h_n} \mu|_{\Delta}; z), \end{cases}$$

locally uniformly in D_{Δ} . Hence, the last asymptotic formula of Theorem 3.1 pulled back to the intervals Δ_n^* as in Lemma 4.7 shows

$$(4.17) \quad G^{-2}(e^{h_n} \mu|_{\Delta_n}; \infty) \int_{\Delta_n^*} P_n^*(x)^2 e^{\theta_n(x)} d\mu(x) = 2 + o_{\mathcal{K}}(1).$$

Define

$$F_n(z) = \exp \left(n \int \log(z - x) d\omega_n(x) \right) \frac{G(e^{h_n} \mu|_{\Delta_n}; \infty)}{G(e^{h_n} \mu|_{\Delta_n}; z)}$$

(where we restricted μ to Δ_n instead of Δ_n^* as in Lemma 4.7). Using identity $\kappa_n(x) \equiv 0$ on Δ_n , we get

$$|F_{n\pm}(x)|^{-2} = G^{-2}(e^{h_n}\mu|_{\Delta_n}; \infty) e^{\theta_n(x)} v_{\Delta_n}(x)$$

almost everywhere on Δ_n by (1.5), (1.7) and (1.8). Therefore,

$$2 + o_K(1) \geq \int_{\Delta_n} |P_n^*(x)/F_{n\pm}(x)|^2 d\omega_{\Delta_n}(x),$$

where we first reduce the interval of integration from Δ_n^* to Δ_n in (4.17) and then drop the singular part of μ . Since the potentials V^{ω_n} are continuous in $\mathbb{C}$, each function P_n^*/F_n is a product of a Szegő function and a bounded analytic function. As such, it belongs to $H^2(D_{\Delta_n})$. Hence, (4.15) gives us

$$|(P_n^*/F_n)(x)|^2 \lesssim_{\Delta, \mathcal{K}} 1/\sqrt{x - \beta_n}, \quad x \in [\beta_n, \beta(\mu)].$$

Observe also that $G(e^{h_n}\mu|_{\Delta_n}; \infty) \lesssim_{\mu, \mathcal{K}} 1$ by the compactness of $\mathcal{K}$ and (4.16).

Now, we are ready to estimate the integrals in the statement of the lemma. We only carry out the estimates on $[\beta_n^*, \beta(\mu)]$ as the estimates on $[\alpha(\mu), \alpha_n^*]$ can be done analogously. Assume that $\beta_n^* < \beta(\mu)$ (otherwise we have nothing to prove). In this case $\beta_n^* = \beta_n + \delta_n$. Using the bounds we just obtained, Lemma 4.8, and (4.4), we get

$$\begin{aligned} \int_{\beta_n^*}^{\beta(\mu)} P_n^*(x)^2 e^{\theta_n(x)} d\mu(x) &= \int_{\beta_n^*}^{\beta(\mu)} \left| \frac{P_n^*(x)}{F_n(x)} \right|^2 \frac{G^2(e^{h_n}\mu|_{\Delta_n}; \infty)}{G^2(e^{h_n}\mu|_{\Delta_n}; x)} e^{2n\kappa_n(x) + h_n(x)} d\mu(x) \\ &\lesssim_{\Delta, \mu, \mathcal{K}} \int_{\beta_n^*}^{\beta(\mu)} \exp \left(-C_{\Delta} n(x - \beta_n)^{\varkappa/2} + \frac{\epsilon_\mu(x - \beta_n)}{\sqrt{x - \beta_n}} \right) \frac{d\mu(x)}{\sqrt{x - \beta_n}}. \end{aligned}$$

Observe that we can always make $\epsilon_\mu(t)$ decay as slowly as we need (this will not affect the inequality in Lemma 4.8). In particular, we can redefine it if necessary to ensure

$$\frac{\epsilon_\mu(t)}{\sqrt{t}} - \frac{1}{2} \log t \leq 2 \frac{\epsilon_\mu(t)}{\sqrt{t}} \quad \Leftrightarrow \quad -\frac{1}{2} \sqrt{t} \log t \leq \epsilon_\mu(t).$$

Thus, by taking $t = x - \beta_n$, we get

$$(4.18) \quad \int_{\beta_n^*}^{\beta(\mu)} P_n^*(x)^2 e^{\theta_n(x)} d\mu(x) \lesssim_{\Delta, \mu, \mathcal{K}} \max_{\delta_n \leq t \leq T} \exp \left(C \left(-nt^{\varkappa/2} + \frac{\epsilon_\mu(t)}{\sqrt{t}} \right) \right),$$

where $T := \sup_n (\beta(\mu) - \beta_n) > 0$. Let

$$(4.19) \quad T_n := (2\epsilon_\mu(T)/n)^{2/(\varkappa+1)}.$$

First, consider the range

$$(4.20) \quad T_n \leq t \leq T.$$

Since $\epsilon_\mu(t)$ is non-decreasing, we get

$$(4.21) \quad \epsilon_\mu(t) \leq \epsilon_\mu(T) \stackrel{(4.19)}{=} (n/2) T_n^{(\varkappa+1)/2} \stackrel{(4.20)}{\leq} (n/2) t^{(\varkappa+1)/2}.$$

For t in that range, we can also write

$$-nt^{\varkappa/2} + \epsilon_\mu(t)/\sqrt{t} \stackrel{(4.21)}{\leq} -(n/2)t^{\varkappa/2} \stackrel{(4.20)}{\leq} -(n/2)T_n^{\varkappa/2} \stackrel{(4.19)}{=} -(n\epsilon_\mu(T)/2)^{1/(\varkappa+1)}.$$

Clearly, the right-hand side above converges to $-\infty$ as $n \rightarrow \infty$. Next, consider t that satisfy

$$(4.22) \quad \delta_n \leq t \leq T_n.$$

Since $\epsilon_\mu(T_n) \rightarrow 0$, we can choose ξ_n in (4.11) so that

$$(4.23) \quad 2\epsilon_\mu(T_n) \leq \xi_n.$$

It again follows from the monotonicity of $\epsilon_\mu(t)$ that

$$(4.24) \quad \epsilon_\mu(t) \leq \epsilon_\mu(T_n) \stackrel{(4.23)}{\leq} (1/2)\xi_n \stackrel{(4.11)}{=} (n/2)\delta_n^{(\varkappa+1)/2} \stackrel{(4.22)}{\leq} (n/2)t^{(\varkappa+1)/2}.$$

Hence, for t satisfying (4.22), we obtain

$$-nt^{\varkappa/2} + \epsilon_\mu(t)/\sqrt{t} \stackrel{(4.24)}{\leq} -(n/2)t^{\varkappa/2} \stackrel{(4.22)}{\leq} -(n/2)\delta_n^{\varkappa/2} \stackrel{(4.11)}{=} -(1/2)(n\xi_n^{\varkappa})^{1/(\varkappa+1)}.$$

To show that the right-hand side of (4.18) converges to zero when $n \rightarrow \infty$ and finish the proof of the lemma, it only remains to choose ξ_n so that

$$(4.25) \quad \lim_{n \rightarrow \infty} \xi_n = 0, \quad \lim_{n \rightarrow \infty} n\xi_n^{\varkappa} = +\infty, \quad \text{and} \quad 2\epsilon_\mu(T_n) \leq \xi_n. \quad \square$$

Lemma 4.10. *Theorem 4.3 holds.*

Proof. Let $F_n^*(z)$ be as in (4.12) and

$$\Omega_{\kappa_n}(z) := \Omega_{\Delta_n^*}(e^{-n\kappa_n}; z),$$

which is an outer function in $H^2(D_{\Delta_n^*})$. Its traces satisfy $|\Omega_{\kappa_n \pm}|^2 = e^{-2n\kappa_n}$ almost everywhere on Δ_n^* , see (1.5). Thus,

$$|(\Omega_{\kappa_n} F_n^*)_{\pm}(x)|^{-2} = G_n^{-2} e^{\theta_n(x)} v_{\Delta_n^*}(x)$$

almost everywhere on Δ_n^* due to (1.5), (1.7) and (1.8), where $G_n := G(e^{h_n} \mu|_{\Delta_n^*}; \infty)$. For brevity, put $P_n = T_n(e^{\theta_n} \mu)$. Recall that $F_n^*(z)/z^n \rightarrow 1$ and $w(z)/z \rightarrow 1$ as $z \rightarrow \infty$,

$$F_{n-}^*(x) = \overline{F_{n+}^*(x)} \quad \text{and} \quad w_{\Delta_n^*+}(x) = -w_{\Delta_n^*-}(x) = i|w_{\Delta_n^*}(x)|, \quad x \in \Delta_n^*.$$

Then, using (1.2) and the Cauchy integral formula, we write

$$\begin{aligned} \frac{P_n(z) - P_n^*(z)}{(\Omega_{\kappa_n} F_n^*)(z) w_{\Delta_n^*}(z)} &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{P_n(s) - P_n^*(s)}{z - s} \frac{1}{(\Omega_{\kappa_n} F_n^*)(s)} \frac{ds}{w_{\Delta_n^*}(s)} \\ &= \int_{\Delta_n^*} \frac{P_n(x) - P_n^*(x)}{z - x} \operatorname{Re} \left(\frac{1}{(\Omega_{\kappa_n} F_n^*)_+(x)} \right) d\omega_{\Delta_n^*}(x), \end{aligned}$$

where Γ_n is any counter-clockwise oriented Jordan curve that separates Δ_n^* and z . Since the absolute value of the real part of a complex number does not exceed its absolute value, the application of the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left| \frac{P_n(z) - P_n^*(z)}{(\Omega_{\kappa_n} F_n^*)(z)} \right|^2 &\leq G_n^{-2} \frac{|w_{\Delta_n^*}(z)|^2}{\operatorname{dist}(z, \Delta_n^*)^2} \int_{\Delta_n^*} (P_n - P_n^*)^2 e^{\theta_n} v_{\Delta_n^*} d\omega_{\Delta_n^*} \\ (4.26) \quad &\leq G_n^{-2} \frac{|w_{\Delta_n^*}(z)|^2}{\operatorname{dist}(z, \Delta_n^*)^2} \int_{\Delta_n^*} (P_n - P_n^*)^2 e^{\theta_n} d\mu. \end{aligned}$$

As was observed in the previous lemma, it follows from the compactness of $\mathcal{K}$ and (4.16) that the sequence G_n^{-2} is uniformly bounded. Because P_n^* is the n -th monic orthogonal polynomials with respect to $e^{\theta_n} \mu|_{\Delta_n^*}$ and P_n is a monic polynomials of degree n , we have that

$$\int_{\Delta_n^*} P_n P_n^* e^{\theta_n} d\mu = \int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu,$$

from which we deduce

$$(4.27) \quad \int_{\Delta_n^*} (P_n - P_n^*)^2 e^{\theta_n} d\mu = \int_{\Delta_n^*} P_n^2 e^{\theta_n} d\mu - \int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu.$$

Recall that the n -th monic orthogonal polynomial P_n satisfies a bound

$$(4.28) \quad \int_{\Delta(\mu)} P_n^2 e^{\theta_n} d\mu \leq \int_{\Delta(\mu)} P^2 e^{\theta_n} d\mu$$

for any monic polynomial P of degree n . Take $P = P_n^*$. Then, (4.27), simple majorization, and (4.28) give us

$$\begin{aligned} \int_{\Delta_n^*} (P_n - P_n^*)^2 e^{\theta_n} d\mu &\leq \int_{\Delta(\mu)} P_n^2 e^{\theta_n} d\mu - \int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu \\ &\stackrel{(4.28)}{\leq} \int_{\Delta(\mu)} (P_n^*)^2 e^{\theta_n} d\mu - \int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu \\ &= \int_{\Delta(\mu) \setminus \Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where the last conclusion is Lemma 4.9. From (1.4), (4.11), and (4.13), we get $\Omega_{\kappa_n}(z) = 1 + o(1)$ locally uniformly in the complement of Δ . Since the intervals Δ_n^* converge to Δ , (4.26), the above estimates, and Lemma 4.7 ensure that

$$P_n(z) = P_n^*(z) + o_{\mathcal{K}}(1)(\Omega_{\kappa_n} F_n^*)(z) = (1 + o_{\mathcal{K}}(1))F_n^*(z)$$

locally uniformly in D_{Δ} . The first claim of Theorem 4.3 now follows from (4.16). It also holds that

$$\int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu \leq \int_{\Delta_n^*} P_n^2 e^{\theta_n} d\mu \leq \int_{\Delta(\mu)} P_n^2 e^{\theta_n} d\mu$$

since P_n^* is the n -th monic orthogonal polynomial with respect $e^{\theta_n} d\mu|_{\Delta_n^*}$ and similarly

$$\int_{\Delta(\mu)} P_n^2 e^{\theta_n} d\mu \leq \int_{\Delta(\mu)} (P_n^*)^2 e^{\theta_n} d\mu = \int_{\Delta_n^*} (P_n^*)^2 e^{\theta_n} d\mu + o(1),$$

where we used Lemma 4.9 for the last equality. The second claim of the theorem is now a consequence of (4.17), the second line of (4.16), and the uniform boundedness of $G^{-2}(e^{h_n} \mu|_{\Delta}; \infty)$. $\square$

5. OPs WITH VARYING WEIGHTS AND STRONGLY SZEGŐ MEASURES

Just like in the previous two sections, we are interested in the strong asymptotics of the monic orthogonal polynomials $T_n(w_N^{2N} \mu_n)$. We again take $\{\mu_n\}$ to be a collection of Szegő measures on an interval Δ , with densities satisfying some mild assumptions and $\{w_N\}$ to be a sequence of smooth weights. The difference is that we now work in the following asymptotic regime:

$$(5.1) \quad N = N(n), \quad n/N \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We think of the measures μ_n as slowly changing with n while the varying behavior comes primarily from the weights w_N^{2N} .

Recall property (1.9) of the weighted extremal measures. In this setting, we let

$$\omega_{n,N} := \omega_{w_N^{N/n}} \quad \text{and} \quad \omega_n := \omega_{n,N(n)}.$$

In this section we place the following assumptions on the weights w_N :

- (i) Each w_N is a three times continuously differentiable function on Δ^2 satisfying

$$w'_N(x) > 0, \quad x < \eta_N, \quad \text{and} \quad w'_N(x) < 0, \quad x > \eta_N,$$

for some $\eta_N \in \Delta$ (if η_N is an endpoint of Δ , one of the above conditions is vacuous).

We assume $w_N(\eta_N) = 1$ and $w_N \geq \delta$ on Δ for some $\delta > 0$ independent of N ;

- (ii) There exists a three times continuously differentiable function w on Δ satisfying

$$w'(x) > 0, \quad x < \eta, \quad \text{and} \quad w'(x) < 0, \quad x > \eta,$$

for some point $\eta \in \Delta$ (again, if η is an endpoint of Δ , one of the above conditions is vacuous). We assume $(w'/w)'(\eta) < 0$ and, in addition, it holds that $w_N^{(k)} \rightarrow w^{(k)}$ uniformly on Δ as $N \rightarrow \infty$ for each $k \in \{0, 1, 2, 3\}$.

²Including the one-sided derivatives at α, β .

As in Section 4, only assuming Szegő condition on a single interval is not sufficient for our purposes. Recall (1.2) and (1.7). Let $\gamma > 0$ be fixed. We say that μ is *strongly Szegő with exponent γ* on a closed interval I if

$$(5.2) \quad \int_{I'} |\log v_{I'}(x)| d\omega_{I'}(x) \lesssim |I'|^{-\gamma}$$

for any closed subinterval $I' \subseteq I$ (in particular, $\mu \in \text{Sz}(I')$ for each such I'), where the constant in the above estimate is independent of I' .

Theorem 5.1. *Let $\{w_N\}$ be a sequence of weights on Δ satisfying conditions (i) and (ii). Let $\{\mu_n\}$ be a sequence of measures on Δ that are strongly Szegő with the same exponent $\gamma \in (\frac{1}{2}, 1)$ on a closed interval $I_\eta \subset \Delta$ that contains η in its interior if η is an interior point of Δ (or as an endpoint if $\eta \in \partial\Delta$). Assume further that masses $|\mu_n|$ as well as constants of proportionality in (5.2) are bounded uniformly in n . Then, given (5.1), the asymptotics*

$$(5.3) \quad T_n(w_N^{2N} \mu_n)(z) = (1 + o(1)) \exp \left(n \int \log(z - x) d\omega_n(x) \right) \quad \text{as } n \rightarrow \infty$$

holds locally uniformly in $\overline{\mathbb{C}} \setminus \{\eta\}$.

The rest of this section is devoted to the proof of Theorem 5.1. In the first subsection below, we discuss implications of the conditions (i) and (ii) for the weighted extremal measures $\omega_{n,N}$. The second subsection contains auxiliary lemmas. The proof of Theorem 5.1 is given in the final subsection.

5.1. Weighted Extremal Measures. Throughout the subsection, we assume that conditions (i) and (ii) of Theorem 5.1 are satisfied. These conditions immediately imply that $w \geq \delta$ on Δ and $w(\eta) = 1$. Furthermore, it is also clear that each w_N reaches its maximum on Δ at η_N , $\eta_N \rightarrow \eta$, and that we can shrink I_η if necessary so that

$$(5.4) \quad \begin{cases} L'_N(x) < 0, & x \in I_\eta, \\ L'(x) < 0, & x \in I_\eta, \end{cases} \quad \text{where} \quad \begin{cases} L_N(x) &:= (w'_N/w_N)(x), \\ L(x) &:= (w'/w)(x). \end{cases}$$

We remark that if $w = \exp(-V^\sigma)$ for some measure σ whose support is disjoint from Δ , then

$$L'(x) = - \int \frac{d\sigma(s)}{(x-s)^2}$$

and the inequalities in (5.4) hold on the entire interval Δ .

The assumptions (i) and (ii) have potential-theoretic implications for the extremal measures $\omega_{n,N}$ and we discuss them below.

Lemma 5.2. *For all n/N small enough, the measure $\omega_{n,N}$ is absolutely continuous and supported on an interval which we denote by $\Delta_{n,N}$. Moreover, $\eta_N \in \Delta_{n,N}$ and $|\Delta_{n,N}| = o(1)$ as $n/N \rightarrow 0$.*

Proof. Let $\nu_{n,N} := (n/N)\omega_{n,N}$. It is known, [11, Theorem I.1.3] that

$$(5.5) \quad V^{\nu_{n,N}}(x) - \log w_N(x) \begin{cases} \geq \ell_{n,N}, & \text{q.e. on } \Delta, \\ \leq \ell_{n,N}, & x \in \text{supp}(\nu_{n,N}), \end{cases}$$

for some constant $\ell_{n,N}$, see (1.9). Let $K \subset \Delta \setminus \{\eta\}$ be a compact set. Then, it follows from the definition of a logarithmic potential and assumptions (i) and (ii) that

$$(5.6) \quad \begin{aligned} V^{\nu_{n,N}}(x) - \log w_N(x) &\geq \frac{n}{N} \log \frac{1}{|\Delta|} - \log w_N(x) \\ &\geq \frac{n}{N} \log \frac{1}{|\Delta|} + 2\varepsilon_K \geq \varepsilon_K, \quad x \in K, \end{aligned}$$

for some $\varepsilon_K > 0$ and all n/N small enough. On the other hand, given an interval $I \subset \Delta$, it follows from the top inequality in (5.5) that

$$\ell_{n,N} \leq \int (V^{\nu_{n,N}} - \log w_N) d\omega_I \leq \frac{n}{N} \log \frac{4}{|I|} - \log \min_I w_N,$$

where we used Fubini-Tonelli's theorem to interchange the order of integration and an estimate $V^{\omega_I}(z) \leq \log(4/|I|)$ for all $z \in \mathbb{C}$ well-known in potential theory. Since $w(\eta) = 1$, we conclude that for any $\varepsilon > 0$ one can choose a small enough interval containing η so that

$$(5.7) \quad \ell_{n,N} \leq \varepsilon$$

for all n/N small enough. Inequalities (5.6) and (5.7) as well as the bottom inequality in (5.5) imply that

$$\text{supp}(\omega_{n,N}) \cap K = \emptyset$$

for all n/N small enough. Our claim that $\eta_N \in \text{supp}(\omega_{n,N})$ follows, for instance, from [11, Theorem IV.1.3] if we take $w_N^{\ell N/n} \cdot 1, \ell \geq 0$ as a required weighted polynomial to test there.

It follows from the first part of the proof and (5.4) that

$$(5.8) \quad \text{supp}(\omega_{n,N}) \subset I_\eta$$

for all n/N small enough. Inequalities (5.5) characterize the weighted extremal measure $\nu_{n,N}$, see [11, Theorem I.3.3]. Hence, $\nu_{n,N}$ remains the weighted extremal measure for the weight w_N even when the minimization problem is posed only on I_η . As $-\log w_N$ is convex on I_η , $\text{supp}(\omega_{n,N})$ must be an interval, see [11, Theorem IV.1.10].

In our case, the support of the weighted extremal measure is an interval while the weight function is absolutely continuous with L^p -integrable derivative for some $p > 1$. Hence, the extremal distribution is itself absolutely continuous with respect to the Lebesgue measure, see [11, Theorem IV.2.4]. The proof of the lemma is completed. $\square$

It immediately follows from Lemma 5.2 that, given (5.1), we have $\cap_n \Delta_{n,N} = \{\eta\}$. In fact, more can be said. Recall (5.4). Write $\Delta = [\alpha, \beta]$. To clarify the statement of the following lemma, observe that

$$(5.9) \quad \begin{cases} \eta_N \in (\alpha, \beta) & \Rightarrow L_N(\eta_N) = 0, L_N(x) > 0, x \in [\alpha, \eta_N), L_N(x) < 0, x \in (\eta_N, \beta], \\ \eta_N = \alpha & \Rightarrow L_N(\alpha) \leq 0, L_N(x) < 0, x \in (\alpha, \beta], \\ \eta_N = \beta & \Rightarrow L_N(\beta) \geq 0, L_N(x) > 0, x \in [\alpha, \beta). \end{cases}$$

Moreover, since $L'_N(x) \rightarrow L'(x)$ uniformly on Δ , (5.4) gives that $|L'_N(\eta_N)| > \delta$ for some positive δ and all N large enough.

Lemma 5.3. *For n/N sufficiently small, we have that*

$$(5.10) \quad |\Delta_{n,N}| \left(\frac{1}{2} |L_N(\eta_N)| + \frac{1}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) \leq \frac{n}{N}.$$

Write $\Delta_{n,N} = [\alpha_{n,N}, \beta_{n,N}]$. If $\alpha < \alpha_{n,N}$ and $\beta_{n,N} < \beta$, then $L_N(\eta_N) = 0$ and

$$(5.11) \quad |\Delta_{n,N}| \left(\frac{1}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) = \frac{n}{N}.$$

On the other hand, if $L_N(\eta_N) \neq 0$, then η_N is an endpoint of Δ and

$$(5.12) \quad |\Delta_{n,N}| \left(\frac{1}{2} |L_N(\eta_N)| + \frac{3}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) = \frac{n}{N}.$$

Furthermore, we have that

$$(5.13) \quad \eta_N \begin{cases} = \\ \leq \\ \geq \end{cases} \left\{ \frac{\alpha_{n,N} + \beta_{n,N}}{2} + \mathcal{O}(|\Delta_{n,N}|^2) \right\},$$

where the first case holds when $\alpha < \alpha_{n,N}$ and $\beta_{n,N} < \beta$, the second one when $\beta_{n,N} < \beta$, and the third one when $\alpha < \alpha_{n,N}$.

Proof. Recall that $\Delta_{n,N} \subset I_\eta$ for all n/N small enough, see (5.8). Since η is an interior point of I_η when it is an interior point of Δ , we can take n/N sufficiently small so that if $\alpha_{n,N}$ and/or $\beta_{n,N}$ is an interior point of Δ , then it is also an interior point of I_η . Thus, it follows from [11, Theorem IV.1.11] applied on I_η that

$$(5.14) \quad \int_{\Delta_{n,N}} L_N(x) \sqrt{\frac{x - \alpha_{n,N}}{\beta_{n,N} - x}} \frac{dx}{\pi} \geq -\frac{n}{N} \quad \text{and} \quad \int_{\Delta_{n,N}} L_N(x) \sqrt{\frac{\beta_{n,N} - x}{x - \alpha_{n,N}}} \frac{dx}{\pi} \leq \frac{n}{N},$$

where the first inequality is replaced by equality when $\beta_{n,N} < \beta$ and the second one becomes equality when $\alpha < \alpha_{n,N}$. Since $|\Delta_{n,N}| = o(1)$ as $n/N \rightarrow 0$, at least one of the conditions $\alpha < \alpha_{n,N}$, or $\beta_{n,N} < \beta$, or possibly both takes place. According to condition (i), it holds that

$$(5.15) \quad L_N(x) = L_N(\eta_N) + L'_N(\eta_N)(x - \eta_N) + \mathcal{O}(|\Delta_{n,N}|^2), \quad x \in \Delta_{n,N},$$

where the error term is uniform in N because $L''_N(x) \rightarrow L''(x)$ uniformly on Δ due to condition (ii). Recall (5.9). Substituting (5.15) into the first inequality in (5.14) and changing signs give

$$(5.16) \quad |\Delta_{n,N}| \left(-\frac{1}{2} L_N(\eta_N) + \frac{3 - 4\xi_{n,N}}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) \leq \frac{n}{N},$$

where $\eta_N =: |\Delta_{n,N}| \xi_{n,N} + \alpha_{n,N}$, $\xi_{n,N} \in [0, 1]$, and the inequality is equality when $\beta_{n,N} < \beta$. In particular, this proves (5.12) when $L_N(\eta_N) < 0$, because in this case $\eta_N = \alpha$ and $\beta_{n,N} < \beta$, so that we have equality in (5.16) with $\xi_{n,N} = 0$. On the other hand, the second inequality in (5.14) and (5.15) give

$$(5.17) \quad |\Delta_{n,N}| \left(\frac{1}{2} L_N(\eta_N) + \frac{4\xi_{n,N} - 1}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) \leq \frac{n}{N},$$

where the inequality is actually an equality when $\alpha < \alpha_{n,N}$. This, respectively, proves (5.12) when $L_N(\eta_N) > 0$, because in this case $\eta_N = \beta$ and $\alpha < \alpha_{n,N}$, so that we have equality in (5.17) with $\xi_{n,N} = 1$. Estimate (5.10) follows by averaging (5.16) and (5.17). It becomes (5.11) when $\alpha < \alpha_{n,N}$ and $\beta_{n,N} < \beta$ because (5.16) and (5.17) are both equalities in this case while $L_N(\eta_N) = 0$. Finally, by taking the difference of (5.16) and (5.17), one of which is equality and the other one is a bound, we get that $\xi_{n,N} \leq 1/2 + \mathcal{O}(|\Delta_{n,N}|)$ when $\beta_{n,N} < \beta$ and that $\xi_{n,N} \geq 1/2 + \mathcal{O}(|\Delta_{n,N}|)$ when $\alpha < \alpha_{n,N}$. Clearly, this is equivalent to the inequalities in (5.13). The equality is an immediate consequence of both inequalities. $\square$

For small enough n/N , this lemma gives

$$(5.18) \quad \frac{n}{N} \lesssim |\Delta_{n,N}| \lesssim \sqrt{\frac{n}{N}} \quad \text{and} \quad |\Delta_{n,N}| = \frac{n}{N} \frac{2 + o(1)}{|L(\eta)|} \quad \text{if} \quad |L(\eta)| > 0.$$

In the next result, we study the densities of the weighted equilibrium measures $\omega_{n,N}$. It is more convenient to do this on a fixed interval. To this end, we denote by $\hat{\omega}_{n,N}$ the pull back of $\omega_{n,N}$ from $\Delta_{n,N}$ to $[0, 1]$. By Lemma 5.2, $\hat{\omega}_{n,N}$ is absolutely continuous with respect to the Lebesgue measure and

$$\hat{\omega}'_{n,N}(s) = |\Delta_{n,N}| \omega'_{n,N}(s|\Delta_{n,N}| + \alpha_{n,N}), \quad s \in [0, 1].$$

Lemma 5.4. *The family of functions $\{\hat{\omega}'_{n,N}\}$ is uniformly equicontinuous on closed subintervals of $(0, 1)$. Moreover,*

$$(5.19) \quad \frac{1}{3\pi} \sqrt{s(1-s)} < \hat{\omega}'_{n,N}(s) < \frac{9}{\pi} s^{\gamma_0} (1-s)^{\gamma_1}, \quad s \in [0, 1],$$

where $\gamma_0 = -\frac{1}{2}$ if $\alpha = \alpha_{n,N}$ and $\gamma_0 = \frac{1}{2}$ otherwise. Similarly, $\gamma_1 = -\frac{1}{2}$ if $\beta_{n,N} = \beta$ and $\gamma_1 = \frac{1}{2}$ otherwise.

Proof. As we have mentioned at the end of the proof of Lemma 5.2, [11, Theorem IV.2.4] implies that $\omega_{n,N}$ is absolutely continuous with respect to the Lebesgue measure. In fact, this theorem also provides an explicit integral formula for its density. We use the symmetry and a trigonometric change of variables to write that formula as

$$(5.20) \quad \hat{\omega}'_{n,N}(s) = \frac{1}{\pi\sqrt{s(1-s)}} \left(1 + |\Delta_{n,N}| \frac{N}{n} \int_0^1 \hat{L}_N(x) \frac{\sqrt{x(1-x)}}{s-x} \frac{dx}{\pi} \right),$$

where $\hat{L}_N(x) := L_N(x|\Delta_{n,N}| + \alpha_{n,N})$ and $\int$ is the integral understood in the sense of the principal value. Set

$$\delta_{n,N} := 1 + |\Delta_{n,N}| \frac{N}{n} \int_0^1 \hat{L}_N(x) \sqrt{\frac{x}{1-x}} \frac{dx}{\pi}.$$

By (5.14), this is a nonnegative quantity and it is equal to 0 when $\beta_{n,N} < \beta$. Similarly to (5.16), plugging (5.15) into the definition of $\delta_{n,N}$ gives

$$(5.21) \quad |\Delta_{n,N}| \left(-\frac{1}{2} L_N(\eta_N) + \frac{3 - 4\xi_{n,N}}{8} |L'_N(\eta_N)| |\Delta_{n,N}| + \mathcal{O}(|\Delta_{n,N}|^2) \right) = \frac{n}{N} (1 - \delta_{n,N}).$$

Hence, by using (5.17) in this identity, we can conclude that

$$(5.22) \quad 2 - \delta_{n,N} \geq |\Delta_{n,N}|^2 \frac{N}{n} \frac{|L'_N(\eta_N)| + \mathcal{O}(|\Delta_{n,N}|)}{4} \Rightarrow \delta_{n,N} \in [0, 2].$$

Now, we deduce from (5.20) and the definition of $\delta_{n,N}$ that

$$(5.23) \quad \hat{\omega}'_{n,N}(s) = \frac{\delta_{n,N}}{\pi\sqrt{s(1-s)}} + \frac{1}{\pi} \sqrt{\frac{1-s}{s}} |\Delta_{n,N}| \frac{N}{n} \int_0^1 \frac{\hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi}.$$

If $\hat{L}_N(x)$ is replaced by 1 in the last formula, the singular integral above is equal to -1 , which follows from applying the Plemelj-Sokhotski formula. Hence, we can write

$$(5.24) \quad \int_0^1 \frac{\hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi} = -\hat{L}_N(s) - \int_0^1 \frac{\hat{L}_N(s) - \hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi}.$$

The mean-value theorem gives

$$\hat{L}_N(s) - \hat{L}_N(x) = |\Delta_{n,N}| L'_N(\xi_{s,x})(s-x) = |\Delta_{n,N}| (L'_N(\eta_N) + \mathcal{O}(|\Delta_{n,N}|))(s-x),$$

where $\xi_{s,x} \in \Delta_{n,N}$. Using the above estimate for $\hat{L}_N(\xi_{n,N}) - \hat{L}_N(s)$ as well, we get that

$$\int_0^1 \frac{\hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi} = -L_N(\eta_N) - |\Delta_{n,N}| L'_N(\eta_N) \left(s - \xi_{n,N} + \frac{1}{2} \right) + \mathcal{O}(|\Delta_{n,N}|^2).$$

Furthermore, by solving (5.21) for $-L_N(\eta_N) - |\Delta_{n,N}| L'_N(\eta_N) \xi_{n,N}$ and recalling that $L'_N(\eta_N) = -|L'_N(\eta_N)|$, we further get

$$|\Delta_{n,N}| \frac{N}{n} \int_0^1 \frac{\hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi} = 2(1 - \delta_{n,N}) + l_{n,N} \left(s - \frac{1}{4} \right) + \mathcal{O}(|\Delta_{n,N}|),$$

where we use $l_{n,N} := |\Delta_{n,N}|^2 (N/n) |L'_N(\eta_N)|$ for shorthand. Plugging the above expression into (5.23) now gives³

$$(5.25) \quad \hat{\omega}'_{n,N}(s) = \frac{\delta_{n,N}}{\pi\sqrt{s(1-s)}} + \frac{1}{\pi} \left[2(1 - \delta_{n,N}) + \mathcal{O}(|\Delta_{n,N}|) + l_{n,N} \left(s - \frac{1}{4} \right) \right] \sqrt{\frac{1-s}{s}}.$$

³The appearance of $s - \frac{1}{4}$ is not entirely surprising since $\int_0^1 (s - \frac{1}{4}) \sqrt{\frac{1-s}{s}} ds = 0$.

Since $l_{n,N} \geq 0$ and $\delta_{n,N} \in [0, 2)$, see (5.22), we get for $s \in [1/4, 1]$ that

$$\hat{\omega}'_{n,N}(s) \geq \frac{1}{\pi} \left(2 - \frac{2}{3}\delta_{n,N} + \mathcal{O}(|\Delta_{n,N}|) \right) \sqrt{s(1-s)} > \frac{1}{3\pi} \sqrt{s(1-s)}.$$

In the other direction, we can rewrite (5.25) as

$$\hat{\omega}'_{n,N}(s) = \frac{1}{\pi} \frac{\delta_{n,N}s + (2 - \delta_{n,N})(1-s)}{\sqrt{s(1-s)}} + \frac{1}{\pi} \left[\mathcal{O}(|\Delta_{n,N}|) + l_{n,N} \left(s - \frac{1}{4} \right) \right] \sqrt{\frac{1-s}{s}}.$$

By (5.10), we have $l_{n,N} \leq 8 + \mathcal{O}(|\Delta_{n,N}|)$ and therefore

$$\hat{\omega}'_{n,N}(s) \leq \frac{2}{\pi} \frac{1}{\sqrt{s(1-s)}} + \frac{6 + \mathcal{O}(|\Delta_{n,N}|)}{\pi} \sqrt{\frac{1-s}{s}} < \frac{9}{\pi} \frac{1}{\sqrt{s(1-s)}}$$

for all $s \in [0, 1]$. Similarly, when $\beta_{n,N} < \beta$, $\delta_{n,N} = 0$ and (5.25) gives

$$\hat{\omega}'_{n,N}(s) \leq \frac{8s + \mathcal{O}(|\Delta_{n,N}|)}{\pi} \sqrt{\frac{(1-s)}{s}} < \frac{9}{\pi} \sqrt{s(1-s)}$$

for $s \in [1/4, 1]$. The last three estimates yield (5.19) on $[1/4, 1]$. The estimates for $s \in [0, 3/4]$ can be obtained analogously.

It only remains to show that the functions $\hat{\omega}'_{n,N}$ form a uniformly equicontinuous family on each closed subinterval of $(0, 1)$. To this end, it is enough to show that the functions $\hat{\omega}''_{n,N}$ are uniformly bounded on each such subinterval. Using (5.24), we get that

$$\frac{d}{ds} \left(\int_0^1 \frac{\hat{L}_N(x)}{s-x} \sqrt{\frac{x}{1-x}} \frac{dx}{\pi} \right) = -\hat{L}'_N(s) + \frac{1}{2} \int_0^1 \hat{L}''_N(\xi_{s,x}) \sqrt{\frac{x}{1-x}} \frac{dx}{\pi}$$

for some $\xi_{s,x} \in \Delta_{n,N}$ by Taylor approximation theorem. Since $L_N^{(k)} \rightarrow L^{(k)}$ uniformly on Δ for $k = 1, 2$, $\hat{L}_N^{(k)}(s) = |\Delta_{n,N}|^k L_N^{(k)}(s|\Delta_{n,N}| + \alpha_{n,N})$, and $|\Delta_{n,N}|^2(N/n) \lesssim 1$ while $\delta_{n,N} \in [0, 2)$, the desired claim readily follows from (5.23). $\square$

5.2. Auxiliary Lemmas. In what follows, we are always assuming that (5.1) takes place. Hence, to simplify the notation, we rename $\Delta_{n,N} = [\alpha_{n,N}, \beta_{n,N}]$ as $\Delta_n = [\alpha_n, \beta_n]$. Also, all constants C_i, C'_i, C''_i , $i \geq 1$, appearing below as well as implicit constants in inequalities $\lesssim, \gtrsim$ are understood to be independent of n .

To prove Theorem 5.1 it will be convenient to rescale the problem so that the equilibrium measures are supported on the interval $[0, 1]$. To this end, given measures μ_n as in Theorem 5.1, we define measures $\hat{\mu}_n$ by

$$(5.26) \quad \hat{\mu}_n(B) = \mu_n^{-1}(|\Delta_n|) \mu_n(\{x|\Delta_n| + \alpha_n : x \in B\})$$

for any set $B \subseteq \hat{\Delta}_n = [\frac{\alpha_n - \alpha_n}{|\Delta_n|}, \frac{\beta_n - \alpha_n}{|\Delta_n|}] =: [\hat{\alpha}_n, \hat{\beta}_n]$.

Lemma 5.5. *There exists a constant $C_1 > 0$ such that*

$$(5.27) \quad |\hat{\mu}_n| \lesssim \exp \left(C_1 |\Delta_n|^{-1/2} \right).$$

Moreover, if we write $d\hat{\mu}_n|_{[0,1]} = \hat{v}_n|_{[0,1]} d\omega_{[0,1]} + d\hat{\mu}_n^s|_{[0,1]}$, where $\hat{\mu}_n^s|_{[0,1]}$ is singular with respect to the Lebesgue measure, then

$$(5.28) \quad \int_0^1 |\log \hat{v}_n|_{[0,1]}(x) d\omega_{[0,1]}(x) \lesssim |\Delta_n|^{-\gamma}.$$

Proof. We recall that $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \eta_n = \eta$. Write $I_\eta = [\alpha_\eta, \beta_\eta]$. Assume that $\eta < \beta$. In this case $\eta < \beta_\eta$. By dropping the singular part and using Jensen's

inequality we get that

$$\begin{aligned}\mu_n(|\Delta_n|) &\geq \int_{\frac{1}{2}(\alpha_n + \beta_n)}^{\beta_n} \mu'_n(x) dx \geq \frac{|\Delta_n|}{2} \exp \left(\frac{2}{|\Delta_n|} \int_{\frac{1}{2}(\alpha_n + \beta_n)}^{\beta_n} \log \mu'_n(x) dx \right) \\ &\geq \frac{|\Delta_n|}{2} \exp \left(-\frac{2}{|\Delta_n|} \int_{\frac{1}{2}(\alpha_n + \beta_n)}^{\beta_n} |\log^- \mu'_n(x)| dx \right).\end{aligned}$$

Since it trivially holds that $\sqrt{(x - \alpha_n)(\beta_n - x)} \leq \sqrt{|I_\eta||\Delta_n|}$ for $x \in [\frac{1}{2}(\alpha_n + \beta_n), \beta_n]$, we have

$$\begin{aligned}\mu_n(|\Delta_n|) &\geq \frac{|\Delta_n|}{2} \exp \left(-2\sqrt{\frac{|I_\eta|}{|\Delta_n|}} \int_{\frac{1}{2}(\alpha_n + \beta_n)}^{\beta_n} |\log^- \mu'_n| d\omega_{[\alpha_n, \beta_n]} \right) \\ &\geq \frac{|\Delta_n|}{2} \exp \left(-2\sqrt{\frac{|I_\eta|}{|\Delta_n|}} \int_{\alpha_n}^{\beta_n} |\log^- \mu'_n| d\omega_{[\alpha_n, \beta_n]} \right).\end{aligned}$$

Since $\alpha_n \rightarrow \eta < \beta_n$ as $n \rightarrow \infty$, we have that $\beta_n - \alpha_n \gtrsim 1$. Thus, (5.2) provides

$$\int_{\alpha_n}^{\beta_n} |\log^- \mu'_n| d\omega_{[\alpha_n, \beta_n]} \leq \int_{\alpha_n}^{\beta_n} |\log \mu'_n| d\omega_{[\alpha_n, \beta_n]} \lesssim 1.$$

Combining these bounds, we get

$$(5.29) \quad \mu_n(|\Delta_n|) \geq \frac{|\Delta_n|}{2} \exp \left(-C'_1 |\Delta_n|^{-1/2} \right) \geq \exp \left(-C_1 |\Delta_n|^{-1/2} \right)$$

for some constants $C'_1, C_1 > 0$. Since the measures μ_n have uniformly bounded masses, the above estimate and (5.26) yield (5.27) in the considered case. If $\eta = \beta$, the above estimate can be easily modified by considering intervals $[\alpha_n, \frac{1}{2}(\alpha_n + \beta_n)]$ and $[\alpha_n, \beta_n]$ to deduce the same conclusion.

Recall that the arcsine distribution is invariant under the linear transformations. Therefore, (5.26) and (5.29) yield

$$\int_0^1 |\log \hat{v}_{n,[0,1]}| d\omega_{[0,1]} \leq \int_{\Delta_n} |\log v_{n,\Delta_n}| d\omega_{\Delta_n} + C_1 |\Delta_n|^{-1/2}.$$

The above estimate and (5.2) readily yield (5.28) finishing the proof. $\square$

Lemma 5.6. *Let $\hat{w}_n(x) := w_N(x|\Delta_n| + \alpha_n)$, $x \in \hat{\Delta}_n = [\hat{\alpha}_n, \hat{\beta}_n]$. Then, there exist a constant $C_2 > 0$ such that*

$$(5.30) \quad \hat{w}_n(x) \begin{cases} \geq e^{-C_2 n/N}, & x \in [0, 1], \\ \leq e^{-C_2 |x - \frac{1}{2}|n/N}, & x \in \hat{\Delta}_n \setminus [0, 1], \end{cases}$$

for all sufficiently large n .

Proof. Using Taylor's theorem at the point η_N and our assumption that $w_N(\eta_N) = 1$, we get that

$$\begin{aligned}\log w_N(x) &= L_N(\eta_N)(x - \eta_N) + \mathcal{O}((x - \eta_N)^2) \\ &\geq -|L_N(\eta_N)||\Delta_n| + \mathcal{O}(|\Delta_n|^2) \stackrel{(5.10)}{\gtrsim} -(n/N + \mathcal{O}(|\Delta_n|^2)) \stackrel{(5.18)}{\gtrsim} -n/N\end{aligned}$$

for $x \in \Delta_n$, which is equivalent to the top line in (5.30).

To prove the upper bound in (5.30), assume first that $1 < \hat{\beta}_n$, i.e., $\beta_n < \beta$. In this case, $L_N(\eta_N) \leq 0$. Recall (5.4). Let $c > 0$ be such that $L'_N(x) \leq -2c$ for $x \in I_\eta$. Recall also that $w_N(\eta_N) = 1$. Therefore, Taylor's theorem applied at the point η_N gives

$$(5.31) \quad \begin{aligned}\log w_N(x) &\leq L_N(\eta_N)(x - \eta_N) - c(x - \eta_N)^2, & \eta_N \leq x \in I_\eta, \\ &\lesssim L_N(\eta_N)(x - \eta_N) - c(x - \eta_N)^2, & \eta_N \leq x \in \Delta,\end{aligned}$$

where the second inequality takes place because w_N decreases on $(\eta_N, \beta]$ and I_η, Δ are fixed intervals independent of n . Consider two cases.

1. First, assume that

$$(5.32) \quad L_N(\eta_N) \leq -\frac{3}{4}|L'_N(\eta_N)||\Delta_n|,$$

in which case $\eta_N = \alpha$ and (5.12) takes place. Then,

$$(5.33) \quad \frac{3}{4}|L'_N(\eta_N)||\Delta_n|^2 + \mathcal{O}(|\Delta_n|^3) \stackrel{(5.32)}{\leq} \frac{1}{2}|L_N(\eta_N)||\Delta_n| + \frac{3}{8}|L'_N(\eta_N)||\Delta_n|^2 + \mathcal{O}(|\Delta_n|^3) \stackrel{(5.12)}{=} \frac{n}{N} + \mathcal{O}(|\Delta_n|^3).$$

Using (5.12) one more time together with the above estimates gives

$$\begin{aligned} L_N(\eta_N)|\Delta_n| &= -2\frac{n}{N} + \frac{3}{4}|L'_N(\eta_N)||\Delta_n|^2 + \mathcal{O}(|\Delta_n|^3) \\ &\stackrel{(5.33)}{\leq} -\frac{n}{N} + \mathcal{O}(|\Delta_n|^3) \leq -\frac{n}{2N} \end{aligned}$$

for all large n . Thus, (5.31) and the above inequality give us

$$w_n(x) \leq \exp(2C'_2 L_N(\eta_N)(x - \eta_N)) \leq \exp\left(-C'_2 \frac{n}{N} \frac{x - \eta_N}{|\Delta_n|}\right), \quad x \in [\beta_n, \beta],$$

for some constant $C'_2 > 0$. Now, notice that our assumption $\beta_n < \beta$ implies $\eta_N \leq (\alpha_n + \beta_n)/2 + \mathcal{O}(|\Delta_n|^2)$ due to (5.13). Hence, after rescaling, we obtain that

$$(5.34) \quad \hat{w}_n(x) \leq \exp\left(-C_2(n/N)|x - \frac{1}{2}|\right), \quad x \in [1, \hat{\beta}_n],$$

for some constant $C_2 > 0$.

2. Suppose now that

$$(5.35) \quad -\frac{3}{4}|L'_N(\eta_N)||\Delta_n| < L_N(\eta_N).$$

Then, since in that case $\beta_n < \beta$, we have equality in (5.16), so

$$\frac{3}{2}|L'_N(\eta_N)||\Delta_n|^2 \stackrel{(5.35)}{\geq} |L_N(\eta_N)||\Delta_n| + \frac{3}{4}|L'_N(\eta_N)||\Delta_n|^2 \stackrel{(5.16)}{\geq} 2\frac{n}{N} + \mathcal{O}(|\Delta_n|^3)$$

for all sufficiently large n . Because $|L'_N(\eta_N)| \rightarrow |L'(\eta)| > 0$, the previous bound gives $|\Delta_n|^2 \gtrsim n/N$. Hence, we keep the second term in (5.31) to get

$$w_n(x) \leq \exp(-cC'_2(x - \eta_N)^2) \leq \exp\left(-C''_2 \frac{n}{N} \left(\frac{x - \eta_N}{|\Delta_n|}\right)^2\right), \quad x \in [\beta_n, \beta],$$

for some constant $C''_2 > 0$. Again, since $\eta_N \leq (\alpha_n + \beta_n)/2 + \mathcal{O}(|\Delta_n|^2)$, we rescale and possibly adjust constant C_2 to deduce

$$(5.36) \quad \hat{w}_n(x) \leq \exp\left(-2C_2(n/N)(x - \frac{1}{2})^2\right), \quad x \in [1, \hat{\beta}_n].$$

Clearly, the bottom line of (5.30) on $[1, \hat{\beta}_n]$ follows from (5.34) and (5.36). The desired estimate for $x < 0$ when $\alpha < \alpha_n$ can be proven analogously. $\square$

Lemma 5.7. *There exists a constant $C_3 > 0$ such that*

$$(5.37) \quad \int_0^1 |P(x)| \hat{w}_n^{2N}(x) d\hat{\mu}_n(x) \geq \exp(-C_3(n + |\Delta_n|^{-\gamma}))$$

for any monic polynomial P of degree at most $2n$.

Proof. Apply the top line in (5.30). We drop the singular part of $\hat{\mu}_n$, use Jensen's inequality and then (5.28) to write

$$\begin{aligned} \int_0^1 |P| \hat{w}_n^{2N} d\hat{\mu}_n &\stackrel{(5.30)}{\geq} e^{-2C_2 n} \int_0^1 |P| \hat{v}_{n,[0,1]} d\omega_{[0,1]} \\ &\stackrel{(\text{Jensen})}{\geq} \exp \left(-2C_2 n + \int_0^1 \log (|P \hat{v}_{n,[0,1]}|) d\omega_{[0,1]} \right) \\ &\stackrel{(5.28)}{\geq} \exp \left(-C'_3 (n + |\Delta_n|^{-\gamma}) + \int_0^1 \log |P| d\omega_{[0,1]} \right) \end{aligned}$$

for some constant $C'_3 > 0$. Recall (1.3). Observe that $z\phi_{[0,1]}(z) \rightarrow 1/4$ as $z \rightarrow \infty$. Then, we get from the mean-value inequality for subharmonic functions that

$$\int_0^1 \log |P| d\omega_{[0,1]} = \int_0^1 \log |P \phi_{[0,1]}^d| d\omega_{[0,1]} \geq \log |P \phi_{[0,1]}^d|(\infty) = -d \log 4,$$

where $d = \deg P$. As $d \leq 2n$, this proves (5.37) with $C_3 = C'_3 + 4 \log 2$. $\square$

Lemma 5.8. *Let $Z(T_n)$ be the set of zeros of the polynomial $T_n := T_n(\hat{w}_n^{2N} \hat{\mu}_n)$. Then, there exists a constant $C_4 > 0$ such that*

$$Z(T_n) \subset \{x : |x - \tfrac{1}{2}| \leq C_4 \max \{1, n^{-1} |\Delta_n|^{-\gamma}\}\}.$$

for all large enough n .

Proof. Let $\{|x - \frac{1}{2}| \leq x_n\}$ be the smallest interval symmetric around the point $\frac{1}{2}$ that contains $Z(T_n)$. Denote by Q the monic polynomial whose zeros are precisely those zeros of T_n that belong to $\{|x - \frac{1}{2}| \leq \frac{1}{2} x_n\}$, counting multiplicities. If there are no such zeros, we let $Q = 1$. Necessarily, $\deg(Q) \leq n - 1$. Then, the orthogonality condition gives us

$$(5.38) \quad \int_{\hat{\Delta}_n} T_n Q \hat{w}_n^{2N} d\hat{\mu}_n = 0.$$

We can assume without loss of generality that $x_n \geq 1$ for infinitely many indices n as otherwise we have nothing to prove. In what follows, we are only interested in such indices. Notice that $(T_n Q)(x)$ has a constant sign on the interval $\{|x - \frac{1}{2}| \leq \frac{1}{2} x_n\}$, which contains $[0, 1]$. Therefore, (5.37) yields

$$(5.39) \quad \left| \int_{\{|x - \frac{1}{2}| \leq \frac{1}{2} x_n\}} T_n Q \hat{w}_n^{2N} d\hat{\mu}_n \right| \geq \exp(-C_3(n + |\Delta_n|^{-\gamma})).$$

On the other hand, in view of (5.30), there exists $x_* > 1$ and $C'_4 > 0$ such that

$$(5.40) \quad 2n \log(4x) + 2N \log \hat{w}_n(x) \leq -C'_4 n(x - \tfrac{1}{2}), \quad x \geq x_*.$$

Again, we only need to consider those indices n for which $x_n + 1 \geq 2x_*$. Since T_n and Q are monic and have all zeros on $[-x_n + \frac{1}{2}, x_n + \frac{1}{2}]$, we can write a bound

$$|T_n(x) Q(x)| \leq |x + x_n - \tfrac{1}{2}|^{2n}, \quad x > \tfrac{1}{2}.$$

Each function $\hat{w}_n(x)$ is decreasing in x on $(1, \infty)$, so (5.27) and (5.40) give us

$$\int_{\frac{1}{2}(x_n+1)}^{x_n+\frac{1}{2}} |(T_n Q)| \hat{w}_n^{2N} d\hat{\mu}_n \lesssim e^{-C'_4 n x_n + C_1 |\Delta_n|^{-1/2}}.$$

Similarly, for every $j \in \{0, 1, 2, \dots\}$ satisfying $x_n + j + \frac{1}{2} < \hat{\beta}_n$, we get

$$\int_{x_n+j+\frac{1}{2}}^{x_n+j+\frac{3}{2}} |T_n Q| \hat{w}_n^{2N} d\hat{\mu}_n \lesssim e^{-C'_4 n(x_n+j) + C_1 |\Delta_n|^{-1/2}}.$$

Analogous estimates can also be derived on the intervals $[\frac{1}{2} - x_n, \frac{1-x_n}{2}]$ and $[-\frac{1}{2} - j - x_n, \frac{1}{2} - j - x_n]$. Combining these bounds, we get

$$(5.41) \quad \int_{\hat{\Delta}_n \setminus \{|x-\frac{1}{2}| \leq \frac{1}{2}x_n\}} |(T_n Q)| \hat{w}_n^{2N} d\hat{\mu}_n \leq e^{-C_4'' n x_n + C_1 |\Delta_n|^{-1/2}}$$

for some constant $C_4'' > 0$. However, if

$$x_n > \frac{C_1 + C_3}{C_4''} \frac{1}{n |\Delta_n|^\gamma},$$

estimates (5.39) and (5.41) contradict (5.38). Thus, x_n is bounded above by the maximum between $2x_* - 1$ and the right-hand side of the above inequality, which is exactly the statement of the lemma. $\square$

Recall (5.5) and that $\text{supp}(\omega_n)$ is an interval Δ_n for all n large enough. Therefore, [11, Theorem I.4.8] implies that V^{ω_n} is continuous everywhere in $\mathbb{C}$. In particular, the first inequality in (5.5) holds everywhere on Δ and not just quasi-everywhere. Thus, in the rescaled variable, we have that

$$(5.42) \quad N \log \hat{w}_n(x) \leq n V^{\hat{\omega}_n}(x) - \hat{\ell}_n, \quad x \in \hat{\Delta}_n,$$

where $\hat{\ell}_n := n \log |\Delta_n| + N \ell_{n,N}$ and the that inequality is actually equality on $[0, 1]$.

Lemma 5.9. *For all large enough n , the bound*

$$(5.43) \quad \int_{\hat{\Delta}_n} |T_n(x)|^2 \hat{w}_n^{2N}(x) d\hat{\mu}_n(x) \lesssim \exp(-2\hat{\ell}_n + C_1 |\Delta_n|^{-1/2})$$

holds.

Proof. Suppose we are given a sequence of probability measures $d\sigma_n(x) = u_n(x)dx$ on $[0, 1]$ such that the set $\{u_n\}$ is uniformly equicontinuous on closed subintervals of $(0, 1)$ and

$$\frac{1}{A} (x(1-x))^{b'} \leq u_n(x) \leq A (x(1-x))^{b''}, \quad x \in (0, 1),$$

for every n and some constants $A > 1$ and $b', b'' > -1$. In [18, Lemma 9.1], it was shown that for sufficiently large n there exist monic polynomials $Q(\sigma_n; z)$ of degree n (see the display right after [18, Equation (9.7)]), such that

$$(5.44) \quad 1 \leq |Q(\sigma_n; x)| e^{n V^{\sigma_n}(x)} \leq \min \left\{ n^B, (x(1-x))^{-B} \right\}, \quad x \in (0, 1),$$

for some constant B depending only on A, b', b'' . We can always take B to be an integer.

According to Lemma 5.4, the measures $\sigma_n = \hat{\omega}_{n+2B}$ satisfy all the above requirements. Set

$$\Psi_n(z) := Q(\sigma_{n-2B}; z) (z(z-1))^B.$$

One can readily see from the upper bound in (5.19) that

$$(5.45) \quad V^{\hat{\omega}_n}(x) < 9 V^{\omega_{[0,1]}}(x) = 9 \log 4, \quad x \in [0, 1].$$

Thus, it follows from (5.44) and (5.45) that

$$|\Psi_n(x)| e^{n V^{\hat{\omega}_n}(x)} \leq \frac{1}{4} e^{2B V^{\hat{\omega}_n}(x)} < \frac{(9 \log 4)^{2B}}{4}, \quad x \in [0, 1].$$

The maximum principle for subharmonic functions now yields that

$$(5.46) \quad |\Psi_n(z)| e^{n V^{\hat{\omega}_n}(z)} \lesssim 1, \quad z \in \mathbb{C}.$$

Because the n -th monic orthogonal polynomial minimizes the L^2 -norm among all monic polynomials of degree n with respect to the measure of orthogonality, we now get from (5.42), (5.46), and (5.27) that

$$\begin{aligned} \int_{\hat{\Delta}_n} |T_n|^2 \hat{w}_n^{2N} d\hat{\mu}_n &\stackrel{(5.42)}{\leq} e^{-2\hat{\ell}_n} \int_{\hat{\Delta}_n} |\Psi_n|^2 e^{2nV\hat{\omega}_n} d\hat{\mu}_n \\ &\stackrel{(5.46)}{\lesssim} e^{-2\hat{\ell}_n} \hat{\mu}_n(\hat{\Delta}_n) \stackrel{(5.27)}{\lesssim} \exp\left(-2\hat{\ell}_n + C_1|\Delta_n|^{-1/2}\right). \quad \square \end{aligned}$$

5.3. Proof of Theorem 5.1. Clearly, to prove the theorem, it is sufficient to show that (5.3) holds uniformly on closed sets of the form $K_{n,\rho} := \{z : \min_{x \in \Delta_n} |z - x| \geq \rho\}$ for each fixed $\rho > 0$. Let T_n be as in Lemma 5.8. Since the monic orthogonal polynomial does not depend on the normalization of the measure of orthogonality, it holds that

$$T_n(w_N^{2N} \mu_n)(z) = |\Delta_n|^n T_n\left(\frac{z - \alpha_n}{|\Delta_n|}\right).$$

Hence, (5.3) is equivalent to saying that the asymptotics

$$(5.47) \quad T_n(z) = (1 + o(1)) \exp\left(n \int \log(z - s) d\hat{\omega}_n(s)\right)$$

holds uniformly on closed set $\hat{K}_{n,\rho} := \{z : \min_{x \in [0,1]} |z - x| \geq \rho/|\Delta_n|\}$ for every $\rho > 0$.

Let $G_n(z) := G(\hat{\mu}_n|_{[0,1]}; z)$ be the Szegő function of $\hat{\mu}_n|_{[0,1]}$, see (1.8). That is, G_n is an outer function in $H^2(D_{[0,1]})$ such that

$$|G_{n\pm}(x)|^2 = \hat{v}_{n,[0,1]}(x) \quad \text{for almost every } x \in (0, 1),$$

see Lemma 5.5. One can readily verify that

$$\left| \frac{\sqrt{z(z-1)}}{z-x} - 1 \right| \lesssim \frac{|\Delta_n|}{\rho}, \quad z \in \hat{K}_{n,\rho},$$

for all sufficiently large n . Hence, it follows from (1.4), (1.6), (1.8), and (5.28) that

$$(5.48) \quad \frac{G_n(z)}{G_n(\infty)} = \exp\left(\frac{1}{2} \int_0^1 \left(\frac{\sqrt{z(z-1)}}{z-x} - 1\right) \log \hat{v}_{n,[0,1]}(x) d\omega_{[0,1]}(x)\right) = 1 + o(1)$$

for $z \in \hat{K}_{n,\rho}$ as $n \rightarrow \infty$. To prove (5.47), define

$$g_n(z) := \frac{T_n(z)}{\|T_n\|_{L^2(\hat{w}_n^{2N} \hat{\mu}_n)}} \exp\left(-\hat{\ell}_n - n \int \log(z - s) d\hat{\omega}_n(s)\right) G_n(z).$$

In view of (5.48), (5.47) will follow if we show that

$$(5.49) \quad g_n(z)/g_n(\infty) = 1 + o(1), \quad z \in \hat{K}_{n,\rho},$$

as $n \rightarrow \infty$. Clearly, each g_n is a function in $H^2(D_{[0,1]})$. In (5.42), we have equality on $[0, 1]$, so

$$\int |g_{n\pm}|^2 d\omega_{[0,1]} = \|T_n\|_{L^2(\hat{w}_n^{2N} \hat{\mu}_n)}^{-2} \int |T_n|^2 \hat{w}_n^{2N} \hat{v}_{n,[0,1]} d\omega_{[0,1]} \leq 1.$$

Moreover, as we already observed above, (1.6) and (5.28) imply that

$$G_n(\infty) = \exp\left(\frac{1}{2} \int \log \hat{v}_{n,[0,1]} d\omega_{[0,1]}\right) \geq \exp(-C_5 |\Delta_n|^{-\gamma})$$

for some constant $C_5 > 0$. Since $\gamma > 1/2$, the bound (5.43) gives

$$g_n(\infty) = \frac{e^{-\hat{\ell}_n} G_n(\infty)}{\|T_n\|_{L^2(\hat{w}_n^{2N} \hat{\mu}_n)}} \geq \exp(-C_6 |\Delta_n|^{-\gamma})$$

for some constant $C_6 > 0$.

Next, we will use a conformal map to the unit disk $\mathbb{D}$ to apply the standard results of the Hardy spaces theory on $\mathbb{D}$. Define

$$\mathbf{g}_n(z) := g_n(\phi_{[0,1]}^{(-1)}(z)), \quad z \in \mathbb{D},$$

where $\phi_{[0,1]}^{(-1)}(z)$ denotes the inverse of $\phi_{[0,1]}(z)$, the conformal map introduced in (1.3). To show (5.49), it is enough to check that

$$(5.50) \quad \mathbf{g}_n(z)/\mathbf{g}_n(0) = 1 + o(1), \quad |z| \leq |\Delta_n|/(2\rho),$$

as $n \rightarrow \infty$. Naturally, $\mathbf{g}_n(z)$ is a function in $H^2(\mathbb{D})$ such that

$$(5.51) \quad \frac{1}{2\pi} \int_{\mathbb{T}} |\mathbf{g}_n(\xi)|^2 |d\xi| \leq 1 \quad \text{and} \quad \mathbf{g}_n(0) \geq \exp(-C_6 |\Delta_n|^{-\gamma}).$$

We can factor $\mathbf{g}_n$ as

$$(5.52) \quad \mathbf{g}_n(z) = \mathbf{w}_n(z) \mathbf{b}_n(z), \quad \mathbf{b}_n(z) = \iota_n \prod_i \frac{z - a_{n,i}}{1 - \overline{a_{n,i}}z},$$

where $\mathbf{w}_n$ is outer and $\iota_n \in \{\pm 1\}$ is chosen so that $\mathbf{b}_n(0) > 0$ (because polynomials T_n are obviously continuous in $\mathbb{C}$ and G_n is outer in $H^2(D_{[0,1]})$, there is no singular inner function in the above decomposition). Observe that the zeros $a_{n,i}$ are all real and are precisely the images of the zeros T_n that lie outside of $[0, 1]$ under the map $\phi_{[0,1]}$.

Since Blaschke products are unimodular on the unit circle, it follows from the first estimate in (5.51) that

$$0 \leq \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |\mathbf{w}_n(\xi)| |d\xi| \leq \frac{1}{4\pi} \int_{\mathbb{T}} \mathbf{w}_n^2(\xi) |d\xi| \leq \frac{1}{2}.$$

On the other hand, since $|\mathbf{b}_n(z)| \leq 1$ in $\mathbb{D}$, it holds that $\mathbf{w}_n(0) \geq e^{-C_6 |\Delta_n|^{-\gamma}}$ by the second estimate in (5.51). In particular, we have that

$$0 \geq \frac{1}{2\pi} \int_{\mathbb{T}} \log^- |\mathbf{w}_n(\xi)| |d\xi| = \log \mathbf{w}_n(0) - \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |\mathbf{w}_n(\xi)| |d\xi| \gtrsim -|\Delta_n|^{-\gamma}.$$

Altogether, we see that

$$0 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |\log |\mathbf{w}_n(\xi)|| |d\xi| \lesssim |\Delta_n|^{-\gamma}.$$

Therefore, similarly to (5.48), it holds that

$$\frac{\mathbf{w}_n(z)}{\mathbf{w}_n(0)} = \exp \left(\frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{\xi + z}{\xi - z} - 1 \right) \log |\mathbf{w}_n(\xi)| |d\xi| \right) = 1 + \mathcal{O}(|\Delta_n|^{1-\gamma})$$

for $|z| < |\Delta_n|/(2\rho)$ and all large enough n . Thus, in view of (5.52) and the above asymptotic formula, to prove (5.50), it is sufficient to show that

$$(5.53) \quad \mathbf{b}_n(z)/\mathbf{b}_n(0) = 1 + o(1), \quad |z| \leq |\Delta_n|/(2\rho), \quad \text{as } n \rightarrow \infty.$$

Let $x_{n,i}$ be the zero of T_n corresponding to $a_{n,i}$. Then, from Lemma 5.8, it follows that

$$|a_{n,i}^{-1} - a_{n,i}| = 4|x_{n,i}(x_{n,i} - 1)|^{1/2} \lesssim \max \{1, n^{-1}|\Delta_n|^{-\gamma}\}.$$

We consider two cases.

1. Assume first that $n^{-1}|\Delta_n|^{-\gamma} \geq 1$. Then, for $|z| \leq |\Delta_n|/(2\rho)$, we get

$$(5.54) \quad |z| \sum_i |a_{n,i}^{-1} - a_{n,i}| \lesssim |\Delta_n|^{1-\gamma}$$

because the sum above has at most n terms.

2. Now, suppose that $n^{-1}|\Delta_n|^{-\gamma} < 1$. In particular, this implies that the zeros $a_{n,i}$ are all separated away from zero by a constant independent of n . Since $|w_n(\xi)| = |g_n(\xi)|$ for $\xi \in \mathbb{T}$, the mean-value inequality and bound (5.51) imply

$$|w_n(0)|^2 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |w_n(\xi)|^2 |d\xi| = \frac{1}{2\pi} \int_{\mathbb{T}} |g_n(\xi)|^2 |d\xi| \stackrel{(5.51)}{\leq} 1.$$

Therefore, $b_n(0) \geq e^{-C_6|\Delta_n|^{-\gamma}}$ by the second estimate in (5.51). Respectively, we have for $|z| \leq |\Delta_n|/(2\rho)$ that

$$\begin{aligned} |z| \sum_i |a_{n,i}^{-1} - a_{n,i}| &\lesssim |z| \sum_i (1 - |a_{n,i}|) \leq -|z| \sum_i \log |a_{n,i}| \\ (5.55) \qquad \qquad \qquad &= -|z| \log b_n(0) \lesssim |\Delta_n|^{1-\gamma}. \end{aligned}$$

Estimates (5.54) and (5.55) justify the first inequality below and give

$$\left| \sum_i \log \left(1 - z \frac{a_{n,i}^{-1} - a_{n,i}}{1 - a_{n,i}z} \right) \right| \lesssim |z| \sum_i |a_{n,i}^{-1} - a_{n,i}| \lesssim |\Delta_n|^{1-\gamma}.$$

It is straightforward to see that the above estimates yield (5.53), which finishes the proof of the theorem.

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