SOBOLEV NORMS OF L^2- SOLUTIONS TO NLS

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Abstract. We apply inverse spectral theory to study Sobolev norms of solutions to the nonlinear Schrödinger equation. For initial datum $q_0 \in L^2(\mathbb{R})$ and $s \in [-1, 0]$, we prove that there exists a conserved quantity which is equivalent to $H^s(\mathbb{R})$-norm of the solution.

1. Introduction

In the last two decades, the theory of polynomials orthogonal on the unit circle (OPUC) has been used to obtain some of the strongest results in the spectral theory (see, e.g., [9, 17, 18]). In [3], the authors of the present paper have applied OPUC techniques to characterize existence and completeness of wave operators for the Dirac evolution on the half-line. One area where scattering theory for Dirac systems finds applications is the so-called inverse scattering approach to the nonlinear Schrödinger equation (NLS). Below, we develop a general framework that enables one to use the theory of Krein systems (a continuous analog of OPUC [13]) in the context of NLS. To illustrate our approach, we study the Sobolev norms of solutions to NLS adding to the area which attracted much attention in recent years [5–7, 15, 19–21, 23]. Our Theorem 1.2 stated below is not new and can be deduced from the results of Koch and Tataru [21] or by using an alternative method of Killip, Visan and Zhang [19]. However, we have developed a new and promising approach to that problem which adapts the technique from [3] to the setting of NLS and shows, in particular, that the sharp regularity class used to characterize scattering in the Dirac system can be studied in the context of Sobolev spaces. Then, we employ our analysis to obtain Theorem 1.2 which represents the first step in applying methods of [3] to NLS. In the current paper, we also develop a convenient language which we hope can be used by the spectral theory community to further study NLS dynamics.

Turning to the actual content of the paper, consider the classical defocusing nonlinear Schrödinger equation (NLS) [14, 26, 30] on the real line,

$$\begin{cases} i\frac{\partial q}{\partial t} = -\frac{\partial^2 q}{\partial \xi^2} + 2|q|^2 q, \\ q\big|_{t=0} = q_0, \end{cases} \qquad \xi \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{1.1}$$

It is known that for sufficiently regular initial datum q_0 the unique classical solution $q = q(\xi, t)$ exists globally in time. For example, if q_0 lies in the Schwartz class $\mathcal{S}(\mathbb{R})$, then $q(\cdot, t) \in \mathcal{S}(\mathbb{R})$ for all $t \in \mathbb{R}$. The long-time asymptotics of q is known [10, 11, 29]. For less regular initial datum q_0, one can define the solution by an approximation argument (see, e.g., [28]):

Theorem 1.1. *Let $q_0 \in L^2(\mathbb{R})$, and let $q_{0,n} \in \mathcal{S}(\mathbb{R})$ converge to q_0 in $L^2(\mathbb{R})$. Denote by $q_n(\xi, t)$ the solution of (1.1) corresponding to $q_{0,n}$. We have*

$$\lim_{n\to+\infty} \|q_n(\cdot, t) - q(\cdot, t)\|_{L^2(\mathbb{R})} = 0, \qquad t \in \mathbb{R},$$

for some function $q(\xi, t) : \mathbb{R}^2 \to \mathbb{R}$ that does not depend on the choice of the sequence $q_{0,n}$.

The function q in Theorem 1.1 is called the L^2–solution of (1.1) corresponding to the initial datum $q_0 \in L^2(\mathbb{R})$. It is clear that such a solution is unique. The total energy of the solution is its $L^2(\mathbb{R})$-norm and it is conserved in time:

$$\|q(\cdot, t)\|_{L^2(\mathbb{R})} = \|q_0\|_{L^2(\mathbb{R})}, \qquad t \in \mathbb{R}.$$

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By Plancherel's formula, it is equal to $\|(\mathcal{F}q)(\cdot,t)\|_{L^2(\mathbb{R})}$ where $\mathcal{F}$ stands for the Fourier transform. In this paper, we work with Sobolev spaces $H^s(\mathbb{R})$, $s\in\mathbb{R}$. The $H^s(\mathbb{R})$-norm of a function $f\in\mathcal{S}(\mathbb{R})$ is defined by

$$\|f\|_{H^s(\mathbb{R})}=\left(\int_{\mathbb{R}}(1+|\eta|^2)^s|(\mathcal{F}f)(\eta)|^2d\eta\right)^{\frac12}. \tag{1.2}$$

The space $H^s(\mathbb{R})$ is the completion of $\mathcal{S}(\mathbb{R})$ with respect to this norm. Equivalently, one can define it by

$$H^s(\mathbb{R})=\{f\in\mathcal{S}'(\mathbb{R}):(1+|\eta|^2)^{\frac{s}{2}}\mathcal{F}f\in L^2(\mathbb{R})\},$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distribution.

In contrast to the linear Schrödinger equation for which all Sobolev norms are conserved, the solutions of NLS can exhibit inflation of Sobolev norm $H^s(\mathbb{R})$ for $s\leqslant-\frac12$ (see, e.g., [8,20] for details). Specifically, given an arbitrarily small positive ε and $s\leqslant-\frac12$, there exists a solution q to (1.1) that satisfies

$$q_0\in\mathcal{S}(\mathbb{R}),\qquad \|q_0\|_{H^s(\mathbb{R})}\leqslant\varepsilon,\qquad \|q(\cdot,\varepsilon)\|_{H^s(\mathbb{R})}\geqslant\varepsilon^{-1}, \tag{1.3}$$

see [8] for that construction. This result is related to the "high-to-low frequency cascade". It occurs when for initial datum $q_0\in\mathcal{S}(\mathbb{R})$, a part of $L^2(\mathbb{R})$-norm of q, when written on the Fourier side, moves from high to low frequencies as time increases. The Sobolev norms with negative index s can be used to capture this phenomenon. Indeed, since $\|q(\cdot,t)\|_{L^2(\mathbb{R})}$ is time-invariant and the weight $(1+\eta^2)^s$ in (1.2) vanishes at infinity when $s<0$, the transfer of L^2-norm from high to low values of frequency η makes the $H^s(\mathbb{R})$-norm grow.

For NLS, the inflation of $H^s(\mathbb{R})$-norm can not happen for $s>-\frac12$. In [21], Koch and Tataru discovered the set of conserved quantities which agree with $H^s(\mathbb{R})$-norm up to a quadratic term for a small value of $\|q_0\|_{H^s(\mathbb{R})}$ and $s>-\frac12$. As a corollary, they obtained the bounds on $\|q(\cdot,t)\|_{H^s(\mathbb{R})}$ that are uniform in time:

$$\|q(\cdot,t)\|_{H^s(\mathbb{R})}\leqslant C(s)\begin{cases}\mathcal{R}+\mathcal{R}^{1+2s}, & s>0,\\ \mathcal{R}+\mathcal{R}^{\frac{1+4s}{1+2s}}, & s\in(-\frac12,0),\end{cases}\qquad \mathcal{R}=\|q_0\|_{H^s(\mathbb{R})}. \tag{1.4}$$

In [19], Killip, Vişan, and Zhang proved a similar estimate using a different method. The estimates on the growth of $H^s(\mathbb{R})$-norms are related to questions of well-posedness and ill-posedness of NLS in Sobolev classes which have been extensively studied previously, see, e.g., [5–7,15,19–21,23].

In our paper, we use some recent results in the inverse spectral theory [1–3] to show that there are conserved quantities of NLS which agree with $H^s(\mathbb{R})$-norm provided that $s\in[-1,0]$ and the value of $\|q_0\|_{L^2(\mathbb{R})}$ is under control. We apply our analysis to prove the following theorem.

Theorem 1.2. *Let $q_0\in L^2(\mathbb{R})$ and let $q=q(\xi,t)$ be the solution of (1.1) corresponding to q_0. Then,*

$$C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{2s}\|q_0\|_{H^s(\mathbb{R})}\leqslant\|q(\cdot,t)\|_{H^s(\mathbb{R})}\leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^{-2s}\|q_0\|_{H^s(\mathbb{R})}, \tag{1.5}$$

where $t\in\mathbb{R}$, $s\in[-1,0]$, and C_1 and C_2 are two positive absolute constants.

This result shows, in particular, that for a given function $q_0:\|q_0\|_{L^2(\mathbb{R})}=1$ whose $L^2(\mathbb{R})$-norm is concentrated on high frequencies, we will never see a significant part of $L^2(\mathbb{R})$-norm of the solution q moving to the low frequencies. That limits the "high-to-low frequency cascade" we discussed above. The close inspection of construction used in [8] shows that the function q_0 in (1.3) has $H^s(\mathbb{R})$-norm smaller than ε but its $L^2(\mathbb{R})$-norm is large when ε is small. Hence, the bounds in Theorem 1.2 do not contradict the estimates in (1.3) when $s\in[-1,-\frac12]$. We do not know whether Theorem 1.2 holds for $s<-1$.

The main idea of the proof of Theorem 1.2 is based on the analysis of the conserved quantity $a(z)$, $\operatorname{Im}z>0$, which is a coefficient in the transition matrix for the Dirac equation with potential $q=q(\cdot,t)$. We take $z=i$ and show that $\log|a(i)|$ is related to a certain quantity $\widetilde{\mathcal{K}}_Q$ (see the Lemma 3.3 below) that characterizes both size and oscillation of q. Using $\widetilde{\mathcal{K}}_Q$ in the context of NLS is the main novelty of our work. We study $\widetilde{\mathcal{K}}_Q$ and show that it is equivalent to $H^{-1}(\mathbb{R})$ norm of q with constants that depend on its $L^2(\mathbb{R})$-norm. That gives the estimate (1.5) for $s=-1$ and the intermediate range of $s\in(-1,0)$ is handled by interpolation. Our analysis relies heavily on the recent results [1–3] that characterize Krein – de Branges canonical systems and the Dirac operators whose spectral measures belong to the Szegő class on the real line. We also establish the framework that allows working with NLS in the context of well-studied Krein systems.

Notation

- The symbol I stands for 2×2 identity matrix $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and symbol J stands for $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Constant matrices σ_3, $\sigma_\pm$, σ are defined in (2.2).
- For a measurable set $S\subset\mathbb{R}$, we say that $f\in L^1_{\rm loc}(S)$ is $f\in L^1(K)$ for every compact $K\subset S$.
- The Fourier transform of a function f is defined by
$$(\mathcal{F}f)(\eta)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\eta x}\,dx.$$
- The symbol C, unless we specify explicitly, denotes the absolute constant which can change the value from formula to formula. If we write, e.g., $C(\alpha)$, this defines a positive function of parameter α.
- For two non-negative functions f_1 and f_2, we write $f_1\lesssim f_2$ if there is an absolute constant C such that $f_1\leqslant Cf_2$ for all values of the arguments of f_1 and f_2. We define $\gtrsim$ similarly and say that $f_1\sim f_2$ if $f_1\lesssim f_2$ and $f_2\lesssim f_1$ simultaneously. If $|f_3|\lesssim f_4$, we will write $f_3=O(f_4)$.
- Symbols $\{e_j\}$ are reserved for the standard basis in $\mathbb{C}^2$: $e_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.
- For matrix A, the symbol $\|A\|_{\rm HS}$ denotes its Hilbert-Schmidt norm: $\|A\|_{\rm HS}=({\rm tr}(A^*A))^{\frac12}$.

2. Preliminaries

Our proof of Theorem 1.1 uses complete integrability of equation (1.1). In that framework, (1.1) can be solved by using the method of inverse scattering which we discuss next following [14].

2.1. **The inverse scattering approach to NLS.** Given a complex-valued function $q\in\mathcal{S}(\mathbb{R})$, define the differential operator
$$L_q=i\sigma_3\frac{d}{d\xi}+i(q\sigma_- -\overline{q}\sigma_+),\tag{2.1}$$
where we borrow notation for constant matrices σ_3, $\sigma_\pm$ from [14]:
$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \sigma_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad \sigma=\sigma_-+\sigma_+=\begin{pmatrix}0&1\\1&0\end{pmatrix}.\tag{2.2}$$
The expression L_q is one of the forms in which the Dirac operator can be written. In Section 3, we will introduce another form and will show how the two are related. Let us also define
$$E(\xi,\lambda)=e^{\frac{\lambda}{2i}\xi\sigma_3}=\begin{pmatrix}e^{\frac{\lambda}{2i}\xi}&0\\0&e^{-\frac{\lambda}{2i}\xi}\end{pmatrix},$$
as in [14]. In the free case when $q=0$, the matrix-function E solves $L_0E=\frac{\lambda}{2}E$, $E(0,\lambda)=I$. Since $q\in\mathcal{S}(\mathbb{R})$, it decays at infinity fast and therefore one can find two solutions $T_\pm=T_\pm(\xi,\lambda)$ such that
$$L_qT_\pm=\frac{\lambda}{2}T_\pm,\qquad T_\pm=E(\xi,\lambda)+o(1),\qquad \xi\to\pm\infty,\tag{2.3}$$
for every $\lambda\in\mathbb{R}$. These solutions are called the *Jost solutions* for L_q. Since both T_+ and T_- solve the same ODE, they must satisfy
$$T_-(\xi,\lambda)=T_+(\xi,\lambda)T(\lambda),\qquad \xi\in\mathbb{R},\qquad \lambda\in\mathbb{R},\tag{2.4}$$
where the matrix $T=T(\lambda)$ does not depend on $\xi\in\mathbb{R}$. One can show that it has the form
$$T(\lambda)=\begin{pmatrix}a(\lambda)&\overline{b(\lambda)}\\b(\lambda)&\overline{a(\lambda)}\end{pmatrix},\qquad \det T=|a|^2-|b|^2=1.\tag{2.5}$$
The matrix T is called the *reduced transition matrix* for L_q, and the ratio $\mathbf{r}_q=b/a$ is called the *reflection coefficient* for L_q. One can obtain T in a different way: let $Z_q=Z_q(\xi,\lambda)$, $\xi\in\mathbb{R}$, $\lambda\in\mathbb{C}$ be the fundamental matrix for L_q, that is,
$$L_qZ_q=\frac{\lambda}{2}Z_q,\qquad Z_q(0,\lambda)=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right).\tag{2.6}$$

Then, we have $Z_q(\xi,\lambda) = T_\pm(\xi,\lambda)T_\pm^{-1}(0,\lambda)$ and the pointwise limits

$$T_\pm^{-1}(0,\lambda) = \lim_{\xi\to\pm\infty} E^{-1}(\xi,\lambda)Z_q(\xi,\lambda) \tag{2.7}$$

exist for every $\lambda\in\mathbb{R}$. Moreover, we have $T(\lambda) = T_+^{-1}(0,\lambda)T_-(0,\lambda)$ on $\mathbb{R}$.

The coefficients a, b, and $\mathbf{r}_q$ were defined for $\lambda\in\mathbb{R}$ and they satisfy $|a|^2 = 1+|b|^2$, $1-|\mathbf{r}_q|^2 = |a|^{-2}$ for these λ. However, one can show that $a(\lambda)$ is the boundary value of the outer function defined in $\mathbb{C}_+ = \{z\in\mathbb{C}: \operatorname{Im} z>0\}$ by the formula (see (6.22) in [14])

$$a(z) = \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}} \frac{1}{\lambda - z}\log|a(\lambda)|\,d\lambda\right), \quad z\in\mathbb{C}_+,$$

which, in view of identity $1-|\mathbf{r}_q|^2 = |a|^{-2}$ on $\mathbb{R}$, can be written as

$$a(z) = \exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{\lambda - z}\log(1-|\mathbf{r}_q(\lambda)|^2)d\lambda\right). \tag{2.8}$$

That shows, in particular, that b defines both a and $\mathbf{r}_q$, and $\mathbf{r}_q$ defines a and b.

The map $q\mapsto\mathbf{r}_q$ is called the direct scattering transform and its inverse is called the inverse scattering transform. These maps are well-studied when $q\in\mathcal{S}(\mathbb{R})$. In particular, we have the following result (see [14] for the proof).

Theorem 2.1. *The map $q\mapsto\mathbf{r}_q$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto the set of complex-valued functions $\{\mathbf{r}\in\mathcal{S}(\mathbb{R}), \|\mathbf{r}\|_{L^\infty(\mathbb{R})}<1\}$.*

The scattering transform has some symmetries:

Lemma 2.1. *If $q\in\mathcal{S}(\mathbb{R})$ and $\lambda\in\mathbb{R}$, then*

$$\begin{aligned}
&\text{(dilation):} && \mathbf{r}_{\alpha q(\alpha\xi)}(\lambda) = \mathbf{r}_{q(\xi)}(\alpha^{-1}\lambda), \quad \alpha>0,\\
&\text{(conjugation):} && \mathbf{r}_{\overline{q}(\xi)}(\lambda) = \overline{\mathbf{r}_{q(\xi)}(-\lambda)},\\
&\text{(translation):} && \mathbf{r}_{q(\xi-\ell)}(\lambda) = \mathbf{r}_{q(\xi)}(\lambda)e^{-i\lambda\ell}, \quad \ell\in\mathbb{R},\\
&\text{(modulation):} && \mathbf{r}_{e^{-i\beta\xi}q(\xi)}(\lambda) = \mathbf{r}_{q(\xi)}(\lambda+\beta), \quad \beta\in\mathbb{R}.\\
&\text{(rotation):} && \mathbf{r}_{\mu q(\xi)}(\lambda) = \mu\mathbf{r}_{q(\xi)}(\lambda), \quad \mu\in\mathbb{C},\ |\mu|=1.
\end{aligned}$$

Proof. Indeed, the direct substitution into (2.3) shows that if $T_\pm(\xi,\lambda)$ are Jost solutions for $q(\xi)$, then

(*a*) $T_\pm(\alpha\xi,\alpha^{-1}\lambda)$ are the Jost solutions for $\alpha q(\alpha\xi)$,
(*b*) $\overline{T}_\pm(\xi,-\lambda)$ are the Jost solutions for $\overline{q(\xi)}$,
(*c*) $T_\pm(\xi-\ell,\lambda)E(\ell,\lambda)$ are the Jost solutions for $q(\xi-\ell)$,
(*d*) $E(-\xi,\beta)T_\pm(\xi,\lambda+\beta)$ are the Jost solutions for $e^{-i\beta\xi}q(\xi)$,
(*e*) $\left(\begin{smallmatrix}1&0\\0&\mu\end{smallmatrix}\right)T_\pm(\xi,\lambda)\left(\begin{smallmatrix}1&0\\0&\bar{\mu}\end{smallmatrix}\right)$ are the Jost solutions for $\mu q(\xi)$, $|\mu|=1$.

Now, it is left to use the formula (2.4) which defines T. A computation using (2.5) shows how a and b change under symmetries (*a*)–(*e*). For example, the translation does not change a and it multiplies b by $e^{-i\lambda l}$. The modulation $e^{-i\beta\xi}q(\xi)$, however, gives $a_{e^{-i\beta\xi}q(\xi)}(\lambda) = a_{q(\xi)}(\lambda+\beta)$. Then, the claim follows from the definition of the reflection coefficient $\mathbf{r}_q = b/a$. □

The next result (see formula (7.5) in [14]), along with the previous theorem, shows how the inverse scattering transform can be used to solve (1.1).

Theorem 2.2. *Let $q_0\in\mathcal{S}(\mathbb{R})$ and let $\mathbf{r}_{q_0} = \mathbf{r}_{q_0}(\lambda)$ be the reflection coefficient of L_{q_0}. Define the family*

$$\mathbf{r}(\lambda,t) = e^{-i\lambda^2 t}\mathbf{r}_{q_0}(\lambda), \qquad \lambda\in\mathbb{R}, \quad t\in\mathbb{R}. \tag{2.9}$$

For each $t\in\mathbb{R}$, let $q = q(\xi,t)$ be the potential in the previous theorem generated by $\mathbf{r}(\lambda,t)$. Then, $q = q(\xi,t)$ is the unique classical solution of (1.1) with the initial datum q_0. Moreover, for every $t\in\mathbb{R}$, the function $\xi\mapsto q(\xi,t)$ lies in $\mathcal{S}(\mathbb{R})$.

The solutions to NLS equation

$$i\frac{\partial q}{\partial t} = -\frac{\partial^2 q}{\partial \xi^2} + 2|q|^2 q \tag{2.10}$$

behave in an explicit way under some transformations. Specifically, we have

- (a) Dilation: if $q(\xi,t)$ solves (2.10), then $\alpha q(\alpha\xi,\alpha^2 t)$ solves (2.10) for every $\alpha\neq 0$.
- (b) Time reversal: if $q(\xi,t)$ solves (2.10), then $\overline{q}(\xi,-t)$ solves (2.10). In particular, if q_0 is real-valued, then $q(\xi,t)=q(\xi,-t)$.
- (c) Translation: if $q(\xi,t)$ solves (2.10), then $q(\xi-\ell,t)$ solves (2.10) for every $\ell\in\mathbb{R}$.
- (d) Modulation or Galilean symmetry: if $q(\xi,t)$ solves (2.10), then $e^{iv\xi-iv^2t}q(\xi-2vt,t)$ solves (2.10) for every $v\in\mathbb{R}$.
- (e) Rotation: if $q(\xi,t)$ solves (2.10), then $\mu q(\xi,t)$ solves (2.10) for every $\mu\in\mathbb{C},|\mu|=1$.

These properties can be checked by direct calculation (see, e.g., formula (1.19) in [15] for (d)) and a simple inspection shows that the bound (1.5) is consistent with all these transformations. The statements of Theorem 2.2 and Lemma 2.1 are consistent with these symmetries as well.

Now, we can explain the idea behind the proof of the Theorem 1.2.

The idea of the proof for Theorem 1.2. One can proceed as follows. First, we assume that $q_0\in\mathcal{S}(\mathbb{R})$ and notice that conservation of $|\mathbf{r}(\lambda,t)|$, $\lambda\in\mathbb{R}$, guaranteed by (2.9), yields that $\log|a(i,t)|$ is conserved, where $a(z,t)$ is defined for $z\in\mathbb{C}_+$ by (2.8). Separately, for every Dirac operator L_q with $q\in L^2(\mathbb{R})$, we show that $\log|a(i)|$ is equivalent to some explicit quantity $\widetilde{\mathcal{K}}_Q$ that involves q. That quantity was introduced and studied in [1–3]: it resembles the matrix Muckenhoupt $A_2(\mathbb{R})$ condition and it is equivalent to $H^{-1}(\mathbb{R})$ norm of q provided that $\|q\|_{L^2(\mathbb{R})}$ is under control, e.g., $\|q\|_{L^2(\mathbb{R})}<C$ with some fixed C. Putting things together, we see that Sobolev $H^{-1}(\mathbb{R})$ norm of $q(\cdot,t)$ does not change much in time provided that the bound $\|q(\cdot,t)\|_{L^2(\mathbb{R})}<C$ holds. Since $\|q(\cdot,t)\|_{L^2(\mathbb{R})}=\|q_0\|_{L^2(\mathbb{R})}$ is time-invariant, we arrive to the statement of Theorem 1.2 for $q_0\in\mathcal{S}(\mathbb{R})$ and $s=-1$. For $s=0$, the claim of Theorem 1.2 is trivial. The intermediate range of $s\in(-1,0)$ is handled by interpolation using Galilean invariance of NLS. The general case when $q_0\in L^2(\mathbb{R})$ follows by a density argument if one uses the stability of L^2-solutions guaranteed by Theorem 1.1.

There are other methods that use conserved quantities that agree with negative Sobolev norms. The paper [19] uses a representation of $\log|a(i)|$ through a perturbation determinant. Then, the analysis of the perturbation series allows the authors of [19] to obtain estimates similar to (1.4). It is conceivable that this approach can provide results along the same lines as Theorem 1.2.

To focus on the Dirac operator with $q\in L^2(\mathbb{R})$, we first consider this operator on half-line $\mathbb{R}_+$ in connection to Krein systems that were introduced in [22].

2.2. **Operator L_q and Krein system.** Let $A:\mathbb{R}_+\to\mathbb{C}$ be a function on the positive half-line $\mathbb{R}_+=[0,+\infty)$ such that
$$\int_0^r|A(\xi)|\,d\xi<\infty,$$
for every $r\geqslant 0$. Recall that we denote the set of such functions by $L^1_{\rm loc}(\mathbb{R}_+)$. The Krein system (see the formula (4.52) in [13]) with the coefficient A has the form
$$\begin{cases}P'(\xi,\lambda)=i\lambda P(\xi,\lambda)-\overline{A(\xi)}P_*(\xi,\lambda), & P(0,\lambda)=1,\\ P_*'(\xi,\lambda)=-A(\xi)P(\xi,\lambda), & P_*(0,\lambda)=1,\end{cases}\tag{2.11}$$
where the derivative is taken with respect to $\xi\in\mathbb{R}_+$ and $\lambda\in\mathbb{C}$. Let also
$$\begin{cases}\widehat{P}'(\xi,\lambda)=i\lambda\widehat{P}(\xi,\lambda)+\overline{A(\xi)}\widehat{P}_*(\xi,\lambda), & \widehat{P}(0,\lambda)=1,\\ \widehat{P}_*'(\xi,\lambda)=A(\xi)\widehat{P}(\xi,\lambda), & \widehat{P}_*(0,\lambda)=1,\end{cases}\tag{2.12}$$
denote the so-called dual Krein system (see Corollary 5.7 in [13]). Set
$$Y(\xi,\lambda)=e^{-i\lambda\xi}\begin{pmatrix}P(2\xi,\lambda) & i\widehat{P}(2\xi,\lambda)\\ P_*(2\xi,\lambda) & -i\widehat{P}_*(2\xi,\lambda)\end{pmatrix}.\tag{2.13}$$
The matrix-function Z_q, which was defined in (2.6) for $q\in\mathcal{S}(\mathbb{R})$, makes sense if we assume that $q\in L^1_{\rm loc}(\mathbb{R})$. In the next lemma, we relate Y to Z_q.

Lemma 2.2. *Let $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, $A(2\xi) = -\overline{q(\xi)}/2$ on $\mathbb{R}_+$, and Y be the corresponding matrix-valued function defined by (2.13). Then, $Z_q(\xi, 2\lambda) = \sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$ for $\xi \geqslant 0$ and $\lambda \in \mathbb{C}$.*

Proof. The proof is a computation. We have

$$\begin{aligned} L_{\bar q} Y(\xi,\lambda) =& \lambda\sigma_3 Y(\xi,\lambda) + i\sigma_3 e^{-i\lambda\xi}\frac{d}{d\xi}\begin{pmatrix} P(2\xi,\lambda) & i\widehat{P}(2\xi,\lambda) \\ P_*(2\xi,\lambda) & -i\widehat{P}_*(2\xi,\lambda)\end{pmatrix} + i(\bar q\sigma_- - q\sigma_+)Y(\xi,\lambda), \\ =& 2i\sigma_3 e^{-i\lambda\xi}\begin{pmatrix} i\lambda P(2\xi,\lambda) - \overline{A(2\xi)}P_*(2\xi,\lambda) & -\lambda\widehat{P}(2\xi,\lambda) + i\overline{A(2\xi)}\widehat{P}_*(2\xi,\lambda) \\ -A(2\xi)P(2\xi,\lambda) & -iA(2\xi)\widehat{P}(2\xi,\lambda)\end{pmatrix} \\ &+ i(\bar q\sigma_- - q\sigma_+ - i\lambda\sigma_3)Y(\xi,\lambda). \end{aligned}$$

Notice, that

$$\begin{aligned} i(\bar q\sigma_- - q\sigma_+ - i\lambda\sigma_3)Y(\xi,\lambda) &= ie^{-i\lambda\xi}\begin{pmatrix} -i\lambda & -q \\ \bar q & i\lambda\end{pmatrix}\begin{pmatrix} P(2\xi,\lambda) & i\widehat{P}(2\xi,\lambda) \\ P_*(2\xi,\lambda) & -i\widehat{P}_*(2\xi,\lambda)\end{pmatrix} \\ &= ie^{-i\lambda\xi}\begin{pmatrix} -i\lambda P(2\xi,\lambda) - qP_*(2\xi,\lambda) & \lambda\widehat{P}(2\xi,\lambda) + iq\widehat{P}_*(2\xi,\lambda) \\ \bar q P(2\xi,\lambda) + i\lambda P_*(2\xi,\lambda) & i\bar q\widehat{P}(2\xi,\lambda) + \lambda\widehat{P}_*(2\xi,\lambda)\end{pmatrix}. \end{aligned}$$

Using relation $2A(2\xi) + \bar q(\xi) = 0$, we obtain

$$L_{\bar q} Y(\xi,\lambda) = ie^{-i\lambda\xi}\begin{pmatrix} i\lambda P(2\xi,\lambda) & -\lambda\widehat{P}(2\xi,\lambda) \\ i\lambda P_*(2\xi,\lambda) & \lambda\widehat{P}_*(2\xi,\lambda)\end{pmatrix} = -\lambda Y(\xi,\lambda).$$

Since $\sigma\sigma_3\sigma = -\sigma_3$ and $\sigma\sigma_\pm\sigma = \sigma_\mp$, one has $\sigma L_{\bar q}\sigma = -L_q$. Therefore,

$$L_q(\sigma Y(\xi,\lambda)\sigma) = \lambda(\sigma Y(\xi,\lambda)\sigma).$$

It follows that matrix-valued functions $Z_q(\xi,2\lambda)$ and $\sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$ solve the same Cauchy problem and thus $Z_q(\xi,2\lambda) = \sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$, as required. □

Lemma 2.3. *Let $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, let $A(2\xi) = q(-\xi)/2$ on $\mathbb{R}_+$, and let Y be the corresponding matrix-valued function defined by (2.13). Then, $Z_q(-\xi, 2\lambda) = Y(\xi,\lambda)Y^{-1}(0,\lambda)$ for $\xi \geqslant 0$ and $\lambda \in \mathbb{C}$.*

Proof. Recall that matrices σ_3, $\sigma_\pm$, σ are defined in (2.2). Using relations $\sigma\sigma_3\sigma = -\sigma_3$ and $\sigma\sigma_\pm\sigma = \sigma_\mp$, we see that $L_{\tilde q}\widetilde{Z}_q = \frac{\lambda}{2}\widetilde{Z}_q$, where $\tilde q(\xi) = -\overline{q(-\xi)}$ and $\widetilde{Z}_q(\xi,\lambda) = \sigma Z_q(-\xi,\lambda)\sigma$. Then, previous lemma applies to $\tilde q$, $Z_{\tilde q}(\xi,2\lambda) = \widetilde{Z}_q(\xi,2\lambda)$ and $A(2\xi) = -\overline{\tilde q(\xi)}/2 = q(-\xi)/2$. It gives $\widetilde{Z}_q(\xi,2\lambda) = \sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$. Returning to Z_q, we get $Z_q(-\xi,2\lambda) = Y(\xi,\lambda)Y^{-1}(0,\lambda)$. □

Given $q \in L^2(\mathbb{R})$, we define the continuous analogs of Wall polynomials (see [16] and Section 7 in [13]) by

$$\mathfrak{A}^\pm = \frac{P_*^\pm + \widehat{P}_*^\pm}{2}, \qquad \mathfrak{A}_*^\pm = \frac{P^\pm + \widehat{P}^\pm}{2}, \qquad \mathfrak{B}^\pm = \frac{P_*^\pm - \widehat{P}_*^\pm}{2}, \qquad \mathfrak{B}_*^\pm = \frac{P^\pm - \widehat{P}^\pm}{2}, \tag{2.14}$$

where $P^\pm$, $P_*^\pm$, $\widehat{P}^\pm$, $\widehat{P}_*^\pm$ are the solutions of systems (2.11), (2.12) for the coefficient $A^+(\xi) = -\overline{q(\xi/2)}/2$ from Lemma 2.2 and the coefficient $A^-(\xi) = q(-\xi/2)/2$ from Lemma 2.3, correspondingly. Functions $P^\pm$, $P_*^\pm$, $\widehat{P}^\pm$, $\widehat{P}_*^\pm$ are continuous analogs of polynomials orthogonal on the unit circle, they depend on two parameters: $\xi \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$ and they satisfy identities (see formula (4.32) in [13]):

$$P_*^\pm(\xi,\lambda) = e^{i\xi\lambda}\overline{P^\pm(\xi,\lambda)}, \qquad \widehat{P}_*^\pm(\xi,\lambda) = e^{i\xi\lambda}\overline{\widehat{P}^\pm(\xi,\lambda)} \tag{2.15}$$

for real λ.

We will use the following result (see Lemma 2 in [12] which contains a stronger statement).

Theorem 2.3. *Let $A \in L^2(\mathbb{R}_+)$, and let P, P_* be the solutions of system (2.11) for the coefficient A. Then, the limit*

$$\Pi(\lambda) = \lim_{\xi\to+\infty} P_*(\xi,\lambda) \tag{2.16}$$

exists for every $\lambda \in \mathbb{C}_+$. That function Π is outer in $\mathbb{C}_+$. If $\lambda \in \mathbb{R}$, the convergence in (2.16) holds in the Lebesgue measure on $\mathbb{R}$ where $\Pi(\lambda)$ now denotes the non-tangential boundary value of Π.

That theorem allows us to define

$$\mathfrak{a}^{\pm}(\lambda)=\lim_{\xi\to+\infty}\mathfrak{A}^{\pm}(\xi,\lambda),\qquad \mathfrak{b}^{\pm}(\lambda)=\lim_{\xi\to+\infty}\mathfrak{B}^{\pm}(\xi,\lambda) \tag{2.17}$$

for every $\lambda\in\mathbb{C}_+$ and for almost every $\lambda\in\mathbb{R}$. Moreover, Corollary 12.2 in [13] gives

$$|\mathfrak{a}^{\pm}(\lambda)|^2=1+|\mathfrak{b}^{\pm}(\lambda)|^2 \tag{2.18}$$

for a.e. $\lambda\in\mathbb{R}$. For every $\lambda\in\mathbb{C}_+$, we define

$$a(\lambda)=\mathfrak{a}^+(\lambda)\mathfrak{a}^-(\lambda)-\mathfrak{b}^+(\lambda)\mathfrak{b}^-(\lambda)\,.$$

Proposition 2.1. *The function a is outer in $\mathbb{C}_+$.*

Proof. We can write

$$a=\mathfrak{a}^+\mathfrak{a}^-(1-s^+s^-),\quad s^{\pm}=\frac{\mathfrak{b}^{\pm}}{\mathfrak{a}^{\pm}}\,.$$

It is known that $\mathfrak{a}^{\pm}$ are outer (see the formulas (12.9) and (12.29) in [13]) and that $s^{\pm}$ satisfy $|s^{\pm}|<1$ in $\mathbb{C}_+$. The function $1-s^+s^-$ has a positive real part in $\mathbb{C}_+$ and so is an outer function. That shows that a is a product of three outer functions and hence it is outer itself. □

Proposition 2.2. *Let $q\in L^2(\mathbb{R})$ and let Z_q be defined by (2.6). Then, the limits in (2.7) exist in the Lebesgue measure on $\mathbb{R}$. The matrix $T(\lambda)=T_+^{-1}(0,\lambda)T_-(0,\lambda)$ has the form (2.5) where*

$$a=\mathfrak{a}^+\mathfrak{a}^--\mathfrak{b}^+\mathfrak{b}^-,\qquad b=\mathfrak{a}^-\overline{\mathfrak{b}^+}-\mathfrak{b}^-\overline{\mathfrak{a}^+}\,, \tag{2.19}$$

and $\mathfrak{a}^{\pm}$, $\mathfrak{b}^{\pm}$ are defined Lebesgue almost everywhere on $\mathbb{R}$ by the convergence in (2.17) in measure.

Proof. If $q\in L^2(\mathbb{R})$, the fundamental matrix Z_q and the continuous Wall polynomials (2.14) are related by the formula

$$Z_q(\xi,2\lambda)=\begin{cases} e^{-i\lambda\xi}\begin{pmatrix}\mathfrak{A}^+(2\xi,\lambda) & \mathfrak{B}^+(2\xi,\lambda)\\ \mathfrak{B}^+_*(2\xi,\lambda) & \mathfrak{A}^+_*(2\xi,\lambda)\end{pmatrix}, & \xi\geqslant 0,\\ e^{i\lambda\xi}\begin{pmatrix}\mathfrak{A}^-_*(-2\xi,\lambda) & \mathfrak{B}^-_*(-2\xi,\lambda)\\ \mathfrak{B}^-(-2\xi,\lambda) & \mathfrak{A}^-(-2\xi,\lambda)\end{pmatrix}, & \xi<0.\end{cases} \tag{2.20}$$

Indeed, it is enough to use Lemma 2.2, Lemma 2.3, and the fact that $Y^{-1}(0,\lambda)=\frac12\left(\begin{smallmatrix}1 & 1\\ -i & i\end{smallmatrix}\right)$. Our next step is to prove that the limit

$$T_+^{-1}(0,2\lambda)=\lim_{\xi\to+\infty}E^{-1}(\xi,2\lambda)Z_q(\xi,2\lambda) \tag{2.21}$$

exists in Lebesgue measure when $\lambda\in\mathbb{R}$. From (2.15), we obtain

$$E^{-1}(\xi,2\lambda)Z_q(\xi,2\lambda)=\begin{pmatrix}1 & 0\\ 0 & e^{-2i\lambda\xi}\end{pmatrix}\begin{pmatrix}\mathfrak{A}^+(2\xi,\lambda) & \mathfrak{B}^+(2\xi,\lambda)\\ \mathfrak{B}^+_*(2\xi,\lambda) & \mathfrak{A}^+_*(2\xi,\lambda)\end{pmatrix}=\begin{pmatrix}\mathfrak{A}^+(2\xi,\lambda) & \mathfrak{B}^+(2\xi,\lambda)\\ \overline{\mathfrak{B}^+}(2\xi,\lambda) & \overline{\mathfrak{A}^+}(2\xi,\lambda)\end{pmatrix},$$

for every $\xi\geqslant 0$ and $\lambda\in\mathbb{R}$. Similarly,

$$E^{-1}(-\xi,2\lambda)Z_q(-\xi,2\lambda)=\begin{pmatrix}e^{-2i\lambda\xi} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathfrak{A}^-_*(2\xi,\lambda) & \mathfrak{B}^-_*(2\xi,\lambda)\\ \mathfrak{B}^-(2\xi,\lambda) & \mathfrak{A}^-(2\xi,\lambda)\end{pmatrix}=\begin{pmatrix}\overline{\mathfrak{A}^-}(2\xi,\lambda) & \overline{\mathfrak{B}^-}(2\xi,\lambda)\\ \mathfrak{B}^-(2\xi,\lambda) & \mathfrak{A}^-(2\xi,\lambda)\end{pmatrix}.$$

Hence, the limits

$$T_{\pm}^{-1}(0,2\lambda)=\lim_{\xi\to\pm\infty}E^{-1}(\xi,2\lambda)Z_q(\xi,2\lambda) \tag{2.22}$$

exist in Lebesgue measure on $\mathbb{R}$ by Theorem 2.3. Moreover,

$$\begin{aligned} T(2\lambda)=T_+^{-1}(0,2\lambda)T_-(0,2\lambda)&=\begin{pmatrix}\mathfrak{a}^+(\lambda) & \mathfrak{b}^+(\lambda)\\ \overline{\mathfrak{b}^+(\lambda)} & \overline{\mathfrak{a}^+(\lambda)}\end{pmatrix}\begin{pmatrix}\overline{\mathfrak{a}^-(\lambda)} & \overline{\mathfrak{b}^-(\lambda)}\\ \mathfrak{b}^-(\lambda) & \mathfrak{a}^-(\lambda)\end{pmatrix}^{-1}\\ &\overset{(2.18)}{=}\begin{pmatrix}\mathfrak{a}^+(\lambda) & \mathfrak{b}^+(\lambda)\\ \overline{\mathfrak{b}^+(\lambda)} & \overline{\mathfrak{a}^+(\lambda)}\end{pmatrix}\begin{pmatrix}\mathfrak{a}^-(\lambda) & -\overline{\mathfrak{b}^-(\lambda)}\\ -\mathfrak{b}^-(\lambda) & \overline{\mathfrak{a}^-(\lambda)}\end{pmatrix}=\begin{pmatrix}a(\lambda) & \overline{b(\lambda)}\\ b(\lambda) & \overline{a(\lambda)}\end{pmatrix}\end{aligned}$$

and the proposition follows. □

We end this section with a few remarks on reflection coefficients of potentials in $L^2(\mathbb{R})$. We have $|a|^2-|b|^2=1$ almost everywhere on $\mathbb{R}$ due to the fact that $\det T_{\pm}(0,\lambda)=1$ almost everywhere on $\mathbb{R}$.

That can also be established directly using (2.18). Proposition 2.2 then allows to define the reflection coefficient $\mathbf{r}_q = b/a$ for every $q \in L^2(\mathbb{R})$. The Lemma 2.1 holds for $\mathbf{r}_q$ in that case as well. However, not all results about scattering transform can be generalized from the case $q \in \mathcal{S}(\mathbb{R})$ to $q \in L^2(\mathbb{R})$. For example, scattering transform is injective on $\mathcal{S}(\mathbb{R})$ by Theorem 2.1, but it is not longer so when extended to $L^2(\mathbb{R})$ (see Example 6.1 in Appendix).

3. Another form of Dirac operator, $q \in L^2(\mathbb{R})$, and the entropy function.

Suppose $q \in L^2(\mathbb{R})$. The alternative to L_q form of writing Dirac operator on the line is given by an expression

$$\mathcal{D}_Q : X \mapsto JX' + QX, \qquad Q = \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix}. \tag{3.1}$$

$\mathcal{D}_Q$ is densely defined self-adjoint operator on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$ of functions $X : \mathbb{R} \to \mathbb{C}^2$ such that $\|X\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} = \int_{\mathbb{R}} \|X(\xi)\|^2_{\mathbb{C}^2}\, d\xi$ is finite. $\mathcal{D}_Q$ and L_q defined in (2.1) are related by a simple formula:

$$\mathcal{D}_Q = \Sigma L_q \Sigma^{-1}, \quad \Sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad \Sigma^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

One way to study $\mathcal{D}_Q$ is to focus on Dirac operators on half-line $\mathbb{R}_+$ first. Given $q \in L^2(\mathbb{R}_+)$, we define $\mathcal{D}_Q^+$ on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ by

$$\mathcal{D}_Q^+ : X \mapsto JX' + QX, \qquad Q = \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix} \tag{3.2}$$

on the dense subset of absolutely continuous functions $X \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ such that $\mathcal{D}_Q^+ X \in L^2(\mathbb{R}_+, \mathbb{C}^2)$, $X(0) = \left(\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right)$. We will call $\mathcal{D}_Q^+$ the Dirac operator defined on the positive half-line with boundary conditions $X(0) = \left(\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right)$ or simply the half-line Dirac operator. Set $A(\xi) = -\overline{q(\xi/2)}/2$ for $\xi \in \mathbb{R}_+$, and let $P(\xi, \lambda)$, $P_*(\xi, \lambda)$ be the solutions of Krein system (2.11) generated by A. The Krein system with coefficient A and Dirac equation (3.2) are related as follows (see the proof of Lemma 6.1 in Appendix): if N_Q solves the Cauchy problem $JN_Q'(\xi, \lambda) + Q(\xi)N_Q(\xi, \lambda) = \lambda N_Q(\xi, \lambda)$, $N_Q(0, \lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then

$$N_Q(\xi, \lambda) = \frac{e^{-i\lambda\xi}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \mathfrak{A}_*^+(2\xi, \lambda) & \mathfrak{B}_*^+(2\xi, \lambda) \\ \mathfrak{B}^+(2\xi, \lambda) & \mathfrak{A}^+(2\xi, \lambda) \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

where the continuous Wall polynomials $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{A}_*^+, \mathfrak{B}_*^+$ were defined in (2.14). The Weyl function of the operator $\mathcal{D}_Q^+$ coincides (see Lemma 6.1 in Appendix) with

$$m_Q(z) = \lim_{\xi \to +\infty} i\frac{\widehat{P}_*(\xi, z)}{P_*(\xi, z)}, \qquad z \in \mathbb{C}_+. \tag{3.3}$$

It is known (see Theorem 7.3 in [13]) that the limit above exists for every $z \in \mathbb{C}_+$ and defines an analytic function of Herglotz-Nevanlinna class in $\mathbb{C}_+$. The latter means that $m_Q(\mathbb{C}_+) \subset \mathbb{C}_+$. In the next theorem, $\operatorname{Im} m_Q(\lambda)$ denotes the nontangential boundary value on $\mathbb{R}$ which exists Lebesgue almost everywhere. It is understood as a nonnegative function $g = \operatorname{Im} m$ on $\mathbb{R}$ and it satisfies $g/(1+\lambda^2) \in L^1(\mathbb{R})$.

Theorem 3.1. *Let $q \in L^2(\mathbb{R}_+)$ and let Q, $\mathcal{D}_Q^+$, m_Q be defined by (3.2), (3.3). Denote by N_Q the solution of the Cauchy problem $JN_Q'(\xi) + Q(\xi)N_Q(\xi) = 0$, $N_Q(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and set $\mathcal{H}_Q = N_Q^* N_Q$. Define also*

$$\mathcal{K}_Q^+ = \log \operatorname{Im} m_Q(i) - \frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_Q(\lambda) \frac{d\lambda}{\lambda^2 + 1}, \tag{3.4}$$

$$\widetilde{\mathcal{K}}_Q^+ = \sum_{k=0}^{+\infty}\left(\det \int_k^{k+2} \mathcal{H}_Q(\xi)\, d\xi - 4\right). \tag{3.5}$$

Then, we have

$$c_1 \mathcal{K}_Q^+ \leqslant \widetilde{\mathcal{K}}_Q^+ \leqslant c_2 e^{c_2 \mathcal{K}_Q^+} \tag{3.6}$$

for some positive absolute constants c_1, c_2.

Proof. Lemma 6.1 in Appendix shows that m_Q coincides with the Weyl function for the canonical system with Hamiltonian $\mathcal{H}_Q$. Then, the bounds in (3.6) follow from the Theorem 1.2 in [2] (see also Corollary 1.4 in [2]). □

The quantity $\mathcal{K}_Q^+$ will be called *the entropy* of the Dirac operator on $\mathbb{R}_+$. We now turn to (3.1) to define the entropy for the Dirac operator on the whole line. Take $q \in L^2(\mathbb{R})$ and let $A^+(\xi) = -\overline{q(\xi/2)}/2$ and $A^-(\xi) = q(-\xi/2)/2$, $\xi \in \mathbb{R}_+$ be the coefficients of Krein systems associated to restrictions of q to the half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$. As in (3.3), the half-line Weyl functions $m_\pm$ satisfy

$$m_\pm(z) = \lim_{\xi\to+\infty} i\frac{\widehat{P}_*^\pm(\xi,z)}{P_*^\pm(\xi,z)}, \qquad z \in \mathbb{C}_+. \tag{3.7}$$

These Weyl functions $m_\pm$ can be used to construct the spectral representation for the Dirac operator. Let

$$m(z) = -\frac{1}{m_+(z)+m_-(z)}\begin{pmatrix} -2m_+(z)m_-(z) & m_+(z)-m_-(z) \\ m_+(z)-m_-(z) & 2 \end{pmatrix}, \qquad z \in \mathbb{C}_+. \tag{3.8}$$

Using $\operatorname{Im} m_\pm(z) > 0$, one can show that $\operatorname{Im} m(z)$ is a positive definite matrix for $z \in \mathbb{C}_+$. In other words, m is the matrix-valued Herglotz function. Therefore, there exists a unique matrix-valued measure ρ taking Borel subsets of $\mathbb{R}$ into 2×2 nonnegative matrices such that

$$m(z) = \alpha + \beta z + \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\right) d\rho(\lambda), \qquad z \in \mathbb{C}_+,$$

where α, β are constant 2×2 real matrices, $\beta \geqslant 0$. The importance of ρ becomes clear when we recall the spectral decomposition for $\mathcal{D}_Q$. Specifically, let $N_Q(\xi,z)$ be the solution of the Cauchy problem

$$J\frac{\partial}{\partial\xi}N_Q(\xi,z) + Q(\xi)N_Q(\xi,z) = zN_Q(\xi,z), \qquad N_Q(0,z) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \quad z \in \mathbb{C}, \quad \xi \in \mathbb{R}. \tag{3.9}$$

Then, the mapping

$$\mathcal{F}_{\mathcal{D}_Q} : X \mapsto \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} N_Q^*(\xi,\lambda)X(\xi)\,d\xi, \qquad \lambda \in \mathbb{R}, \tag{3.10}$$

initially defined on the set of compactly supported smooth functions $X : \mathbb{R} \to \mathbb{C}^2$, extends (see Appendix) to the unitary operator between the Hilbert spaces $L^2(\mathbb{R},\mathbb{C}^2)$ and $L^2(\rho)$,

$$L^2(\rho) = \Big\{Y : \mathbb{R} \to \mathbb{C}^2 : \ \|Y\|^2_{L^2(\rho)} = \int_{\mathbb{R}} Y^*(\lambda)\,d\rho(\lambda)\,Y(\lambda) < \infty\Big\}.$$

Moreover, $\mathcal{D}_Q$ is unitary equivalent to the operator of multiplication by the independent variable in $L^2(\rho)$ and the unitary equivalence is given by the operator $\mathcal{F}_Q$. In fact, these properties of ρ will not be used in the paper, we mention them only to motivate the following definition. Let us define the *entropy function* $\mathcal{K}_Q(z)$ by

$$\mathcal{K}_Q(z) = -\frac{1}{\pi}\int_{\mathbb{R}} \log(\det\rho_{ac}(\lambda))\frac{\operatorname{Im} z}{|\lambda - z|^2}\,d\lambda, \qquad z \in \mathbb{C}_+, \tag{3.11}$$

where ρ_{ac} denotes the absolutely continuous part of the spectral measure ρ and it satisfies $\rho_{ac}(\lambda) = \lim_{\varepsilon\to 0,\varepsilon>0}\operatorname{Im} m(\lambda+i\varepsilon)$ for a.e. $\lambda \in \mathbb{R}$. The quantity $\mathcal{K}_Q$ will play a crucial role in our considerations. We first relate it to the coefficient a of the reduced transition matrix T which was introduced in Proposition 2.2.

Lemma 3.1. *We have* $\det\rho_{ac}(\lambda) = |a(\lambda)|^{-2}$ *for almost all* $\lambda \in \mathbb{R}$. *In particular,* $\mathcal{K}_Q(z) = 2\log|a(z)|$ *for all* $z \in \mathbb{C}_+$.

Proof. From the definition (or see page 59 in [24]), one has

$$\det\operatorname{Im} m(z) = 4\frac{\operatorname{Im} m_+(z)\operatorname{Im} m_-(z)}{|m_+(z)+m_-(z)|^2}, \qquad z \in \mathbb{C}_+. \tag{3.12}$$

Substituting expressions for

$$m_\pm(z) = \lim_{\xi\to+\infty} i\frac{\widehat{P}_*^\pm(\xi,z)}{P^\pm(\xi,z)} = i\frac{\mathfrak{a}^\pm(z) - \mathfrak{b}^\pm(z)}{\mathfrak{a}^\pm(z) + \mathfrak{b}^\pm(z)}, \qquad z \in \mathbb{C}_+,$$

into (3.12), we obtain

$$\begin{aligned}\det \rho_{ac}(\lambda) &= \lim_{\varepsilon\to+0} \det \operatorname{Im} m(\lambda+i\varepsilon)\\ &= \lim_{\varepsilon\to+0} \frac{(|\mathfrak{a}^+(\lambda+i\varepsilon)|^2-|\mathfrak{b}^+(\lambda+i\varepsilon)|^2)(|\mathfrak{a}^-(\lambda+i\varepsilon)|^2-|\mathfrak{b}^-(\lambda+i\varepsilon)|^2)}{|\mathfrak{a}^+(\lambda+i\varepsilon)\mathfrak{a}^-(\lambda+i\varepsilon)-\mathfrak{b}^+(\lambda+i\varepsilon)\mathfrak{b}^-(\lambda+i\varepsilon)|^2}\\ &= \frac{1}{|a(\lambda)|^2},\end{aligned}$$

for almost every $\lambda\in\mathbb{R}$ and the first claim of the lemma follows. Then, the second claim is immediate because a is an outer function as we showed in Proposition 2.1. □

Consider again the half-line entropy functions

$$\mathcal{K}_Q^{\pm}(z) = \log \operatorname{Im} m_{\pm}(z) - \frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_{\pm}(\lambda)\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda, \qquad z\in\mathbb{C}_+.$$

We see that $\mathcal{K}_Q^+(i)$ coincides with the entropy (3.4) for the restriction of Q to $\mathbb{R}_+$ (that explains why we use the same notation for the two objects), and $\mathcal{K}_Q^-(z) = \mathcal{K}_{Q_-}^+(z)$ for the potential

$$Q_-(\xi) = \begin{pmatrix} -\operatorname{Im} q(-\xi) & \operatorname{Re} q(-\xi)\\ \operatorname{Re} q(-\xi) & \operatorname{Im} q(-\xi)\end{pmatrix}, \qquad \xi\in\mathbb{R}_+.$$

Our plan now is to relate $\mathcal{K}_Q^{\pm}(i)$ with $\mathcal{K}_Q(i)$ and then use the fact that the full line entropy $\mathcal{K}_Q(i)$ is conserved, see Lemma 3.1. That will eventually lead to the proof of Theorem 1.2.

Lemma 3.2. *Let $q\in L^2(\mathbb{R})$ and let $q_\ell(\xi) = q(\xi-\ell)$, where $\ell\in\mathbb{R}$ and $\xi\in\mathbb{R}$. Denote by Q_ℓ the matrix-function in (3.1) corresponding to q_ℓ. Then, $\mathcal{K}_{Q_\ell}^+(z)\to\mathcal{K}_Q(z)$, $\mathcal{K}_{Q_\ell}^-(z)\to 0$ as $\ell\to+\infty$ for every $z\in\mathbb{C}_+$.*

Proof. Take $z\in\mathbb{C}_+$. We have

$$\mathcal{K}_{Q_\ell}^+(z)+\mathcal{K}_{Q_\ell}^-(z) = \log\Bigl(\operatorname{Im} m_{\ell,+}(z)\operatorname{Im} m_{\ell,-}(z)\Bigr) - \frac{1}{\pi}\int_{\mathbb{R}}\log\Bigl(\operatorname{Im} m_{\ell,+}(\lambda)\operatorname{Im} m_{\ell,-}(\lambda)\Bigr)\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda,$$

for the corresponding Weyl functions $m_{\ell,\pm}$. We also have

$$\log|m_{\ell,+}(z)+m_{\ell,-}(z)|^2 = \frac{1}{\pi}\int_{\mathbb{R}}\log|m_{\ell,+}(\lambda)+m_{\ell,-}(\lambda)|^2\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda$$

by the mean value theorem for harmonic functions. From (3.12), it follows that

$$\mathcal{K}_{Q_\ell}^+(z)+\mathcal{K}_{Q_\ell}^-(z) = \log\left(4\frac{\operatorname{Im} m_{\ell,+}(z)\operatorname{Im} m_{\ell,-}(z)}{|m_{\ell,+}(z)+m_{\ell,-}(z)|^2}\right)+\mathcal{K}_{Q_\ell}(z).$$

Notice that $\mathcal{K}_{Q_\ell}(z)$ does not depend on $\ell\in\mathbb{R}$ because the coefficient a in Lemma 3.1 for the potential Q_ℓ does not depend on ℓ. So, we only need to show that

$$\mathcal{K}_{Q_\ell}^-(z)\to 0 \quad\text{ and }\quad \log\left(4\frac{\operatorname{Im} m_{\ell,+}(z)\operatorname{Im} m_{\ell,-}(z)}{|m_{\ell,+}(z)+m_{\ell,-}(z)|^2}\right)\to 0,$$

when $\ell\to+\infty$ and $z\in\mathbb{C}_+$. The second relation follows from $m_{\ell,+}(z)\to i$, $m_{\ell,-}(z)\to i$, which hold because q_ℓ tends to zero weakly in $L^2(\mathbb{R})$ as $\ell\to+\infty$ and $\|q_\ell\|_{L^2(\mathbb{R})} = \|q\|_{L^2(\mathbb{R})}$ (see Lemma 6.2 in Appendix). Moreover, relation $m_{\ell,-}(z)\to i$ implies that $\mathcal{K}_{Q_\ell}^-(z)\to 0$ if and only if

$$\frac{1}{\pi}\int_{\mathbb{R}}\log\operatorname{Im} m_{\ell,-}(\lambda)\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda\to 0\,. \tag{3.13}$$

In the rest of the proof, we will show (3.13). Let $\mathfrak{a}_\ell^-$, $\mathfrak{b}_\ell^-$ be the limits of continuous Wall polynomials corresponding to Q_ℓ^-. Consider $s_\ell^- = \mathfrak{b}_\ell^-/\mathfrak{a}_\ell^-$. The formula (12.57) in [13] gives

$$s_\ell^-(z) = \frac{1+im_{\ell,-}(z)}{1-im_{\ell,-}(z)}, \qquad m_{\ell,-}(z) = i\frac{\mathfrak{a}_\ell^-(z)-\mathfrak{b}_\ell^-(z)}{\mathfrak{a}_\ell^-(z)+\mathfrak{b}_\ell^-(z)}.$$

It implies that $\operatorname{Im} m_{\ell,-}(\lambda) = |\mathfrak{a}_\ell^-(\lambda) + \mathfrak{b}_\ell^-(\lambda)|^{-2}$ when $\lambda \in \mathbb{R}$ and that $s_\ell^-(z) \to 0$ when $\ell \to +\infty$ and $z \in \mathbb{C}_+$. Now, we can write

$$\frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_{\ell,-}(\lambda) \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda = \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1}{|\mathfrak{a}_\ell^-(\lambda) + \mathfrak{b}_\ell^-(\lambda)|^2}\right) \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda$$
$$= \log \frac{1}{|\mathfrak{a}_\ell^-(z) + \mathfrak{b}_\ell^-(z)|^2} = \log \frac{1}{|\mathfrak{a}_\ell^-(z)|^2} + \log \frac{1}{|1 + s_\ell^-(z)|^2}\,.$$

So, it remains to show that $|\mathfrak{a}_\ell^-(z)|^2 \to 1$ as $\ell \to +\infty$. That holds because $\|q_{\ell,-}\|_{L^2(\mathbb{R}_+)} \to 0$ as $\ell \to +\infty$ and

$$\|q_{\ell,-}\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{\pi}\int_{\mathbb{R}} \log |\mathfrak{a}_\ell^-(\lambda)|^2\, d\lambda \geqslant \frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}} \log |\mathfrak{a}_\ell^-(\lambda)|^2 \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda = \operatorname{Im} z \cdot \log |\mathfrak{a}_\ell^-(z)|^2 \geqslant 0,$$

where the first equality follows from $\|q_{\ell,-}\|^2_{L^2(\mathbb{R}_+)} = 2\|A_{\ell,-}\|^2_{L^2(\mathbb{R}_+)}$ and the formula (12.2) in [13]. Thus, (3.13) holds and we are done. □

As an immediate corollary of Theorem 3.1 and Lemma 3.2, we get the following estimate.

Lemma 3.3. *Let $q \in L^2(\mathbb{R})$. Denote by N_Q the solution of the Cauchy problem $JN_Q'(\xi) + Q(\xi)N_Q(\xi) = 0$, $N_Q(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and set $\mathcal{H}_Q = N_Q^* N_Q$. Consider*

$$\mathcal{K}_Q = \mathcal{K}_Q(i), \qquad \widetilde{\mathcal{K}}_Q = \sum_{k\in\mathbb{Z}} \left(\det \int_k^{k+2} \mathcal{H}_Q(\xi)\, d\xi - 4\right). \tag{3.14}$$

Then, we have

$$c_1 \mathcal{K}_Q \leqslant \widetilde{\mathcal{K}}_Q \leqslant c_2 \mathcal{K}_Q e^{c_2 \mathcal{K}_Q} \tag{3.15}$$

for some positive absolute constants c_1, c_2.

Proof. By Lemma 3.2, we have $\mathcal{K}_Q = \lim_{\ell\to+\infty} \mathcal{K}^+_{Q_\ell}$. It remains to substitute Q_ℓ into the estimate (3.6) and take the limit as $\ell \to +\infty$ for $\ell \in \mathbb{Z}$. □

4. Proof of Theorem 1.2

The following result will play a crucial role in what follows. We postpone its proof to the next section.

Theorem 4.1. *Suppose $q \in L^2(\mathbb{R})$ and let N_Q satisfy $JN_Q' + QN_Q = 0, N_Q(0) = I$, where $Q = \left(\begin{smallmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{smallmatrix}\right)$. Then,*

$$e^{-C_1 R}\|q\|^2_{H^{-1}(\mathbb{R})} \lesssim \widetilde{\mathcal{K}}_Q \lesssim e^{C_2 R}\|q\|^2_{H^{-1}(\mathbb{R})}\,, \tag{4.1}$$

where $R = \|q\|_{L^2(\mathbb{R})}$ and C_1, C_2 are two positive absolute constants.

Proof of Theorem 1.2 in the case $s = -1$. First, assume that $q_0 \in \mathcal{S}(\mathbb{R})$ and let $q(\xi, t)$ be the solution of (1.1) with the initial datum q_0. We want to prove that

$$C_1(1 + \|q_0\|_{L^2(\mathbb{R})})^{-2}\|q_0\|_{H^{-1}(\mathbb{R})} \leqslant \|q(\cdot, t)\|_{H^{-1}(\mathbb{R})} \leqslant C_2(1 + \|q_0\|_{L^2(\mathbb{R})})^2\|q_0\|_{H^{-1}(\mathbb{R})}\,. \tag{4.2}$$

We have $\|q(\cdot, t)\|_{L^2(\mathbb{R})} = \|q_0\|_{L^2(\mathbb{R})}$ for all t, see formula (4.33) in [14]. Let $a(z, t)$ denote the coefficient in the matrix (2.5) given by $q(\xi, t)$. For each $t \in \mathbb{R}$, define Q by (3.2). Let $\widetilde{\mathcal{K}}_Q(t)$ be as in Lemma 3.3 and $\mathcal{K}_Q(z, t)$ be defined by (3.11). Formulas (2.8) and (2.9) show that $a(z, t)$ is constant in t and Lemma 3.1 says that $\mathcal{K}_Q(z, t)$ is constant in t as well. The bound (3.15) yields

$$c_1 \mathcal{K}_Q(i, 0) \leqslant \widetilde{\mathcal{K}}_Q(t) \leqslant c_2 \mathcal{K}_Q(i, 0) e^{c_2 \mathcal{K}_Q(i,0)}\,. \tag{4.3}$$

Assume first that $R = \|q_0\|_{L^2(\mathbb{R})} \leqslant 1$. Taking $t = 0$ in (4.3) and applying (4.1) to q_0, we get $\mathcal{K}_Q(i, 0) \lesssim 1$ since $\|q_0\|_{H^{-1}(\mathbb{R})} \leqslant R \leqslant 1$. Hence, in that case (4.3) can be written as $\widetilde{\mathcal{K}}_Q(t) \sim \mathcal{K}_Q(i, 0)$. By (4.1), $\|q(\cdot, t)\|^2_{H^{-1}(\mathbb{R})} \sim \widetilde{\mathcal{K}}_Q(t)$, and so $\|q(\cdot, t)\|^2_{H^{-1}(\mathbb{R})} \sim \|q(\cdot, 0)\|^2_{H^{-1}(\mathbb{R})}$.

If $R = \|q_0\|_{L^2(\mathbb{R})} > 1$, we use dilation. Consider $q_\alpha(\xi, t) = \alpha q(\alpha\xi, \alpha^2 t)$ which solves the same equation and notice that $\|q_\alpha\|_{L^2(\mathbb{R})} = \alpha^{\frac{1}{2}} R$.

Let $\alpha=\alpha_c=R^{-2}<1$ making $\|q_{\alpha_c}\|_{L^2(\mathbb{R})}=1$. Then, for the Sobolev norm, we get

$$\|q_\alpha(\cdot,t)\|_{H^{-1}(\mathbb{R})}=\alpha^{\frac{1}{2}}\left(\int_{\mathbb{R}}\frac{1}{1+\alpha^2\eta^2}|(\mathcal{F}q)(\eta,\alpha^2t)|^2d\eta\right)^{\frac{1}{2}}. \tag{4.4}$$

Since

$$\frac{1}{1+\eta^2}\leqslant\frac{1}{1+\alpha_c^2\eta^2}\leqslant\frac{1}{\alpha_c^2(1+\eta^2)}, \tag{4.5}$$

one has

$$\alpha_c^{\frac{1}{2}}\|q(\cdot,\alpha_c^2t)\|_{H^{-1}(\mathbb{R})}\leqslant\|q_{\alpha_c}(\cdot,t)\|_{H^{-1}(\mathbb{R})}\leqslant\alpha_c^{-\frac{1}{2}}\|q(\cdot,\alpha_c^2t)\|_{H^{-1}(\mathbb{R})}.$$

In particular, at $t=0$ we get

$$\alpha_c^{\frac{1}{2}}\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})}\leqslant\|q_{\alpha_c}(\cdot,0)\|_{H^{-1}(\mathbb{R})}\leqslant\alpha_c^{-\frac{1}{2}}\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})}.$$

Since $\|q_{\alpha_c}(\cdot,0)\|_{L^2(\mathbb{R})}=1$, one can apply the previous bounds to obtain

$$\|q_{\alpha_c}(\cdot,t)\|_{H^{-1}(\mathbb{R})}\sim\|q_{\alpha_c}(\cdot,0)\|_{H^{-1}(\mathbb{R})}.$$

Then,

$$\alpha_c^{\frac{1}{2}}\|q(\cdot,\alpha_c^2t)\|_{H^{-1}(\mathbb{R})}\lesssim\|q_{\alpha_c}(\cdot,0)\|_{H^{-1}(\mathbb{R})}\lesssim\alpha_c^{-\frac{1}{2}}\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})},$$

$$\alpha_c^{-\frac{1}{2}}\|q(\cdot,\alpha_c^2t)\|_{H^{-1}(\mathbb{R})}\gtrsim\|q_{\alpha_c}(\cdot,0)\|_{H^{-1}(\mathbb{R})}\gtrsim\alpha_c^{\frac{1}{2}}\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})}.$$

Recalling that $\alpha_c=R^{-2}$, we obtain

$$R^{-2}\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})}\lesssim\|q(\cdot,t)\|_{H^{-1}(\mathbb{R})}\lesssim R^2\|q(\cdot,0)\|_{H^{-1}(\mathbb{R})}$$

for all $t\in\mathbb{R}$. Finally, having proved (4.2) for $q_0\in\mathcal{S}(\mathbb{R})$, it is enough to use Theorem 1.1 to extend (4.2) to $q_0\in L^2(\mathbb{R})$. □

Our next goal is to prove the estimate

$$C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{2s}\|q_0\|_{H^s(\mathbb{R})}\leqslant\|q(\cdot,t)\|_{H^s(\mathbb{R})}\leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^{-2s}\|q_0\|_{H^s(\mathbb{R})}, \tag{4.6}$$

where $t\in\mathbb{R}$, $s\in(-1,0]$, and C_1 and C_2 are positive absolute constants. For $s=0$, this bound is trivial. To cover $s\in(-1,0)$, we will need some auxiliary results first. One of the basic properties of NLS which we discussed in the Introduction has to do with modulation: if $q(\xi,t)$ solves (2.10), then $\widetilde{q}_v(\xi,t)=e^{iv\xi-iv^2t}q(\xi-2vt,t)$ solves (2.10) for every $v\in\mathbb{R}$.

Lemma 4.1. *Let $q_0\in L^2(\mathbb{R})$, $t\in\mathbb{R}$. Then,*

$$\|\tilde{q}_v(\cdot,t)\|^2_{H^{-1}(\mathbb{R})}=\int_{\mathbb{R}}\frac{|(\mathcal{F}q)(\eta,t)|^2}{1+(\eta+v)^2}\,d\eta.$$

Proof. It is clear that $\|e^{-iv^2t}f\|_{H^{-1}(\mathbb{R})}=\|f\|_{H^{-1}(\mathbb{R})}$ for every $f\in H^{-1}(\mathbb{R})$ and $t\in\mathbb{R}$, because e^{-iv^2t} is a unimodular constant. We have $\mathcal{F}(e^{iv\xi}q(\xi-2vt,t))(\eta)=(\mathcal{F}q(\xi,t))(\eta-v)e^{-2ivt(\eta-v)}$, $\eta\in\mathbb{R}$. Since $|e^{-2ivt(\eta-v)}|=1$, it only remains to change the variable of integration in

$$\|\widetilde{q}_v\|_{H^{-1}(\mathbb{R})}=\int_{\mathbb{R}}\frac{|(\mathcal{F}q(\xi,t))(\eta-v)|^2}{1+\eta^2}\,d\eta$$

to get the statement of the lemma. □

The next result is a standard property of convolutions.

Lemma 4.2. *Let $\gamma\in(-\frac{1}{2},1]$ and set $a_k=\frac{1}{(1+k^2)^\gamma}$ for $k\in\mathbb{Z}$. We have*

$$\sum_{k\in\mathbb{Z}}\frac{a_k}{1+(\eta-k)^2}\sim C_\gamma\frac{1}{(1+\eta^2)^\gamma},\qquad\eta\in\mathbb{R}.$$

Proof. After comparing the sum to an integral, it is enough to show that

$$\int_{\mathbb{R}}\frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)}\sim C_\gamma\frac{1}{(1+\eta^2)^\gamma}.$$

The function on the left-hand side is even and continuous in η and γ, so we can assume that $\eta > 1$. Then,

$$\int_{|\eta-u|<0.5\eta} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \sim \frac{1}{(1+\eta^2)^\gamma}, \quad \int_{|\eta-u|>0.5\eta} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \lesssim \mathcal{I}_1 + \mathcal{I}_2 \,,$$

where

$$\mathcal{I}_1 = \int_{u<-\eta/2,\, u>3\eta/2} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \lesssim \int_{-\infty}^{-\eta/2} \frac{du}{u^{2+2\gamma}} + \int_{3\eta/2} \frac{du}{u^{2+2\gamma}} \leqslant C_\gamma \eta^{-1-2\gamma} \,,$$

$$\mathcal{I}_2 = \int_{|u|<\eta/2} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \lesssim \eta^{-2} \int_{|u|<\eta/2} \frac{du}{(1+u^2)^\gamma} \lesssim \eta^{-2\gamma} \,.$$

Combining these bounds proves the lemma. □

Proof of Theorem 1.2, the case $s \in (-1, 0)$**.** We can again assume that $q_0 \in \mathcal{S}(\mathbb{R})$. Recall the estimate (1.5) for $s = -1$:

$$C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{-2}\|q_0\|_{H^{-1}(\mathbb{R})} \leqslant \|q(\cdot,t)\|_{H^{-1}(\mathbb{R})} \leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^2\|q_0\|_{H^{-1}(\mathbb{R})} \,. \tag{4.7}$$

According to Lemma 4.1, we have

$$\|\tilde{q}_v(\cdot,t)\|^2_{H^{-1}(\mathbb{R})} = \int_{\mathbb{R}} \frac{|(\mathcal{F}q(\cdot,t))(\eta)|^2}{1+(v+\eta)^2} d\eta \tag{4.8}$$

for $\tilde{q}_v(\xi,t) = e^{iv\xi - iv^2 t} q(\xi - 2vt, t)$. Let a_k, $k \in \mathbb{Z}$, be the coefficients from Lemma 4.2 with $\gamma = -s$. Then, (4.8) and Lemma 4.2 imply

$$\sum_{k\in\mathbb{Z}} a_k \|\tilde{q}_k(\cdot,t)\|^2_{H^{-1}(\mathbb{R})} \sim C_s \|q(\cdot,t)\|^2_{H^s(\mathbb{R})} \,. \tag{4.9}$$

In particular, taking $t = 0$ gives

$$\sum_{k\in\mathbb{Z}} a_k \|\tilde{q}_k(\cdot,0)\|^2_{H^{-1}(\mathbb{R})} \sim C_s \|q_0\|^2_{H^s(\mathbb{R})} \,. \tag{4.10}$$

We now apply (4.7) to $\tilde{q}_k$ and use (4.9) and (4.10) to get

$$C_1(s)(1+\|q_0\|_{L^2(\mathbb{R})})^{-2}\|q_0\|_{H^s(\mathbb{R})} \leqslant \|q(\cdot,t)\|_{H^s(\mathbb{R})} \leqslant C_2(s)(1+\|q_0\|_{L^2(\mathbb{R})})^2\|q_0\|_{H^s(\mathbb{R})} \,. \tag{4.11}$$

If $R = \|q_0\|_{L^2(\mathbb{R})} \leqslant 1$, we have the statement of our theorem. If $R = \|q_0\|_{L^2(\mathbb{R})} > 1$, we use dilation transformation like in the previous proof for $s = -1$. Consider $q_\alpha(\xi,t) = \alpha q(\alpha\xi, \alpha^2 t)$ which solves the same equation and notice that $\|q_\alpha\|_{L^2(\mathbb{R})} = \alpha^{\frac12} R$. Let $\alpha = \alpha_c = R^{-2} < 1$ making $\|q_{\alpha_c}\|_{L^2(\mathbb{R})} = 1$. Then, for the Sobolev norm, we have

$$\|q_\alpha(\cdot,t)\|_{H^s(\mathbb{R})} = \alpha^{\frac12} \left(\int_{\mathbb{R}} \frac{1}{(1+\alpha^2\eta^2)^{|s|}} |(\mathcal{F}q)(\eta, \alpha^2 t)|^2 d\eta \right)^{\frac12} .$$

From (4.5),

$$\frac{1}{(1+\eta^2)^{|s|}} \leqslant \frac{1}{(1+\alpha_c^2\eta^2)^{|s|}} \leqslant \frac{1}{\alpha_c^{2|s|}(1+\eta^2)^{|s|}} \,.$$

Then, one has

$$\alpha_c^{\frac12}\|q(\cdot,\alpha_c^2 t)\|_{H^s(\mathbb{R})} \leqslant \|q_{\alpha_c}(\cdot,t)\|_{H^s(\mathbb{R})} \leqslant \alpha_c^{\frac12-|s|}\|q(\cdot,\alpha_c^2 t)\|_{H^s(\mathbb{R})} \,.$$

In particular, taking $t = 0$ gives us

$$\alpha_c^{\frac12}\|q(\cdot,0)\|_{H^s(\mathbb{R})} \leqslant \|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})} \leqslant \alpha_c^{\frac12-|s|}\|q(\cdot,0)\|_{H^s(\mathbb{R})} \,.$$

Now $\|q_{\alpha_c}(\cdot,0)\|_{L^2(\mathbb{R})} = 1$ and we can apply the previous bounds to get

$$\|q_{\alpha_c}(\cdot,t)\|_{H^s(\mathbb{R})} \sim \|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})} \,.$$

Then,

$$\alpha_c^{\frac12}\|q(\cdot,\alpha_c^2 t)\|_{H^s(\mathbb{R})} \lesssim \|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})} \lesssim \alpha_c^{\frac12-|s|}\|q(\cdot,0)\|_{H^s(\mathbb{R})},$$
$$\alpha_c^{\frac12-|s|}\|q(\cdot,\alpha_c^2 t)\|_{H^s(\mathbb{R})} \gtrsim \|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})} \gtrsim \alpha_c^{\frac12}\|q(\cdot,0)\|_{H^s(\mathbb{R})}.$$

Recalling that $\alpha_c = R^{-2} = \|q_0\|^{-2}_{L^2(\mathbb{R})}$, we obtain

$$\|q_0\|^{-2|s|}_{L^2(\mathbb{R})}\|q(\cdot,0)\|_{H^s(\mathbb{R})} \lesssim \|q(\cdot,t)\|_{H^s(\mathbb{R})} \lesssim \|q_0\|^{2|s|}_{L^2(\mathbb{R})}\|q(\cdot,0)\|_{H^s(\mathbb{R})}$$

for all $t \in \mathbb{R}$. □

Our approach also provides the bounds for some positive Sobolev norms. The following proposition slightly improves (1.4) when $s \in [0, \frac{1}{2})$, $\|q_0\|_{H^s(\mathbb{R})}$ is large, and $\|q_0\|_{L^2(\mathbb{R})}$ is much smaller than $\|q_0\|_{H^s(\mathbb{R})}$.

Proposition 4.1. *Let $q_0 \in \mathcal{S}(\mathbb{R})$ and let $q = q(\xi, t)$ be the solution of (1.1) corresponding to q_0. Then, for each $s \in [0, \frac{1}{2})$, we get*

$$\|q(\cdot,t)\|_{H^s(\mathbb{R})} \sim C_s\|q_0\|_{H^s(\mathbb{R})} \tag{4.12}$$

if $\|q_0\|_{L^2(\mathbb{R})} \leqslant 1$ and

$$\|q(\cdot,t)\|_{H^s(\mathbb{R})} \lesssim C_s(\|q_0\|^{1+2s}_{L^2(\mathbb{R})} + \|q_0\|_{H^s(\mathbb{R})}) \tag{4.13}$$

if $\|q_0\|_{L^2(\mathbb{R})} > 1$.

Proof. In the case when $\|q\|_{L^2(\mathbb{R})} \leqslant 1$, the proof of proposition repeats the arguments given above to get (4.11) except that the constants in the inequalities depend on s and can blow up when $s \to \frac{1}{2}$. Suppose $\|q\|_{L^2(\mathbb{R})} \geqslant 1$. Then, for the Sobolev norm, we have

$$\|q_\alpha(\cdot,t)\|_{H^s(\mathbb{R})} = \alpha^{\frac{1}{2}}\left(\int_{\mathbb{R}} (1+\alpha^2\eta^2)^s|(\mathcal{F}q)(\eta,\alpha^2 t)|^2 d\eta\right)^{\frac{1}{2}}.$$

Take $\alpha = \alpha_c$ and write the following estimate for the integral above:

$$\int_{\mathbb{R}}\left(1+\frac{\eta^2}{R^4}\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta \sim \int_{-R^2}^{R^2}|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta + R^{-4s}\int_{|\eta|>R^2}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta$$

$$\lesssim R^2 + R^{-4s}\int_{\mathbb{R}}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta\,.$$

We use $\|q_{\alpha_c}(\cdot,t)\|_{L^2(\mathbb{R})} = 1$ and (4.11) to get $\|q_{\alpha_c}(\cdot,t)\|_{H^s(\mathbb{R})} \sim C_s\|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})}$. The previous estimate for $t = 0$ yields $\|q_{\alpha_c}(\cdot,0)\|_{H^s(\mathbb{R})} \lesssim 1 + R^{-1-2s}\|q(\cdot,0)\|_{H^s(\mathbb{R})}$. Hence, $\|q_{\alpha_c}(\cdot,t)\|_{H^s(\mathbb{R})} \leqslant C_s(1 + R^{-1-2s}\|q(\cdot,0)\|_{H^s(\mathbb{R})})$. We can write a lower bound

$$\|q_{\alpha_c}(\cdot,t)\|^2_{H^s(\mathbb{R})} = R^{-2}\int_{\mathbb{R}}\left(1+\frac{\eta^2}{R^4}\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta \gtrsim R^{-2-4s}\int_{|\eta|>R^2}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta$$

so

$$\int_{|\eta|>R^2}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta \leqslant C_s(R^{2+4s} + \|q(\cdot,0)\|^2_{H^s(\mathbb{R})}).$$

Writing the integral as a sum of two:

$$\int_{\mathbb{R}}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta = \int_{|\eta|>R^2}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta + \int_{|\eta|<R^2}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta$$

and estimating each of them, we get a bound which holds for all t:

$$\int_{\mathbb{R}}\left(1+\eta^2\right)^s|(\mathcal{F}q)(\eta,\alpha_c^2 t)|^2 d\eta \leqslant C_s(R^{2+4s} + \|q(\cdot,0)\|^2_{H^s(\mathbb{R})})\,.$$

That is the required upper bound (4.13). □

5. Oscillation and Sobolev space $H^{-1}(\mathbb{R})$.

In this part of the paper, our goal is to prove the Theorem 4.1. Let us recall its statement.

Theorem 5.1. *Suppose that $q \in L^2(\mathbb{R})$ and let N_Q satisfy $JN_Q' + QN_Q = 0, N_Q(0) = I$, where $Q = \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q\end{pmatrix}$. Then,*

$$e^{-C_1 R}\|q\|^2_{H^{-1}(\mathbb{R})} \lesssim \widetilde{\mathcal{K}}_Q \lesssim e^{C_2 R}\|q\|^2_{H^{-1}(\mathbb{R})}\,, \tag{5.1}$$

where $R = \|q\|_{L^2(\mathbb{R})}$ and C_1, C_2 are two positive absolute constants.

Theorem 5.1 is of independent interest in the spectral theory of Dirac operators. For example, Lemma 3.3 shows that $\|q\|_{L^2(\mathbb{R})}$ and $\|q\|_{H^{-1}(\mathbb{R})}$ control the size of $\mathcal{K}_Q$.

The strategy of the proof is the following. In the next subsection, we show that $H^{-1}(\mathbb{R})$-norm of any function can be characterized through BMO-like condition for its "antiderivative". In Subsection 5.2, we consider solution to Cauchy problem $JN' + QN = 0, N(0) = I$ on the interval $[0,1]$ where zero-trace symmetric Q and study the quantity $\det \int_0^1 N^*N dx$, which represents a single term in the sum for $\widetilde{\mathcal{K}}_Q$. The results in Subsection 5.3 show that small value of $\widetilde{\mathcal{K}}_Q$ guarantees that the "local" H^{-1} norm of Q is also small. This rough estimate is used in the proof of Theorem 4.1 which is contained in Subsection 5.4.

5.1. **One property of Sobolev space $H^{-1}(\mathbb{R})$.** Observe that a function $f \in L^2(\mathbb{R})$ belongs to the Sobolev space $H^{-1}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)\chi_{\mathbb{R}_+}(x-y)e^{-(x-y)}\, dy \right|^2 dx < \infty. \tag{5.2}$$

Moreover, the last integral is equal to $\|f\|^2_{H^{-1}(\mathbb{R})}$. Indeed, recall that $\mathcal{F}f$ stands for the Fourier transform of f:

$$(\mathcal{F}f)(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\eta x}\, dx\,.$$

Then, from Plancherel's identity and formula

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} e^{-x}e^{-ix\eta}dx = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\eta},$$

we obtain

$$\|f\|^2_{H^{-1}(\mathbb{R})} = 2\pi\|(\mathcal{F}f)\mathcal{F}(\chi_{\mathbb{R}_+}e^{-x})\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \frac{|(\mathcal{F}f)(\eta)|^2}{1+\eta^2}\, d\eta$$

by properties of convolutions. We will need the following proposition.

Proposition 5.1. *Suppose that $f \in L^1_{\rm loc}(\mathbb{R}) \cap H^{-1}(\mathbb{R})$. Let g be an absolutely continuous function on $\mathbb{R}$ such that $g' = f$ almost everywhere on $\mathbb{R}$. Then,*

$$c_1\|f\|^2_{H^{-1}(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} \int_{I_k} |g - \langle g\rangle_{I_k}|^2\, dx \leqslant c_2\|f\|^2_{H^{-1}(\mathbb{R})}, \tag{5.3}$$

where $I_k = [k, k+2]$, $\langle g\rangle_I = \frac{1}{|I|}\int_I g(x)\, dx$, and the positive constants c_1 and c_2 are universal.

Proof. Take a function $f \in L^1_{\rm loc}(\mathbb{R}) \cap H^{-1}(\mathbb{R})$, and let g be an absolutely continuous function on $\mathbb{R}$ such that $g' = f$ almost everywhere on $\mathbb{R}$. Assume first that f has a compact support. The integral under the sum does not change if we add a constant to g, so we can suppose without loss of generality that

$$g(x) = \int_{-\infty}^{x} f(s)\, ds, \qquad x \in \mathbb{R}.$$

Upper bound. Given f, define o_f by

$$o_f(x) = e^{-x}\int_{-\infty}^{x} f(y)e^y dy$$

and recall (see (5.2)) that

$$\|f\|_{H^{-1}(\mathbb{R})} = \|o_f\|_{L^2(\mathbb{R})}. \tag{5.4}$$

Moreover,

$$o_f' + o_f = f\,. \tag{5.5}$$

For each interval I_k, we use (5.5) for the corresponding term in the sum (5.3):

$$\int_{I_k}\left|\int_k^x f dx_1-\frac{1}{2}\int_k^{k+2}\left(\int_k^{x_1}f(x_2)dx_2\right)dx_1\right|^2 dx$$
$$=\int_{I_k}\left|\int_k^x o(x_1)dx_1+o(x)-\frac{1}{2}\int_k^{k+2}\left(o(x_1)+\int_k^{x_1}o(x_2)dx_2\right)dx_1\right|^2 dx$$
$$\lesssim\int_{I_k}|o|^2dx$$

after the Cauchy-Schwarz inequality is applied. Summing these estimates in $k\in\mathbb{Z}$ and using (5.4), we get the upper bound in (5.3) for compactly supported f.

Lower bound. Integration by parts gives

$$\int_{-\infty}^x f(y)e^{-(x-y)}\,dy=\int_{-\infty}^x f(s)\,ds-\int_{-\infty}^x\left(\int_{-\infty}^y f(s)\,ds\right)e^{-(x-y)}\,dy$$
$$=g(x)-\int_{-\infty}^x g(y)\,e^{-(x-y)}\,dy$$
$$=\int_{-\infty}^x (g(x)-g(y))\,e^{-(x-y)}\,dy.$$

Therefore,

$$\int_{\mathbb{R}}\left|\int_{-\infty}^x f(y)e^{-(x-y)}\,dy\right|^2 dx\leqslant\sum_{k\in\mathbb{Z}}\int_k^{k+2}\left(\int_{-\infty}^x|g(x)-g(y)|^2e^{-(x-y)}\,dy\right)dx$$
$$\lesssim\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}\int_k^{k+2}\int_j^{j+2}|g(x)-g(y)|^2\,dx\,dy\,.$$

Using the inequality $(x+y+z)^2\leqslant 3(x^2+y^2+z^2)$, we continue the estimate:

$$\ldots\lesssim\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}\left(\int_{I_k}|g-\langle g\rangle_{I_k}|^2dx+|\langle g\rangle_{I_j}-\langle g\rangle_{I_k}|^2+\int_{I_j}|g-\langle g\rangle_{I_j}|^2dx\right).$$

Since

$$\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}\left(\int_{I_k}|g-\langle g\rangle_{I_k}|^2dx+\int_{I_j}|g-\langle g\rangle_{I_j}|^2dx\right)\lesssim\sum_{k\in\mathbb{Z}}\int_{I_k}|g-\langle g\rangle_{I_k}|^2dx\,,$$

we are left with estimating

$$\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}|\langle g\rangle_{I_j}-\langle g\rangle_{I_k}|^2.$$

Applying the Cauchy-Schwarz inequality for the telescoping sum

$$\langle g\rangle_{I_k}-\langle g\rangle_{I_j}=\sum_{s=j+1}^{k}\left(\langle g\rangle_{I_s}-\langle g\rangle_{I_{s-1}}\right),$$

we get

$$|\langle g\rangle_{I_j}-\langle g\rangle_{I_k}|^2\leqslant(k-j)\sum_{j\leqslant s\leqslant k-1}|\langle g\rangle_{I_s}-\langle g\rangle_{I_{s+1}}|^2.$$

Then,

$$\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}(k-j)\sum_{j\leqslant s\leqslant k-1}|\langle g\rangle_{I_s}-\langle g\rangle_{I_{s+1}}|^2=\sum_{s\in\mathbb{Z}}|\langle g\rangle_{I_s}-\langle g\rangle_{I_{s+1}}|^2\sum_{k,j:\,j\leqslant s\leqslant k-1}(k-j)e^{-(k-j)}.$$

We have

$$\sum_{k,j:\,j\leqslant s\leqslant k-1}(k-j)e^{-(k-j)}=\sum_{j\leqslant s}\sum_{m\geqslant 1}(s+m-j)e^{-(s+m-j)}$$
$$=\sum_{j\leqslant 0}\sum_{m\geqslant 1}(m-j)e^{-(m-j)}=\sum_{j\geqslant 0}\sum_{m\geqslant 1}(m+j)e^{-(m+j)}.$$

The last sum is finite and does not depend on index s. Now, the estimate

$$|\langle g\rangle_{I_s}-\langle g\rangle_{I_{s+1}}|^2=\int_{I_s\cap I_{s+1}}|\langle g\rangle_{I_s}-g+g-\langle g\rangle_{I_{s+1}}|^2dx\leqslant 2\int_{I_s}|g-\langle g\rangle_{I_s}|^2dx+2\int_{I_{s+1}}|g-\langle g\rangle_{I_{s+1}}|^2dx$$

proves that

$$\sum_{k\in\mathbb{Z}}\sum_{j\leqslant k}e^{-(k-j)}|\langle g\rangle_{I_j}-\langle g\rangle_{I_k}|^2\lesssim\sum_{s\in\mathbb{Z}}\int_{I_s}|g-\langle g\rangle_{I_s}|^2dx\,.$$

Hence, the lower bound in (5.3) holds for compactly supported f.

Now, take any $f\in L^1_{\rm loc}(\mathbb{R})\cap H^{-1}(\mathbb{R})$. The definition (1.2) of $H^{-1}(\mathbb{R})$ implies that $\mathcal{F}f$ can be written as $(1+i\eta)(\mathcal{F}o)$ for some function $o\in L^2(\mathbb{R})$. Moreover, this map $f\mapsto o$ is a bijection between $H^{-1}(\mathbb{R})$ and $L^2(\mathbb{R})$ and $\|f\|_{H^{-1}(\mathbb{R})}=\|o\|_{L^2(\mathbb{R})}$. Taking the inverse Fourier transform of identity $\mathcal{F}f=(1+i\eta)(\mathcal{F}o)$, one gets a formula $f=o+o'$ where o' is understood as a derivative in $\mathcal{S}'(\mathbb{R})$. Since $f\in L^1_{\rm loc}(\mathbb{R})$ and $o\in L^2(\mathbb{R})$, we have $o'\in L^1_{\rm loc}(\mathbb{R})$ and, therefore, o is absolutely continuous on $\mathbb{R}$ with the derivative equal to $f-o$. Now, take $o_n(x)=o(x)\mu_n(x)$ and define the corresponding $f_n=o_n+o_n'$. Here, $\mu_n(x)$ is even and

$$\mu_n(x)=\begin{cases}1, & 0\leqslant x<n,\\ n+1-x, & x\in[n,n+1),\\ 0, & x\geqslant n+1\,.\end{cases}$$

Then, $\{o_n\}\to o$ in $L^2(\mathbb{R})$ and so $\{f_n\}\to f$ in $H^{-1}(\mathbb{R})$ because the mapping $f\mapsto o$ is unitary from $H^{-1}(\mathbb{R})$ onto $L^2(\mathbb{R})$. Also, each f_n is compactly supported and $\{f_n\}$ converges to f uniformly on every finite interval. Define $g_n=\int_0^x f_nds, g=\int_0^x fds$, and write (5.3) for f_n. The estimate on the right gives

$$\sum_{|k|\leqslant N}\int_{I_k}|g_n-\langle g_n\rangle_{I_k}|^2\,dx\leqslant c_2\|f_n\|^2_{H^{-1}(\mathbb{R})}$$

for each $N\in\mathbb{N}$. Sending $n\to\infty$, the bound

$$\sum_{|k|\leqslant N}\int_{I_k}|g-\langle g\rangle_{I_k}|^2\,dx\leqslant c_2\|f\|^2_{H^{-1}(\mathbb{R})}$$

appears. Taking $N\to\infty$, one has the right estimate in (5.3). In particular, it shows that the sum in (5.3) converges. By construction,

$$\sum_{k\in\mathbb{Z}}\int_{I_k}|g_n-\langle g_n\rangle_{I_k}|^2\,dx=\sum_{-n\leqslant k\leqslant n-2}\int_{I_k}|g-\langle g\rangle_{I_k}|^2\,dx+\epsilon_n\,,$$

where ϵ_n is a sum of integrals over $I_{-n-2},I_{-n-1},I_{n-1},I_n$. Since $o\in L^2(\mathbb{R})$,

$$\lim_{n\to\infty}\int_{I_k}|g_n-\langle g_n\rangle_{I_k}|^2\,dx=0,\quad k\in\{-n-2,-n-1,n-1,n\}\,.$$

Hence, $\lim_{n\to\infty}\epsilon_n=0$ and, taking $n\to\infty$ in inequality

$$c_1\|f_n\|^2_{H^{-1}(\mathbb{R})}\leqslant\sum_{k\in\mathbb{Z}}\int_{I_k}|g_n-\langle g_n\rangle_{I_k}|^2\,dx\,,$$

one gets the left bound in (5.3). Since all antiderivatives are different by a constant and the integral in (5.3) does not change if we add a constant to g, the proof is finished. □

5.2. **Auxiliary perturbative results for a single interval.** Notice that for any real symmetric 2×2 matrix Q with zero trace, we have that $V=JQ$ is also real, symmetric and has zero trace. The converse statement is true as well. Hence, the equation $JN_Q'+QN_Q=0$ in Theorem 5.1, which is equivalent to $N_Q'=JQN_Q$, can be written as $N_Q'=VN_Q$ with V having the same properties as Q. Let $U_+(x,y)$ denote the solution to

$$\frac{d}{dx}U_+(x,y)=V(x)U_+(x,y),\quad U_+(y,y)=I$$

and $U_-(x,y)$ denote the solution to

$$\frac{d}{dx}U_-(x,y) = -V(x)U_-(x,y), \quad U_-(y,y) = I\,.$$

Lemma 5.1. *Suppose $N' = VN, N(0) = I$, where V is real-valued, $V \in L^1[0,1]$, $V = V^*$, and $\operatorname{tr} V = 0$. Then, for $\mathcal{H} = N^*N$, we have*

$$\det \int_0^1 \mathcal{H}(\xi)\,d\xi = \frac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\Big(U_+^*(x,y)U_+(x,y)\Big)\,dxdy = \frac{1}{2}\int_0^1\int_0^1 \|U_+(x,y)\|_{\mathrm{HS}}^2\,dxdy, \tag{5.6}$$

$$\det \int_0^1 \mathcal{H}(\xi)\,d\xi - 1 = \frac{1}{2}\int_0^1\int_0^1 \Big\|\Big(U_+(x,y) - U_-(x,y)\Big)e_1\Big\|^2\,dxdy\,. \tag{5.7}$$

Proof. Notice that $N, U_+, U_- \in \mathrm{SL}(2,\mathbb{R})$ and that every matrix $A \in \mathrm{SL}(2,\mathbb{R})$ satisfies

$$JA^* = A^{-1}J, \qquad AJ = J(A^*)^{-1}\,. \tag{5.8}$$

Also, for any real 2×2 matrix B, we have

$$\det B = \langle JBe_1, Be_2\rangle = -\langle JBe_2, Be_1\rangle\,.$$

Hence,

$$\mathcal{I} := \det\int_0^1 \mathcal{H}(\xi)d\xi = \int_0^1\int_0^1 \langle JN^*(x)N(x)e_1, N^*(y)N(y)e_2\rangle dxdy$$
$$= -\int_0^1\int_0^1 \langle JN^*(x)N(x)e_2, N^*(y)N(y)e_1\rangle dxdy\,.$$

For the second integrand, we have

$$\langle JN^*(x)N(x)e_1, N^*(y)N(y)e_2\rangle = \langle N^*(y)N(y)JN^*(x)N(x)e_1, e_2\rangle\,.$$

Then, identities (5.8) imply

$$N^*(y)N(y)JN^*(x)N(x) = N^*(y)J(N^*(y))^{-1}N^*(x)N(x) = J(N(y))^{-1}(N^*(y))^{-1}N^*(x)N(x)$$

and, since $Je_1 = e_2$ and $J^* = -J$,

$$\langle JN^*(x)N(x)e_1, N^*(y)N(y)e_2\rangle = \langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)N(x)e_1, e_1\rangle\,.$$

Similarly, $\langle JN^*(x)N(x)e_2, N^*(y)N(y)e_1\rangle = -\langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)N(x)e_2, e_2\rangle$. Hence,

$$\mathcal{I} = \frac{1}{2}\int_0^1\int_0^1 \sum_{j=1}^{2}\langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)N(x)e_j, e_j\rangle dxdy =$$

$$\frac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\Big((N(y))^{-1}(N^*(y))^{-1}N^*(x)N(x)\Big)dxdy = \frac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\Big((N^*(y))^{-1}N^*(x)N(x)(N(y))^{-1}\Big)dxdy\,.$$

Now, we use the formula $N(x)(N(y))^{-1} = U_+(x,y)$ to rewrite the last expression as

$$\mathcal{I} = \frac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\Big(U_+^*(x,y)U_+(x,y)\Big)dxdy\,.$$

Finally, (5.7) follows from $U_+(x,y) \in \mathrm{SL}(2,\mathbb{R})$ by direct inspection after one uses the identities $JU_+(x,y)J = -U_-(x,y)$ and $\operatorname{tr}(A^*A) - 2 = \|(A + JAJ)e_1\|^2$, which holds for every $A \in \mathrm{SL}(2,\mathbb{R})$. $\square$

Remark. The integrand in (5.6) is symmetric: $\operatorname{tr}\Big(U_+^*(x,y)U_+(x,y)\Big) = \operatorname{tr}\Big(U_+^*(y,x)U_+(y,x)\Big)$ because $U_+(x,y) = U_+^{-1}(y,x)$ and $U_+(x,y) \in \mathrm{SL}(2,\mathbb{R})$. Notice also, that

$$\operatorname{tr}\Big(U_+^*(x,y)U_+(x,y)\Big) = \lambda_{x,y}^2 + \lambda_{x,y}^{-2} \geqslant 2\,,$$

where $\lambda_{x,y}$ is an eigenvalue of $U_+^*(x,y)U_+(x,y)$ which explains why the left-hand side in (5.7) is nonnegative.

Lemma 5.2. *Suppose real-valued matrix-function $V = \left(\begin{smallmatrix} v_1 & v_2 \\ v_2 & -v_1\end{smallmatrix}\right)$ is defined on $[0,1]$ and satisfies $\|V\|_{L^1[0,1]} < \infty$. Consider $\mathcal{H} = N^*N$, where N: $N' = VN, N(0) = I$. Then,*

$$\det\int_0^1 \mathcal{H}\,dx - 1 \lesssim \|V\|_{L^1[0,1]}^2 \exp(C\|V\|_{L^1[0,1]})\,. \tag{5.9}$$

Proof. The integral equation for N is

$$N = I + \int_0^x VNds\,. \tag{5.10}$$

By Gronwall's inequality,

$$\|N(x)\| \leqslant \exp\left(\int_0^x \|V(s)\|ds\right) \leqslant \exp(\|V\|_{L^1[0,1]}). \tag{5.11}$$

Iteration of (5.10) gives

$$N = I + \int_0^x Vdx_1 + \int_0^x V(x_1)\left(\int_0^{x_1} V(x_2)N(x_2)dx_2\right)dx_1\,.$$

Then,

$$\int_0^1 N^*Ndx = I + 2\int_0^1 \left(\int_0^x V(x_1)dx_1\right)dx + R, \ \ \|R\| \lesssim \|V\|^2_{L^1[0,1]} \exp(C\|V\|_{L^1[0,1]})\,.$$

Since $\operatorname{tr} V = 0$, the identity $\det(I+A) = 1 + \operatorname{tr} A + \det A$, which holds for all 2×2 matrices A, gives

$$\det \int_0^1 \mathcal{H}dx - 1 \lesssim \|V\|^2_{L^1[0,1]} \exp(C\|V\|_{L^1[0,1]})\,.$$

□

Lemma 5.3. *Suppose real-valued symmetric matrix-functions V and O are defined on $[0,1]$ and satisfy*

$$V = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix} = O + O', O = O^* = \begin{pmatrix} o_1 & o_2 \\ o_2 & -o_1 \end{pmatrix}, \tag{5.12}$$

$$\delta := \|O\|_{L^2[0,1]} < \infty\,, \tag{5.13}$$

$$d := \|O'\|_{L^2[0,1]} < \infty\,. \tag{5.14}$$

Consider $\mathcal{H} = N^*N$ where $N' = VN, N(0) = I$. Then, we have

$$\det \int_0^1 \mathcal{H}\,dx - 1 = 4\sum_{j=1}^2 \int_0^1 |g_j - \langle g_j\rangle|^2\,dx + r,\ |r| \lesssim \delta^{2.5}\exp(C(d+\delta))\,, \tag{5.15}$$

where

$$g_j := \int_0^x v_j\,dx \tag{5.16}$$

and C is an absolute positive constant. An analogous result holds if O and V are related by $V = O - O'$.

Proof. We will use the formula (5.7) for our analysis. Fix $y \in [0,1]$ and take $U_+(x,y)$ and $U_-(x,y)$ which solve $\frac{d}{dx}U_+(x,y) = V(x)U_+(x,y), U_+(y,y) = I$ and $\frac{d}{dx}U_-(x,y) = -V(x)U_-(x,y), U_-(y,y) = I$. Iterating the corresponding integral equations, one gets

$$U_+(x,y) = I + \int_y^x Vdx_1 + \int_y^x V\int_y^{x_1} Vdx_2dx_1 + \int_y^x V\int_y^{x_1} V\int_y^{x_2} Vdx_3dx_2dx_1 +$$
$$\int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vdx_4dx_3dx_2dx_1 + \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vf_+dx_4dx_3dx_2dx_1,$$
$$f_+(x_4) = \int_y^{x_4} V(s)U_+(s,y)ds\,.$$

$$U_-(x,y) = I - \int_y^x Vdx_1 + \int_y^x V\int_y^{x_1} Vdx_2dx_1 - \int_y^x V\int_y^{x_1} V\int_y^{x_2} Vdx_3dx_2dx_1 +$$
$$\int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vdx_4dx_3dx_2dx_1 - \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vf_-dx_4dx_3dx_2dx_1,$$
$$f_-(x_4) = \int_y^{x_4} V(s)U_-(s,y)ds\,.$$

Taking $U_+(x,y)-U_-(x,y)$ as in (5.7) leaves us with

$$\frac{U_+(x,y)-U_-(x,y)}{2}=\int_y^x Vdx_1+\mathcal{I}_1+\mathcal{I}_2\,, \tag{5.17}$$

$$\mathcal{I}_1=\int_y^x V\int_y^{x_1} V\int_y^{x_2} Vdx_3dx_2dx_1, \tag{5.18}$$

$$\mathcal{I}_2=\int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} V(f_++f_-)dx_4dx_3dx_2dx_1\,. \tag{5.19}$$

Recall that $V=O+O'$ where O satisfies (5.13) and (5.14). These assumptions are to be used in the following proposition. On $\mathbb{R}^2_+$, we define the partial order

$$\begin{bmatrix} x_1\\ x_2\end{bmatrix}\leqslant\begin{bmatrix} y_1\\ y_2\end{bmatrix}$$

by requiring that $x_1\leqslant y_1$ and $x_2\leqslant y_2$.

Proposition 5.2. *Suppose a matrix-function O is defined on $[0,1]$ and denote*

$$\delta=\|O\|_{L^2[0,1]},\ d=\|O'\|_{L^2[0,1]}. \tag{5.20}$$

Let an operator $G_{(y)}$ be given by: $F\mapsto (G_{(y)}F)(x)=\int_y^x(O+O')Fds$ where $y\in[0,1]$ and a matrix-function F, defined on $[0,1]$, satisfies $\|F\|_{L^\infty[0,1]}<\infty$ and $\|F'\|_{L^2[0,1]}<\infty$. Then,

$$\begin{bmatrix} \|G_{(y)}F\|_{L^\infty[0,1]}\\ \|(G_{(y)}F)'\|_{L^2[0,1]}\end{bmatrix}\leqslant C\mathcal{M}\begin{bmatrix} \|F\|_{L^\infty[0,1]}\\ \|F'\|_{L^2[0,1]}\end{bmatrix},\quad \mathcal{M}=\begin{pmatrix}\delta+\sqrt{\delta d} & \delta\\ \delta+d & 0\end{pmatrix}, \tag{5.21}$$

where C is an absolute positive constant, the norms and derivatives are computed with respect to x.

Proof. Let $b=\|F\|_{L^\infty[0,1]}, c=\|F'\|_{L^2[0,1]}$. Write

$$O^*(x)O(x)-O^*(y)O(y)=\int_y^x((O^*)'O+O^*O')ds\,. \tag{5.22}$$

Then,

$$\|O(x)\|^2=\max_{\|\xi\|_{\mathbb{C}^2}\leqslant 1}\|O(x)\xi\|^2_{\mathbb{C}^2}=\max_{\|\xi\|_{\mathbb{C}^2}\leqslant 1}\langle O^*(x)O(x)\xi,\xi\rangle\overset{(5.22)}{\leqslant}\|O(y)\|^2+2\int_0^1\|O'(s)\|\cdot\|O(s)\|ds\,.$$

Applying Cauchy-Schwarz inequality to the integral, integrating in y from 0 to 1 and maximizing in x gives

$$\|O\|_{L^\infty[0,1]}\lesssim\delta+(d\delta)^{\frac12}\,. \tag{5.23}$$

Then,

$$(G_{(y)}F)(x)=\int_y^x OFds+O(x)F(x)-O(y)F(y)-\int_y^x OF'ds$$

and the estimate for the first coordinate in (5.21) follows from Cauchy-Schwarz inequality and (5.23). Since $(G_{(y)}F)'=(O+O')F$, we get $\|(G_{(y)}F)'\|_{L^2[0,1]}\leqslant(\|O\|_{L^2[0,1]}+\|O'\|_{L^2[0,1]})\|F\|_{L^\infty[0,1]}$ and the bound for the second coordinate in (5.21) is obtained. □

Continuation of the proof of Lemma 5.3. We apply the proposition to $\mathcal{I}_1$ three times with the initial choice of F: $F=I$. That gives rise to taking the third power of matrix $\mathcal{M}$: $\mathcal{M}^3$, applying it to $(1,0)^t$, and looking at the first coordinate. As the result, one has $\|\mathcal{I}_1\|_{L^\infty[0,1]}\lesssim\delta^{\frac32}(\delta+d)^{\frac32}$. Therefore,

$$\|\mathcal{I}_1e_1\|_{L^\infty([0,1]^2)}\lesssim\delta^{\frac32}\exp(\delta+d). \tag{5.24}$$

Similarly, we consider $\mathcal{I}_2$ and use the previous proposition four times making the first choice of F as $F=f_++f_-$. Applying the bound (5.11) to U_+ and U_-, we get $\|f_++f_-\|_{L^\infty[0,1]}\lesssim(\delta+d)\exp(\delta+d),\|f'_++f'_-\|_{L^2[0,1]}\lesssim(\delta+d)\exp(\delta+d)$. This time, we compute the fourth power of matrix $\mathcal{M}:\mathcal{M}^4$, apply it to vector $(\delta+d)\exp(\delta+d)(1,1)^t$, and look at the first coordinate. In the end, one has

$$\|\mathcal{I}_2e_1\|_{L^\infty([0,1]^2)}\lesssim\delta^2\exp(C(d+\delta))\,. \tag{5.25}$$

The first term in (5.17) can be written as

$$\int_y^x Vds=\int_y^x Ods+O(x)-O(y)$$

and

$$\left\|\int_y^x O ds + O(x) - O(y)\right\|_{L^2([0,1]^2)} \lesssim \delta\,. \tag{5.26}$$

For any three vectors v_1, v_2 and v_3 in $\mathbb{R}^2$, we have an estimate

$$\big|\|v_1+v_2+v_3\| - \|v_1\|\big| \leqslant \|v_2+v_3\| \leqslant \|v_2\| + \|v_3\|\,,$$

which follows from the triangle inequality. Multiplying with

$$\|v_1+v_2+v_3\| + \|v_1\| \leqslant 2\|v_1\| + \|v_2\| + \|v_3\|\,,$$

we get

$$\big|\|v_1+v_2+v_3\|^2 - \|v_1\|^2\big| \leqslant 2\|v_1\|(\|v_2\|+\|v_3\|) + (\|v_2\|+\|v_3\|)^2\,.$$

Applying it to (5.17) gives

$$\left|\frac14 \|(U_+(x,y)-U_-(x,y))e_1\|^2 - \left\|\left(\int_y^x V ds\right)e_1\right\|^2\right|$$
$$\lesssim \left\|\left(\int_y^x V ds\right)e_1\right\| \cdot (\|\mathcal{I}_1 e_1\| + \|\mathcal{I}_2 e_1\|) + \|\mathcal{I}_1 e_1\|^2 + \|\mathcal{I}_2 e_1\|^2\,.$$

Taking $L^1([0,1]^2)$ norm in variables x and y of both sides and using (5.24), (5.25), (5.26) and the Cauchy-Schwartz inequality gives

$$\frac14\int_0^1\int_0^1 \|(U_+(x,y)-U_-(x,y))e_1\|^2 dxdy = \int_0^1\int_0^1 \left\|\left(\int_y^x V ds\right)e_1\right\|^2 dxdy + r,\ \ |r| \lesssim \delta^{2.5}\exp(C(d+\delta))\,.$$

Recalling the definition (5.16), we get

$$\left\|\left(\int_y^x V ds\right)e_1\right\|^2 = \sum_{j=1}^2 (g_j(x)-g_j(y))^2$$

so

$$\frac12\int_0^1\int_0^1 \|(U_+(x,y)-U_-(x,y))e_1\|^2 dxdy = 4\sum_{j=1}^2\int_0^1 |g_j - \langle g_j\rangle|^2 dx + r,\ \ |r| \lesssim \delta^{2.5}\exp(C(d+\delta))\,.$$

Lemma 5.3 is proved. □

Remark. All statements in this subsection can be easily adjusted to any interval but the constants in the inequalities will depend on the size of that interval.

5.3. Rough bound when $\widetilde{\mathcal{K}}_Q$ is small.

Lemma 5.4. *Suppose an absolutely continuous function f is defined on $[0,1]$ and satisfies*

$$f \in L^2[0,1], \quad f' = l_1 + l_2, \quad l_1 \in L^1[0,1], \quad l_2 \in L^2[0,1]\,. \tag{5.27}$$

Then, $\|f\|_{L^\infty[0,1]} \leqslant \sqrt{\delta^2 + 2(\delta\tau + \epsilon(\tau+\epsilon+\delta))}$, *where* $\delta = \|f\|_{L^2[0,1]}$, $\epsilon = \|l_1\|_{L^1[0,1]}$, $\tau = \|l_2\|_{L^2[0,1]}$.

Proof. There is $\xi \in [0,1]$ such that $|f(\xi)| \leqslant \delta$ and

$$|f(x) - f(\xi)| \leqslant \left|\int_\xi^x f' ds\right| \leqslant \tau + \epsilon\,.$$

Thus, $\|f\|_{L^\infty[0,1]} \leqslant \tau + \epsilon + \delta$. Then, writing

$$f^2(x) - f^2(y) = 2\int_y^x f f' ds\,,$$

integrating in y and maximizing in x, we get

$$\|f\|^2_{L^\infty[0,1]} \leqslant \delta^2 + 2(\delta\tau + \epsilon(\tau+\epsilon+\delta))\,.$$

□

Suppose Q is real-valued, symmetric matrix-function on $\mathbb{R}$ with zero trace and $\|Q\|_{L^2(\mathbb{R})} < \infty$. Define $\mathcal{H}_Q = N^*N$, where $N : N' = JQN, N(0) = I$. Notice that $\det\int_n^{n+2} S^*\mathcal{H}_Q S dx = \det\int_n^{n+2}\mathcal{H}_Q dx$

for every constant matrix $S \in \mathrm{SL}(2,\mathbb{R})$. Therefore, we can apply Lemma 5.2 to each interval $[n, n+2]$ by choosing $S = N^{-1}(n)$ and get an estimate which explains how $\|Q\|_{L^2(\mathbb{R})}$ controls $\widetilde{\mathcal{K}}_Q$:

$$\widetilde{\mathcal{K}}_Q = \sum_{n\in\mathbb{Z}}\left(\det\int_n^{n+2}\mathcal{H}_Q dx - 4\right) \lesssim \sum_{n\in\mathbb{Z}}\|Q\|^2_{L^2[n,n+2]}\exp(C\|Q\|_2) \lesssim \|Q\|^2_{L^2(\mathbb{R})}\exp(C\|Q\|_2)\,.$$

The next lemma shows that $\widetilde{\mathcal{K}}_Q$ controls the convolution of Q with the exponential.

Lemma 5.5. *Suppose Q is real-valued, symmetric 2×2 matrix-function on $\mathbb{R}$ with zero trace and entries in $L^2(\mathbb{R})$. Define $\mathcal{H}_Q = N^*N$ where $N : N' = JQN, N(0) = I$ and assume that $\widetilde{\mathcal{K}}_Q < \infty$. If $O := e^x\int_x^\infty e^{-s}Qds$, then $\|O\|_{L^\infty(\mathbb{R})} \lesssim \exp(C(\|Q\|_{L^2(\mathbb{R})} + \widetilde{\mathcal{K}}_Q))\widetilde{\mathcal{K}}_Q^{\frac14}$ where C is a positive absolute constant.*

Proof. Let $R = \|Q\|_{L^2(\mathbb{R})}$ and $E = \widetilde{\mathcal{K}}_Q$. We split the proof into several steps.

1. Bound for a single interval $[0,1]$**.** The definitions (3.5) and (3.14) imply that $\widetilde{\mathcal{K}}_Q^+ \leqslant E$. From Theorem 1.2 and Theorem 3.2 in [2], we know that $\mathcal{H}_Q$ admits the following factorization on $\mathbb{R}_+$: $\mathcal{H}_Q = G^*WG$ where G and W satisfy conditions:

$$G' = J(v_1+v_2)G, \quad \|v_1\|_{L^1(\mathbb{R}_+)} \lesssim E, \quad \|v_2\|_{L^2(\mathbb{R}_+)} \lesssim E^{\frac12}\,, \tag{5.28}$$

$$\det G = 1, \quad v_1 + v_2 = (v_1+v_2)^*\,, \tag{5.29}$$

and

$$W \geqslant 0, \quad \det W = 1, \quad \|\mathrm{tr}\, W - 2\|_{L^1(\mathbb{R}_+)} \lesssim E\,.$$

Since $\|\mathrm{tr}\,W - 2\|_{L^1[0,1]} \lesssim E$, we have $\|\lambda + \lambda^{-1} - 2\|_{L^1[0,1]} \lesssim E$, where λ is the largest eigenvalue of W. If one denotes $p = \mathrm{tr}\,W - 2 = \lambda + \lambda^{-1} - 2$, then

$$\lambda = \frac{2+p+\sqrt{4p+p^2}}{2}, \quad \lambda^{-1} = \frac{2+p-\sqrt{4p+p^2}}{2}\,. \tag{5.30}$$

In particular, that yields

$$\int_0^1 \|W\|\,dx \lesssim 1 + E\,. \tag{5.31}$$

The given conditions on Q and (5.11) yield

$$\|N(x)\|, \|N^{-1}(x)\| \lesssim \exp(CR), \quad x\in[0,1]\,,$$

where the second estimate follows from the first since $\det N = 1$. The Hamiltonian $\mathcal{H}_Q = N^*N$ is absolutely continuous on $\mathbb{R}_+$ and

$$0 < \exp(-CR)I \lesssim \mathcal{H}_Q(x) \lesssim \exp(CR)I \tag{5.32}$$

on $[0,1]$. We claim that $\|G(0)\| \lesssim \exp(C(R+E))$ and that $\|G^{-1}(0)\| \lesssim \exp(C(R+E))$. Indeed, if X satisfies $X' = J(v_1+v_2)X$ and $X(0) = I$, then $G = XG(0)$. Moreover, given conditions on v_1 and v_2 and $\det X = 1$, we have

$$\|X(x)\| \lesssim \exp(CE), \qquad \|X^{-1}(x)\| \lesssim \exp(CE) \tag{5.33}$$

uniformly on $[0,1]$. Identity $\mathcal{H}_Q = G^*(0)X^*WXG(0)$ yields

$$(G^*(0))^{-1}\mathcal{H}_Q(G(0))^{-1} = X^*WX\,.$$

Taking an arbitrary $\xi\in\mathbb{C}^2$ with $\|\xi\|_{\mathbb{C}^2} = 1$, we get

$$\begin{aligned}\|G^{-1}(0)\xi\|^2 &\overset{(5.32)}{\lesssim} \exp(CR)\int_0^1\langle\mathcal{H}_QG^{-1}(0)\xi, G^{-1}(0)\xi\rangle dx\\ &= \exp(CR)\int_0^1\langle WX\xi, X\xi\rangle dx \overset{(5.31)+(5.33)}{\lesssim} \exp(C(R+E))\,,\end{aligned}$$

which implies $\|G^{-1}(0)\| \lesssim \exp(C(R+E))$. We also have $\|G(0)\| \lesssim \exp(C(R+E))$ since $\det G = 1$ and the claim is proved. Finally, we have

$$\|G(x)\| \lesssim \exp(C(R+E)), \quad \|G^{-1}(x)\| \lesssim \exp(C(R+E))$$

for $x\in[0,1]$ since $G = XG(0)$.

Next, let us study W and W'. Since $W = (G^*)^{-1}N^*NG^{-1}$, one has $\|W\| \lesssim \exp(C(R+E))$ on $x \in [0,1]$. Recall that $W \geqslant 0$ and $\det W = 1$, so

$$\exp(-C(R+E))I \lesssim W \lesssim \exp(C(R+E))I, \quad x \in [0,1]\,.$$

Since λ is the largest eigenvalue of W and $\lambda \lesssim \exp(C(R+E))$, then (5.30) yields $\|\lambda - 1\|_{L^2[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E))$ and $\|\lambda^{-1}-1\|_{L^2[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E))$. Introduce $\Upsilon = W - I$. The matrix Υ is unitarily equivalent to $\begin{pmatrix} \lambda-1 & 0 \\ 0 & 1/\lambda-1 \end{pmatrix}$ and that gives

$$\|\Upsilon\|_{L^2[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E))\,. \tag{5.34}$$

We need to study Υ', which is equal to W'. To do so, notice that

$$2N^*JQN = \mathcal{H}'_Q = G^*J(v_1+v_2)WG + G^*WJ(v_1+v_2)G + G^*W'G\,. \tag{5.35}$$

Hence,

$$\Upsilon' = W' = F_1 + F_2\,,$$

where

$$F_1 = -J(v_1+v_2)W - WJ(v_1+v_2), \quad F_2 = 2(G^*)^{-1}N^*JQNG^{-1}.$$

The previously obtained estimates give us

$$\|F_1\|_{L^1[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E)), \quad \|F_2\|_{L^2[0,1]} \lesssim \exp(C(R+E))\,. \tag{5.36}$$

Now, we use (5.34), (5.36) to apply the previous lemma to each component of Υ to obtain

$$\|\Upsilon\|_{L^\infty[0,1]} \lesssim E^{\frac{1}{4}}\exp(C(R+E))\,. \tag{5.37}$$

The formula (5.35) also gives an expression for Q:

$$Q = -J(H_1+H_2)\,,$$

where

$$H_1 = 0.5(N^*)^{-1}(G^*J(v_1+v_2)WG + G^*WJ(v_1+v_2)G)N^{-1}$$

and

$$H_2 = 0.5(N^*)^{-1}(G^*\Upsilon'G)N^{-1}\,.$$

Since $\|H_1\|_{L^1[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E))$, we have

$$\left\| e^x\int_x^1 e^{-s}H_1ds \right\|_{L^\infty[0,1]} \lesssim E^{\frac{1}{2}}\exp(C(R+E))\,.$$

For smooth matrix-functions u_1, u_2, u_3, we have

$$\int_x^1 u_1u_2'u_3\,ds = u_1u_2u_3\Big|_x^1 - \int_x^1 u_1'u_2u_3\,ds - \int_x^1 u_1u_2u_3'\,ds.$$

Then,

$$2e^x\int_x^1 e^{-s}H_2ds =$$
$$e^x\left(e^{-1}(N^*(1))^{-1}G^*(1)\Upsilon(1)G(1)(N(1))^{-1} - e^{-x}(N^*(x))^{-1}G^*(x)\Upsilon(x)G(x)(N(x))^{-1}\right)$$
$$-e^x\int_x^1 (e^{-s}(N^*(s))^{-1}G^*)'\Upsilon GN^{-1}ds - e^x\int_x^1 e^{-s}(N^*(s))^{-1}G^*\Upsilon(GN^{-1})'ds\,.$$

Since $\|(N^{-1})'\|_{L^2[0,1]} \lesssim \exp(C(R+E))$ and $\|G'\|_{L^1[0,1]} \lesssim \exp(C(R+E))$, we have

$$\left\| e^x\int_x^1 e^{-s}H_2ds \right\|_{L^\infty[0,1]} \lesssim \|\Upsilon\|_{L^\infty[0,1]}\exp(C(R+E)) \overset{(5.37)}{\lesssim} E^{\frac{1}{4}}\exp(C(R+E))\,.$$

Summing up, we get

$$\left\| e^x\int_x^1 e^{-s}Qds \right\|_{L^\infty[0,1]} \lesssim E^{\frac{1}{4}}\exp(C(R+E))\,. \tag{5.38}$$

2. Handling all intervals $[n,n+1], n\in\mathbb{Z}$**.** Take any $n\in\mathbb{Z}$. Our immediate goal is to show the bound

$$\left\|e^x\int_x^{n+1}e^{-s}Qds\right\|_{L^\infty[n,n+1]}\lesssim E^{\frac14}\exp(C(R+E)) \tag{5.39}$$

analogous to (5.38) but written for interval $[n,n+1]$. To this end, take the Hamiltonian $\mathcal{H}^{(n)}(x):=\mathcal{H}_Q(x+n)$ defined on $\mathbb{R}_+$. For the corresponding $\widetilde{\mathcal{K}}^+_{(n)}$, we get $\widetilde{\mathcal{K}}^+_{(n)}\leqslant E$ as follows from its definition. Since the $\widetilde{\mathcal{K}}$-characteristics of the Hamiltonians $\mathcal{H}$ and $S^*\mathcal{H}S$ are equal for every constant matrix $S\in\mathrm{SL}(2,\mathbb{R})$, we can instead consider $\widehat{\mathcal{H}}^{(n)}=\widehat{N}^*\widehat{N}$ where $\widehat{N}'=JQ(x+n)\widehat{N}, \widehat{N}(0)=I$. Using the arguments in step 1 for $\widehat{\mathcal{H}}^{(n)}$, we get (5.39).

3. Summing up. Denote $O_n(x)=e^x\int_x^\infty e^{-s}Q\cdot\chi_{n<s<n+1}ds$ and notice that $O=\sum_{n\in\mathbb{Z}}O_n$. Then, since $O_n(x)=0$ for $x>n+1$ and $\|O_n(x)\|\lesssim e^{x-n}\|O_n\|_{L^\infty[n,n+1]}$ for $x<n$, we get

$$\|O(x)\|\leqslant\sum_{n\in\mathbb{Z}}\|O_n(x)\|\lesssim E^{\frac14}\exp(C(R+E))\sum_{n\geqslant 0}e^{-n}\sim E^{\frac14}\exp(C(R+E))$$

as follows from (5.39). That finishes the proof of Lemma 5.5. □

5.4. **Proof of Theorem 4.1.** Denote $E=\widetilde{\mathcal{K}}_Q$, $O=e^x\int_x^\infty e^{-s}Qds$, and recall that $\|O\|_{L^2(\mathbb{R})}\sim\|Q\|_{H^{-1}(\mathbb{R})}\leqslant\|Q\|_{L^2(\mathbb{R})}$.

1. Lower bound. Define $\delta_n=\|O\|_{L^2[n,n+1]}$. By Lemma 5.5, we know that $\sup_n\delta_n\lesssim E^{\frac14}\exp(C(R+E))$. Next, we apply Lemma 5.3 to each interval $[n,n+2]$. The remainder r_n in that lemma allows the estimate

$$r_n\lesssim(\delta_n+\delta_{n+1})^{2.5}\exp(C(\delta_n+\delta_{n+1}+R)),\quad n\in\mathbb{Z}\,.$$

For each $R>0$ and $\eta>0$, we can find a positive $E_0(R,\eta)$ such that $E\in(0,E_0(R,\eta))$ implies that the remainder r_n is smaller than $\eta(\delta_n^2+\delta_{n+1}^2)$ uniformly in all n. For example, one can take

$$E_0(R,\eta)\sim e^{-C_\eta R}, \tag{5.40}$$

where C_η is a sufficiently large positive number that depends on η. Therefore, for such E and some positive constant c independent of η, we have

$$\sum_{n\in\mathbb{Z}}(c-\eta)\delta_n^2\lesssim\sum_{n\in\mathbb{Z}}\left(\det\int_n^{n+2}\mathcal{H}_Qdx-4\right)\lesssim\sum_{n\in\mathbb{Z}}(c+\eta)\delta_n^2,$$

where the Proposition 5.1 has been applied to the terms $\int_n^{n+2}|g_j-\langle g_j\rangle|^2\,dx$ in the right-hand side of (5.15), adjusted to the interval $[n,n+2]$, to show that they are comparable to $\delta_n^2+\delta_{n+1}^2$. Taking $\eta=c/2$, we see that

$$E=\sum_{n\in\mathbb{Z}}\left(\det\int_n^{n+2}\mathcal{H}_Qdx-4\right)\sim\sum_{n\in\mathbb{Z}}\delta_n^2\sim\|O\|^2_{L^2(\mathbb{R})},$$

in the case $E\leqslant E_0(R,\frac c2)$. If $E>E_0(R,\frac c2)$, one uses inequality $\|O\|_{L^2(\mathbb{R})}\lesssim R$ to get

$$e^{-CR}\|O\|^2_{L^2(\mathbb{R})}\lesssim\frac{E_0(R,\frac c2)}{1+R^2}\|O\|^2_{L^2(\mathbb{R})}\lesssim E\,, \tag{5.41}$$

which holds for some positive absolute constant C due to (5.40). That provides the required lower bound.

2. Upper bound. Let $\delta_n=\|O\|_{L^2[n,n+1]}$. For given value of R, apply Lemma 5.3 and Proposition 5.1 to each interval $[n,n+2]$. That gives

$$E\lesssim\sum_{n\in\mathbb{Z}}\delta_n^2e^{C(R+\delta_n)}$$

with an absolute constant C. Since $\sum_{n\in\mathbb{Z}}\delta_n^2\sim\|q\|^2_{H^{-1}(\mathbb{R})}$ and $\|q\|_{H^{-1}(\mathbb{R})}\lesssim R$, one has

$$E\lesssim\|q\|^2_{H^{-1}(\mathbb{R})}e^{C(R+\|q\|_{H^{-1}(\mathbb{R})})}\lesssim\|q\|^2_{H^{-1}(\mathbb{R})}e^{C_2R}\,.$$

□

6. Appendix

Here we collect some auxiliary results used in the main text.

1. We start with an example that shows that the scattering transform is not injective when defined on $q \in L^2(\mathbb{R})$. This is an analog of Lemma 17 in [27].

Example 6.1. *There exist potentials $q_1, q_2 \in L^2(\mathbb{R})$ such that $q_1 \neq q_2$ in $L^2(\mathbb{R})$ but we have $\mathbf{r}_{q_1} = \mathbf{r}_{q_2}$ a.e. on $\mathbb{R}$ for their reflection coefficients. In other words, the scattering transform $q \mapsto \mathbf{r}_q$ is not injective on $L^2(\mathbb{R})$.*

Proof. Let us consider

$$\mathfrak{a}_1^+ = 1, \quad \mathfrak{b}_1^+ = 0, \quad \mathfrak{a}_1^- = \mathfrak{a}, \quad \mathfrak{b}_1^- = \mathfrak{b},$$

and

$$\mathfrak{a}_2^+ = \mathfrak{a}, \quad \mathfrak{b}_2^+ = \mathfrak{b}, \quad \mathfrak{a}_2^- = 1, \quad \mathfrak{b}_2^- = 0,$$

where $\mathfrak{a} = 1 + i/x$, $\mathfrak{b} = i/x$. Note that

$$\int_{\mathbb{R}} \log(1 - |s_k^\pm(x)|^2)\, dx > -\infty, \qquad s_k^\pm = \frac{\mathfrak{b}^\pm}{\mathfrak{a}_k^\pm}, \qquad k = 1, 2.$$

Theorem 12.11 in [13] says that for every contractive analytic function s on $\mathbb{C}_+$ whose boundary values on $\mathbb{R}$ satisfy $\log(1 - |s|^2) \in L^1(\mathbb{R})$ there exists a unique coefficient $A \in L^2(\mathbb{R}_+)$ such that $s = \lim_{\xi \to +\infty} \frac{\mathfrak{B}(\xi,\lambda)}{\mathfrak{A}(\xi,\lambda)}$, $\lambda \in \mathbb{C}_+$ for the continuous Wall polynomials generated by A. Moreover, we have

$$2\pi \|A\|^2_{L^2(\mathbb{R}_+)} = \|\log(1 - |s|^2)\|_{L^1(\mathbb{R}_+)}. \tag{6.1}$$

Applying this result, we see that there exist functions $A_1^\pm, A_2^\pm \in L^2(\mathbb{R}_+)$ such that $\mathfrak{a}_{1,2}^\pm$, $\mathfrak{b}_{1,2}^\pm$ are the limits of their continuous Wall polynomials. Now define potentials $q_{1,2} \in L^2(\mathbb{R})$ by relations

$$A_{1,2}^+(\xi) = -\overline{q_{1,2}(\xi/2)}/2, \qquad A_{1,2}^-(\xi) = q_{1,2}(-\xi/2)/2, \qquad \xi \in \mathbb{R}_+.$$

From Proposition 2.2, we conclude that the coefficients $a_{1,2}$, $b_{1,2}$ for these potentials satisfy

$$a_1 = \mathfrak{a} = a_2, \qquad b_1 = -\mathfrak{b} = \overline{\mathfrak{b}} = b_2,$$

on $\mathbb{R} \setminus \{0\}$. Hence, $\mathbf{r}_{q_1} = \mathbf{r}_{q_2}$ on $\mathbb{R} \setminus \{0\}$. On the other hand, we have $A_1^+ = 0$ and $A_2^- = 0$ by construction. It follows that $\operatorname{supp} q_1 \subset (-\infty, 0]$ and $\operatorname{supp} q_2 \subset [0, +\infty)$. Since q_1 and q_2 are nonzero (they have a nonzero $L^2(\mathbb{R})$-norm as follows from (6.1)), that yields $q_1 \neq q_2$ in $L^2(\mathbb{R})$. □

2. Next, we outline how to prove that the spectral representation for the Dirac operator $\mathcal{D}_Q$, defined by relation (3.1), is given by the Weyl-Titchmarsh transform (3.10). To this end, we will use the corresponding result for canonical Hamiltonian systems proved in [24].

At first, we note that if $\mathcal{H}_Q = N_Q^* N_Q$ is the Hamiltonian from Theorem 3.1, then $\det \mathcal{H}_Q = 1$ on $\mathbb{R}$ and the operator $V : X \mapsto N_Q^{-1} X$ is unitary from $L^2(\mathbb{R}, \mathbb{C}^2)$ onto the Hilbert space

$$L^2(\mathcal{H}_Q) = \Big\{ Y : \mathbb{R} \to \mathbb{C}^2 : \ \|Y\|^2_{L^2(\mathcal{H}_Q, \mathbb{R})} = \int_{\mathbb{R}} \langle \mathcal{H}_Q(\xi) Y(\xi), Y(\xi) \rangle_{\mathbb{C}^2}\, d\xi < \infty \Big\}.$$

Moreover, $V \mathcal{D}_Q V^{-1}$ coincides with the operator $\mathcal{D}_{\mathcal{H}_Q} : Y \mapsto \mathcal{H}^{-1} J Y'$ of the canonical Hamiltonian system generated by the Hamiltonian $\mathcal{H}_Q$. Thus, the operator $\mathcal{D}_Q$ on $L^2(\mathbb{R}, \mathbb{C}^2)$ is unitary equivalent to the operator $\mathcal{D}_{\mathcal{H}_Q}$ on $L^2(\mathcal{H}_Q)$. Let $\widetilde{M}$ be the solution of Cauchy problem

$$J \widetilde{M}'(\xi, z) = z \mathcal{H}_Q(\xi) \widetilde{M}(\xi, z), \qquad \widetilde{M}(0, z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{6.2}$$

where $z \in \mathbb{C}$, $\xi \in \mathbb{R}$, and the differentiation is taken with respect to $\xi \in \mathbb{R}$. The Weyl-Titchmarsh transform for $\mathcal{D}_{\mathcal{H}_Q}$ is defined by

$$\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}} : Y \mapsto \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \widetilde{M}(\xi, \lambda)^* \mathcal{H}_Q(\xi) Y(\xi)\, d\xi$$

on a dense subset of $L^2(\mathcal{H}_Q)$ of smooth compactly supported functions. This operator is unitary from $L^2(\mathcal{H}_Q)$ onto the space $L^2(\rho)$ defined in the same way as at the beginning of Section 3. Specifically, we let $m_\pm$ be the half-line Weyl functions of $\mathcal{H}_Q$ and define ρ as the representing measure for the matrix-valued Herglotz function m in (3.8). It was proved in Theorem 3.21 in [24] that $\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}} \mathcal{D}_{\mathcal{H}_Q} \mathcal{F}^{-1}_{\mathcal{D}_{\mathcal{H}_Q}}$

coincides with the operator of multiplication by the independent variable in $L^2(\rho)$. We also have

$$\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}}(VX) = \mathcal{F}_{\mathcal{D}_Q}X, \qquad X \in L^2(\mathbb{R}, \mathbb{C}^2).$$

Thus, we only need to check that the Weyl functions $m_\pm$ used in Section 3 coincide with the half-line Weyl functions of the Hamiltonian $\mathcal{H}_Q$. For the $\mathbb{R}_+$-Weyl functions this follows from Lemma 6.1 below. Comparing the formulas for A^+, A^- in the beginning of Section 3, we see that the Weyl function m_- for $\mathcal{D}_Q$ corresponds to the Weyl function m_+ for $\mathcal{D}_{\widetilde{Q}}$ where $\widetilde{Q}(\xi) = \sigma_3 Q(-\xi)\sigma_3$. Similarly, in the setting of canonical Hamiltonian systems, the Weyl function m_- for $\mathcal{D}_{\mathcal{H}_Q}$ coincides with the Weyl function m_+ of $\mathcal{D}_{\widetilde{\mathcal{H}}_Q}$, $\widetilde{\mathcal{H}}_Q(\xi) = \sigma_3\mathcal{H}(-\xi)\sigma_3$. Therefore, the statement for A^- follows from Lemma 6.1 below and from the relation $\widetilde{\mathcal{H}}_Q = \sigma_3\mathcal{H}_Q\sigma_3 = (\sigma_3 N_Q^*\sigma_3)(\sigma_3 N_Q\sigma_3) = \mathcal{H}_{\widetilde{Q}}$.

Lemma 6.1. *Let $q \in L^2(\mathbb{R}_+)$. Define*

$$Q(\xi) = \begin{pmatrix} -\operatorname{Im} q(\xi) & -\operatorname{Re} q(\xi) \\ -\operatorname{Re} q(\xi) & \operatorname{Im} q(\xi)\end{pmatrix}, \qquad A(\xi) = -\overline{q(\xi/2)}/2, \qquad \xi \in \mathbb{R}_+.$$

Let N_Q be defined by $JN_Q'(\xi,\lambda) + Q(\xi)N_Q(\xi,\lambda) = \lambda N_Q(\xi,\lambda)$, $N_Q(0,\lambda) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Consider the Hamiltonian $\mathcal{H}_Q = N_Q^*(\xi,0)N_Q(\xi,0)$ on $\mathbb{R}_+$ and let $\widetilde{M} = \left(\begin{smallmatrix}\widetilde{M}_{11} & \widetilde{M}_{12}\\ \widetilde{M}_{21} & \widetilde{M}_{22}\end{smallmatrix}\right)$ be defined by $J\widetilde{M}'(\xi,z) = z\mathcal{H}_Q(\xi)\widetilde{M}(\xi,z)$, $\widetilde{M}(0,z) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Let, finally, P, P_*, $\widehat{P}$, $\widehat{P}_*$ be the solutions to Krein systems (2.11), (2.12) for the coefficient A on $\mathbb{R}_+$. Then,

$$\lim_{\xi\to+\infty}\frac{\widetilde{M}_{22}(\xi,z)}{\widetilde{M}_{21}(\xi,z)} = \lim_{\xi\to+\infty}\frac{(N_Q)_{22}(\xi,z)}{(N_Q)_{21}(\xi,z)} = \lim_{\xi\to+\infty} i\frac{\widehat{P}_*(\xi,z)}{P_*(\xi,z)}, \qquad z \in \mathbb{C}_+. \tag{6.3}$$

In other words, the function m_+ in (3.7) is the half-line Weyl function for the operators $\mathcal{D}_{\mathcal{H}_Q}$, $\mathcal{D}_Q$.

Proof. The formula

$$\lim_{\xi\to+\infty}\frac{\widetilde{M}_{22}(\xi,z)}{\widetilde{M}_{21}(\xi,z)} = \lim_{\xi\to+\infty}\frac{(N_Q)_{22}(\xi,z)}{(N_Q)_{21}(\xi,z)}$$

for $\mathcal{D}_Q$ and $\mathcal{D}_{\mathcal{H}_Q}$ is well-known and can be derived from the analysis of Weyl circles by using identity $N_Q(\xi,\lambda) = N_Q(\xi,0)\widetilde{M}(\xi,\lambda)$ and the invariance of Weyl circles under transforms generated by J-unitary matrices (in our setting, the J-unitary matrix is $N_Q(\xi,0)$: we have $N_Q^*(\xi,0)JN_Q(\xi,0) = J$ on $\mathbb{R}$). See, e.g., [4] or Section 8 in [25] for more details on Weyl circles for canonical Hamiltonian systems. Thus, we focus on the second identity in (6.3) and define

$$X(\xi,z) = e^{-i\xi z}\begin{pmatrix} \dfrac{P(2\xi,z)+P_*(2\xi,z)}{2} & \dfrac{\widehat{P}(2\xi,z)-\widehat{P}_*(2\xi,z)}{2i} \\ \dfrac{P_*(2\xi,z)-P(2\xi,z)}{2i} & \dfrac{\widehat{P}(2\xi,z)+\widehat{P}_*(2\xi,z)}{2}\end{pmatrix}, \qquad \xi\in\mathbb{R}, \quad z \in \mathbb{C}.$$

Differentiating, one obtains $JX' + QX = zX$, $X(0,z) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. It follows that $X(\xi,z) = N_Q(\xi,z)$. In particular, we have

$$(N_Q)_{22} = e^{-i\xi z}\frac{\widehat{P}(2\xi,z)+\widehat{P}_*(2\xi,z)}{2}, \qquad (N_Q)_{21} = e^{-i\xi z}\frac{P_*(2\xi,z)-P(2\xi,z)}{2i}.$$

Since $P(\xi,z) \to 0$, $P_*(\xi,z) \to \Pi(z) \neq 0$ as $\xi \to +\infty$ (see Theorem 12.1 in [13]), and analogous relations hold for $\widehat{P}$ and $\widehat{P}_*$, we have

$$\lim_{\xi\to+\infty}\frac{(N_Q)_{22}(\xi,z)}{(N_Q)_{21}(\xi,z)} = \lim_{\xi\to+\infty} i\frac{\widehat{P}_*(\xi,z)}{P_*(\xi,z)}, \qquad z \in \mathbb{C}_+.$$

The lemma is proved. □

3. Lemma 6.1 and some known results for canonical systems can be used to show that weak convergence of potentials of the Dirac operator implies convergence of the corresponding Weyl functions.

Lemma 6.2. *Suppose $\{q_\ell\}_{\ell>0}$ is a bounded sequence in $L^2(\mathbb{R}_+)$ which converges to zero weakly. Let Q_ℓ be the associated matrix-functions defined as in Lemma 6.1. Then, the sequence of corresponding Weyl functions $\{m_{\ell,+}\}$ converges to i locally uniformly in $\mathbb{C}_+$ when $\ell \to +\infty$.*

Proof. For $\ell > 0$, denote by $\mathcal{H}_{Q_\ell}$ the Hamiltonian generated by Q_ℓ as in Lemma 6.1. Then, $m_{\ell,+}$ is the Weyl function for the half-line operators $\mathcal{D}_{\mathcal{H}_{Q_\ell}}$ and $\mathcal{D}_{Q_\ell}$. Since $\sup_{\ell>0} \|q_\ell\|_{L^2(\mathbb{R}_+)} < \infty$ and q_ℓ converge to zero weakly in $L^2(\mathbb{R}_+)$ as $\ell \to +\infty$, the Hamiltonians $\mathcal{H}_{Q_\ell}$ tend to the identity matrix $\mathcal{H}_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ uniformly on compact subsets on $\mathbb{R}_+$. Then, their Weyl functions $m_{+,\ell}$ tend to the Weyl function $m_+ = i$ of the Hamiltonian $\mathcal{H}_0$ locally uniformly in $\mathbb{C}_+$ by Theorem 5.7 (b) in [24]. □

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