

CONVERGENCE OF SPARSE SQUARE-SUMMABLE NLFT

SERGEY A. DENISOV

ABSTRACT. We study convergence of $SU(1,1)$ and $SU(2)$ NLFT with sparse $\ell^2(\mathbb{Z})$ data. The asymptotics of the associated polynomials orthogonal on the unit circle is obtained as a corollary.

1. THE $SU(1,1)$ AND $SU(2)$ NLFT AND MAIN RESULTS.

This note is a companion to [4] where the $SU(1,1)$ nonlinear Fourier transform (NLFT) with general square-summable coefficients was studied and we will use some results and notation from that work. Denote $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{Z}$ will stand for the set of all integers. Let $F = \{F_n\}_{n \in \mathbb{Z}}$ be a double-infinite sequence of complex numbers and $|F_n| < 1, \forall n \in \mathbb{Z}$. Taking $N \in \mathbb{Z}^+$, we define a compactly supported $F^{\langle N \rangle}$ by truncation: $F_n^{\langle N \rangle} = F_n \cdot \chi_{|n| \leq N}, n \in \mathbb{Z}$, where χ_E denotes the characteristic function of a set E . Consider 2×2 matrices $\tilde{X}_n(z, F^{\langle N \rangle}), n \in \mathbb{Z}$ defined by the recursion

$$(1.1) \quad \tilde{X}_n = \frac{1}{(1 - |F_n^{\langle N \rangle}|^2)^{\frac{1}{2}}} \begin{pmatrix} 1 & \overline{F_n^{\langle N \rangle}} z^{-n} \\ F_n^{\langle N \rangle} z^n & 1 \end{pmatrix} \tilde{X}_{n-1}, \quad z \in \mathbb{C}$$

with $\tilde{X}_n = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $n < -N$. One has $\tilde{X}_n \in SU(1,1)$ for $z \in \mathbb{T}$ and a more careful study of this recursion gives (see p.5 in [13])

$$\tilde{X}_n(z, F^{\langle N \rangle}) =: \begin{pmatrix} \mathbf{a}_n(z, F^{\langle N \rangle}) & \mathbf{b}_n^{(*)}(z, F^{\langle N \rangle}) \\ \mathbf{b}_n(z, F^{\langle N \rangle}) & \mathbf{a}_n^{(*)}(z, F^{\langle N \rangle}) \end{pmatrix},$$

where

$$(1.2) \quad f^{(*)}(z) := \overline{f(\bar{z}^{-1})}.$$

Clearly, $f^{(*)}(z) = \overline{f(z)}$ if $z \in \mathbb{T}$. We also have

$$(1.3) \quad |\mathbf{a}_n|^2 = |\mathbf{b}_n|^2 + 1, \quad z \in \mathbb{T}.$$

We put forward a definition:

Definition. For $F^{\langle N \rangle}$, define the $SU(1,1)$ nonlinear Fourier transform (NLFT) on $\mathbb{Z}$ as the map

$$F^{\langle N \rangle} \mapsto \widetilde{F_{su(1,1)}^{\langle N \rangle}} := \begin{pmatrix} \mathbf{a}(z, F^{\langle N \rangle}) \\ \mathbf{b}(z, F^{\langle N \rangle}) \end{pmatrix},$$

where $\mathbf{a}(z, F^{\langle N \rangle}) := \mathbf{a}_\infty(z, F^{\langle N \rangle})$ and $\mathbf{b}(z, F^{\langle N \rangle}) = \mathbf{b}_\infty(z, F^{\langle N \rangle})$.

In fact, since $F_n^{\langle N \rangle} = 0$ for $n > N$, we get

$$\begin{pmatrix} \mathbf{a}(z, F^{\langle N \rangle}) \\ \mathbf{b}(z, F^{\langle N \rangle}) \end{pmatrix} = \begin{pmatrix} \mathbf{a}_n(z, F^{\langle N \rangle}) \\ \mathbf{b}_n(z, F^{\langle N \rangle}) \end{pmatrix}, \quad \forall n \geq N.$$

We refer the reader to [4, 6, 7, 9–11, 13] for additional references and discussion about the NLFT. Our first result is the following theorem.

Theorem 1.1. Suppose $\{n_j\}_{j=0}^\infty$ is a subsequence in $\mathbb{N}$ that satisfies

$$(1.4) \quad \frac{n_{j+1}}{n_j} \geq q, \quad \forall j \in \mathbb{Z}^+, \quad n_0 = 1,$$

This research was supported by NSF grant DMS-2450716, the Simons Fellowship in Mathematics, the Simons Travel Support for Mathematicians Award, and the Van Vleck Professorship Research Award.

where $q \geq 2$. Take $\{\delta_j\} \in \ell^2(\mathbb{Z}^+)$ such that $\|\{\delta_j\}\|_{\ell^\infty(\mathbb{Z}^+)} < 1$. Let $F_{n_j} = \delta_j$, $\forall j \in \mathbb{Z}^+$ and set $F_n = 0$ for all other $n \in \mathbb{Z}$. For any $p \in [2, \infty)$, the limit

$$\lim_{N \rightarrow \infty} \overbrace{F^{\langle N \rangle}}_{su(1,1)}$$

exists in $L^p(\mathbb{T}, m)$.

Remark. Clearly, in our theorem, the sparse sequence F can be taken infinite in both directions, e.g., $\{n_j\} = \{\pm 2^s\}$, $s \in \mathbb{Z}^+$ vs. the standard $n_j = 2^j$, $j \in \mathbb{Z}^+$. The pointwise convergence of $SU(1, 1)$ NLFT with sparse ℓ^2 data has been studied in [11]. In particular, the a.e. convergence was established there. Later, we will comment on how that can be obtained from our argument.

Remark. Taking $\delta_j > 0$, $\{\delta_j\} \notin \ell^1(\mathbb{Z}^+)$ and $z = 1$ in Theorem 1.1 shows that the $L^\infty(\mathbb{T}, m)$ norm of NLFT can grow infinitely.

We make a few observations before starting with the proof of Theorem 1.1. Without loss of generality, we can assume that $\|\{\delta_j\}\|_{\ell^2} \leq \frac{1}{2}$. Consider $Y_n(z, F)$ that solves the recursion

$$(1.5) \quad Y_n = \begin{pmatrix} 1 & \overline{F_n} z^{-n} \\ F_n z^n & 1 \end{pmatrix} Y_{n-1},$$

with $Y_0 = I$. Clearly, $\tilde{X}_n = \prod_{j=0}^n (1 - |F_j|^2)^{-\frac{1}{2}} Y_n$ for $n \leq N$ and, therefore, it is enough to show that $\lim_{n \rightarrow \infty} Y_n(z, F)$ converges for a.e. $z \in \mathbb{T}$. We denote

$$Y_n(z, F) =: \begin{pmatrix} \tilde{\mathbf{a}}_n(z, F) & \tilde{\mathbf{b}}_n^{(*)}(z, F) \\ \tilde{\mathbf{b}}_n(z, F) & \tilde{\mathbf{a}}_n^{(*)}(z, F) \end{pmatrix}.$$

Notice that $Y_n = Y_{n-1}$ if $F_n = 0$. We will show that $\lim_{n \rightarrow \infty} \tilde{\mathbf{b}}_n$ and $\lim_{n \rightarrow \infty} \tilde{\mathbf{a}}_n^{(*)}$ converge for a.e. $z \in \mathbb{T}$ but we need some notation first. In what follows, m will denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$, i.e., $dm = (2\pi)^{-1} d\theta$, $\theta \in [0, 2\pi)$. Also, for the function $f \in L^1(\mathbb{T}, m)$, we write $\hat{f}_n = \int_{\mathbb{T}} f e^{-in\theta} dm$, $n \in \mathbb{Z}$ for the Fourier transform. Similarly, given a sequence $\{G_n\}$, we will write $\check{G} = \sum_{n \in \mathbb{Z}} G_n e^{in\theta}$ for the inverse Fourier transform whenever the sum makes sense.

Lemma 1.2. Assume that $\|\{\delta_j\}\|_{\ell^2} \leq \frac{1}{2}$, then

$$\sup_n \|\tilde{\mathbf{a}}_n^{(*)}(z, F)\|_{L^2(\mathbb{T}, m)}^2 = 1 + O(\|\{\delta_j\}\|_{\ell^2}^2), \quad \sup_n \|\tilde{\mathbf{b}}_n(z, F)\|_{L^2(\mathbb{T}, m)}^2 = O(\|\{\delta_j\}\|_{\ell^2}^2).$$

Proof. Denote for shorthand $\alpha_j := \tilde{\mathbf{a}}_{n_j}^{(*)}$, $\beta_j := \tilde{\mathbf{b}}_{n_j}$ for $j \in \mathbb{Z}^+$ and $\alpha_{-1} := 1$, $\beta_{-1} := 0$. Then, for $z \in \mathbb{T}$, we have

$$(1.6) \quad \alpha_j = \alpha_{j-1} + \delta_j z^{n_j} \overline{\beta_{j-1}}, \quad \beta_j = \beta_{j-1} + \delta_j z^{n_j} \overline{\alpha_{j-1}}.$$

Due to $\text{supp } \hat{\alpha}_{j-1} \subset [0, n_{j-1} - 1]$ and $\text{supp } \hat{\beta}_{j-1} \subset [1, n_{j-1}]$ (see [13], p.6, Lemma 2), we get the $L^2(\mathbb{T}, m)$ -orthogonality of the terms due to (1.4). Then, for $j \in \mathbb{Z}^+$, we get

$$(1.7) \quad \|\alpha_j\|_{L^2(\mathbb{T}, m)}^2 = \|\alpha_{j-1}\|_{L^2(\mathbb{T}, m)}^2 + |\delta_j|^2 \|\beta_{j-1}\|_{L^2(\mathbb{T}, m)}^2, \quad \|\alpha_{-1}\|_{L^2(\mathbb{T}, m)} = 1,$$

$$(1.8) \quad \|\beta_j\|_{L^2(\mathbb{T}, m)}^2 = \|\beta_{j-1}\|_{L^2(\mathbb{T}, m)}^2 + |\delta_j|^2 \|\alpha_{j-1}\|_{L^2(\mathbb{T}, m)}^2, \quad \|\beta_{-1}\|_{L^2(\mathbb{T}, m)} = 0.$$

Since $\{\delta_j\} \in \ell^2(\mathbb{Z}^+)$, we can add these two identities and iterate to get

$$\|\alpha_j\|_{L^2(\mathbb{T}, m)}^2 + \|\beta_j\|_{L^2(\mathbb{T}, m)}^2 = \prod_{s \leq j} (1 + |\delta_s|^2).$$

Hence, for $j \in \mathbb{N}$, we have

$$(1.9) \quad \|\alpha_j\|_{L^2(\mathbb{T}, m)}^2 = \|\alpha_{j-1}\|_{L^2(\mathbb{T}, m)}^2 + O\left(|\delta_j|^2 \prod_{s \leq j-1} (1 + |\delta_s|^2)\right), \quad \|\alpha_{-1}\|_{L^2(\mathbb{T}, m)} = 1,$$

$$(1.10) \quad \|\beta_j\|_{L^2(\mathbb{T}, m)}^2 = \|\beta_{j-1}\|_{L^2(\mathbb{T}, m)}^2 + O\left(|\delta_j|^2 \prod_{s \leq j-1} (1 + |\delta_s|^2)\right), \quad \|\beta_{-1}\|_{L^2(\mathbb{T}, m)} = 0.$$

Iterating these formulas, we get our result. $\square$

It will help to work with α_j and β_j on the Fourier side. For $j \in \mathbb{Z}^+$, we let $\Lambda_j = \text{supp } \hat{\alpha}_j$ and $\Sigma_j = \text{supp } \hat{\beta}_j$. As we already noted before,

$$\Lambda_j \subset [0, n_j - 1], \quad \Sigma_j \subset [1, n_j].$$

We will need more notation: given $N \in \mathbb{Z}$ and a set $S \subset \mathbb{Z}$, we write $N - S$ as a shorthand for a set $\{N - s, s \in S\}$. From (1.6), we get for $j \in \mathbb{N}$:

$$(1.11) \quad \Lambda_j \subset \Lambda_{j-1} \cup (n_j - \Sigma_{j-1}), \quad \Sigma_j \subset \Sigma_{j-1} \cup (n_j - \Lambda_{j-1})$$

and $\Lambda_0 = \{0\}$ and $\Sigma_0 \subset \{1\}$. Clearly, the sets in the above unions are disjoint due to the sparseness of $\{n_j\}$.

Lemma 1.3. *If $\alpha_j =: \sum_{0 \leq s} x_s^{(j)} z^s$, $\beta_j =: \sum_{0 \leq s} y_s^{(j)} z^s$, then for fixed s we get $x_s^{(j)} = x_s$ and $y_s^{(j)} = y_s$ for large enough j . Moreover, $x := \{x_s\} \in \ell^2(\mathbb{Z}^+)$ and $y := \{y_s\} \in \ell^2(\mathbb{Z}^+)$.*

Proof. The first claim follows from (1.6) and (1.11) which show that $\{x_s\}$ and $\{y_s\}$ are defined recursively. The second claim follows from the previous lemma. $\square$

The following result was proved in [11]. It also comes as a corollary from our arguments above.

Corollary 1.4. *Given assumptions of Theorem 1.1, the limit*

$$(1.12) \quad \lim_{N \rightarrow \infty} \overbrace{F_{su(1,1)}^{(N)}}^{\text{exists for a.e. } z \in \mathbb{T}}.$$

exists for a.e. $z \in \mathbb{T}$.

Proof. By the Carleson theorem on convergence of Fourier series [3], the series

$$\sum_{s=0}^{\infty} x_s z^s, \quad \sum_{s=0}^{\infty} y_s z^s$$

converge for a.e. $z \in \mathbb{T}$. Since $\{\tilde{\mathbf{a}}_n^{(*)}\}$ and $\{\tilde{\mathbf{b}}_n\}$ are partial sums of these series, we get (1.12). $\square$

Remark. In [11], it is proved that, provided that $q \geq 3$, the convergence of NLFT on a set of positive Lebesgue measure implies that the sparse NLFT data must be in $\ell^2(\mathbb{Z})$.

Let σ be a probability measure on $\mathbb{T}$ with the infinite support (in the sense of cardinality). Denote the monic orthogonal polynomials by $\{\Phi_n(z, \sigma)\}$ and orthonormal polynomials by $\{\varphi_n(z, \sigma)\}$, we use OPUC for both as shorthand. That is,

$$\int_{\mathbb{T}} \Phi_n(z, \sigma) z^{-j} d\sigma = 0, \quad \int_{\mathbb{T}} \varphi_n(z, \sigma) z^{-j} d\sigma = 0, \quad \forall j \in \{0, \dots, n-1\}, \quad \int_{\mathbb{T}} |\varphi_n(z, \sigma)|^2 d\sigma = 1, \\ \deg(\Phi_n) = \deg(\varphi_n) = n, \quad \text{coeff}_n(\Phi_n) = 1, \quad \text{coeff}_n(\varphi_n) > 0,$$

where $\text{coeff}_n(Q)$ denotes the n -th coefficient of the polynomial Q . For any polynomial Q , we let $Q^* := z^n \overline{Q(\bar{z}^{-1})}$. Notice that such $*$ -operation is different from the $(*)$ -operation in (1.2) and it depends on $n \in \mathbb{Z}^+$. In fact, $Q^* = z^n \overline{Q^{(*)}}$. Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$. The polynomials $\{\Phi_n(z, \sigma)\}$ satisfy a recursion ([12], see (1.5.10) on p. 56)

$$\Phi_{n+1} = z\Phi_n - \overline{\gamma_n} \Phi_n^*, \quad \Phi_0 = 1, \quad n \in \mathbb{Z}^+,$$

where the coefficients $\{\gamma_n\}_{n=0}^{\infty} \in \mathbb{D}^{\infty}$ are called recursion (aka Schur or Verblunsky) parameters. It is known that the set of measures σ and the set of sequences $\{\gamma_n\}_{n=0}^{\infty} \in \mathbb{D}^{\infty}$ are bijectively equivalent. We get the following two results.

Corollary 1.5. *Suppose $\{n_j\}$ and $\{\delta_j\}$ are taken as in the previous theorem. If the recurrence coefficients $\{\gamma_n(\sigma)\}$ of measure σ satisfy $\gamma_{n_j} = \delta_j, \forall j \in \mathbb{N}$ and $\gamma_n = 0$ for all other $n \in \mathbb{Z}^+$, then the limits $\lim_{n \rightarrow \infty} \varphi_n^*(z, \sigma)$ and $\lim_{n \rightarrow \infty} \Phi_n^*(z, \sigma)$ exist for a.e. $z \in \mathbb{T}$.*

Proof. The proof is immediate if one uses the connection between $SU(1,1)$ NLFT and the OPUC ([4], Section 3 or [13], Lecture 5). $\square$

Next, we will discuss the $SU(2)$ NLFT. Take any sequence of complex numbers $G = \{G_n\}_{n \in \mathbb{Z}}$. For any $N \in \mathbb{Z}^+$, consider 2×2 matrices $Z_n(z, G^{\langle N \rangle})$, $n \in \mathbb{Z}$ defined by recursion

$$(1.13) \quad Z_n = \frac{1}{(1 + |G_n^{\langle N \rangle}|^2)^{\frac{1}{2}}} \begin{pmatrix} 1 & -\overline{G_n^{\langle N \rangle}} z^{-n} \\ G_n^{\langle N \rangle} z^n & 1 \end{pmatrix} Z_{n-1},$$

with $Z_n = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $n < -N$. We have $Z_n \in SU(2)$ for $z \in \mathbb{T}$. The matrix Z_n takes the form (we use the same notation as for the $SU(1, 1)$ case but these are different quantities):

$$Z_n(z, G^{\langle N \rangle}) =: \begin{pmatrix} \mathbf{a}_n(z, G^{\langle N \rangle}) & -\mathbf{b}_n^{(*)}(z, G^{\langle N \rangle}) \\ \mathbf{b}_n(z, G^{\langle N \rangle}) & \mathbf{a}_n^{(*)}(z, G^{\langle N \rangle}) \end{pmatrix}.$$

Following [1, 2, 8], we give a definition:

Definition. For $G^{\langle N \rangle}$, define $SU(2)$ nonlinear Fourier transform (NLFT) on $\mathbb{Z}$ as the map

$$G^{\langle N \rangle} \mapsto \widehat{G_{su(2)}^{\langle N \rangle}} := \begin{pmatrix} \mathbf{a}(z, G^{\langle N \rangle}) \\ \mathbf{b}(z, G^{\langle N \rangle}) \end{pmatrix},$$

where $\mathbf{a}(z, G^{\langle N \rangle}) := \mathbf{a}_\infty(z, G^{\langle N \rangle})$ and $\mathbf{b}(z, G^{\langle N \rangle}) = \mathbf{b}_\infty(z, G^{\langle N \rangle})$. The proof of the following result repeats the proof of Corollary 1.4 word for word.

Corollary 1.6. Suppose $\{n_j\}_{j=0}^\infty$ is a subsequence of natural numbers that satisfies

$$(1.14) \quad \frac{n_{j+1}}{n_j} \geq q, \quad \forall j \in \mathbb{Z}^+, \quad n_0 = 1,$$

where $q \geq 2$. Take $\{\delta_n\} \in \ell^2(\mathbb{Z}^+)$ and let $G_{n_j} = \delta_j$, $\forall j \in \mathbb{Z}^+$ and set $G_n = 0$ for all other $n \in \mathbb{Z}$. Then,

$$\widehat{G_{su(2)}^{\langle N \rangle}} \text{ converges for a.e. } z \in \mathbb{T} \text{ when } N \rightarrow \infty.$$

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It will be sufficient to establish the asymptotics of matrices Y_n . By interpolation, it is enough to consider $p = 2\ell$, $\ell \in \mathbb{N}$ where the case $\ell = 1$ follows from our arguments above. To explain the idea better, we start by taking $\ell = 2$ which gives $p = 4$.

Recall our definitions of α_j and β_j , and the coefficients $\{x_s\}$ and $\{y_s\}$ from Lemma 1.3. We can partition the sequences x and y into blocks using the rapid growth of n_j and write

$$x := \{x_s\} = \sum_{j \geq 0} P_j, \quad y := \{y_s\} = \sum_{j \geq 0} Q_j,$$

where, see (1.6), $\check{P}_0 = 1$, $\check{Q}_0 = 0$ and, for $j \in \mathbb{N}$, we have

$$(1.15) \quad \check{P}_j := \delta_j z^{n_j} \bar{\beta}_{j-1}, \quad \check{Q}_j := \delta_j z^{n_j} \bar{\alpha}_{j-1}.$$

Moreover, $\text{supp } P_{j_1} \cap \text{supp } P_{j_2} = \emptyset$ and $\text{supp } Q_{j_1} \cap \text{supp } Q_{j_2} = \emptyset$ if $j_1 \neq j_2$. Hence, $\alpha_j = \check{\mathbf{a}}_{n_j}^{(*)} = \sum_{s \leq j} \check{P}_s$ and $\beta_j = \check{\mathbf{b}}_{n_j} = \sum_{s \leq j} \check{Q}_s$. We have $\|\check{P}_j\|_{L^q(\mathbb{T}, m)} = |\delta_j| \|\beta_{j-1}\|_{L^q(\mathbb{T}, m)}$, $\|\check{Q}_j\|_{L^q(\mathbb{T}, m)} = |\delta_j| \|\alpha_{j-1}\|_{L^q(\mathbb{T}, m)}$ for every $q \geq 1$. Take $\mu \in \mathbb{N}$ as a large parameter to be specified later. Define

$$x^{(d)} := \sum_{j \geq 0} P_j^{(d)}, \quad d \in \{0, \dots, \mu-1\}, \quad P_j^{(d)} := P_{j\mu+d},$$

$$y^{(d)} := \sum_{j \geq 0} Q_j^{(d)}, \quad d \in \{0, \dots, \mu-1\}, \quad Q_j^{(d)} := Q_{j\mu+d},$$

and

$$\Delta_j := \left(\sum_{|s-j\mu| \leq 3\mu} |\delta_s|^2 \right)^{\frac{1}{2}}.$$

Notice that $\sum_j \Delta_j^2 \lesssim \mu \sum_j |\delta_j|^2$. We write the sums

$$(1.16) \quad f_j^{(d)} := \sum_{s \leq j} \check{P}_s^{(d)} = f_{j-1}^{(d)} + \check{P}_j^{(d)}, \quad g_j^{(d)} := \sum_{s \leq j} \check{Q}_s^{(d)} = g_{j-1}^{(d)} + \check{Q}_j^{(d)}$$

so, for $d \in \{1, \dots, \mu - 1\}$,

$$(1.17) \quad \sum_{s \leq \mu j + d - 1} \check{Q}_j = g_j^{(0)} + \dots + g_j^{(d-1)} + g_{j-1}^{(d)} + \dots + g_{j-1}^{(\mu-1)}$$

and, for $d = 0$,

$$(1.18) \quad \sum_{s \leq \mu j - 1} \check{Q}_s = g_{j-1}^{(0)} + \dots + g_{j-1}^{(\mu-1)}.$$

Analogous formulas hold for the sums of $\check{P}_j$. Notice that

$$(1.19) \quad \sup_j \|\alpha_j\|_{L^q(\mathbb{T}, m)} \leq \sum_{d=0}^{\mu-1} \sup_j \|f_j^{(d)}\|_{L^q(\mathbb{T}, m)}, \quad \sup_j \|\beta_j\|_{L^q(\mathbb{T}, m)} \leq \sum_{d=0}^{\mu-1} \sup_j \|g_j^{(d)}\|_{L^q(\mathbb{T}, m)}$$

by the triangle inequality. Our goal is to estimate $\|f_j^{(d)}\|_{L^4(\mathbb{T}, m)}$ and $\|f_{j-1}^{(d)}\|_{L^4(\mathbb{T}, m)}$. To this end, we take the square of both sides in (1.16):

$$(1.20) \quad (f_j^{(d)})^2 = (f_{j-1}^{(d)})^2 + (\check{P}_j^{(d)})^2 + 2f_{j-1}^{(d)}\check{P}_j^{(d)} =: I_1 + I_2 + I_3,$$

$$(1.21) \quad (g_j^{(d)})^2 = (g_{j-1}^{(d)})^2 + (\check{Q}_j^{(d)})^2 + 2g_{j-1}^{(d)}\check{Q}_j^{(d)} =: J_1 + J_2 + J_3.$$

Notice that $\text{supp } \hat{I}_1 \subset [0, 2n_{(j-1)\mu+d}]$ and $\text{supp } \hat{I}_2 \subset [2(n_{j\mu+d} - n_{j\mu+d-1}), 2n_{j\mu+d}]$ and $\text{supp } \hat{I}_3 \subset [(n_{j\mu+d} - n_{j\mu+d-1}), 2n_{j\mu+d}]$. Now, we choose μ large enough to make sure that $I_1 \perp I_2$ and $I_1 \perp I_3$. This is possible due to (1.4). Similarly, we can guarantee that $J_1 \perp J_2$ and $J_1 \perp J_3$ by taking μ large. Therefore,

$$(1.22) \quad \int |f_j^{(d)}|^4 dm = \int |f_{j-1}^{(d)}|^4 dm + \int |\check{P}_j^{(d)}|^4 dm + O\left(\int |f_{j-1}^{(d)}\check{P}_j^{(d)}|^2 dm\right) + O\left(\int |f_{j-1}^{(d)}| \cdot |\check{P}_j^{(d)}|^3 dm\right).$$

Now, we can use recursion (1.15) to estimate the second, third, and the fourth terms as follows. If $d \in \{1, \dots, \mu - 1\}$, we use Young's inequality: $ab \leq a^p/p + b^q/q$, $a, b \geq 0$, $p^{-1} + q^{-1} = 1$, $p \in (1, \infty)$, along with (1.17) to get

$$(1.23) \quad \int |\check{P}_j^{(d)}|^4 dm \lesssim |\Delta_j|^4 \int |g_j^{(0)} + \dots + g_j^{(d-1)} + g_{j-1}^{(d)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm,$$

$$(1.24) \quad \int |f_{j-1}^{(d)}\check{P}_j^{(d)}|^2 dm \lesssim$$

$$(1.25) \quad |\Delta_j|^2 \left(\int |g_j^{(0)} + \dots + g_j^{(d-1)} + g_{j-1}^{(d)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm + \int |f_{j-1}^{(d)}|^4 dm \right),$$

$$(1.26) \quad \int |f_{j-1}^{(d)}| \cdot |\check{P}_j^{(d)}|^3 dm \lesssim$$

$$(1.27) \quad |\Delta_j|^3 \left(\int |g_j^{(0)} + \dots + g_j^{(d-1)} + g_{j-1}^{(d)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm + \int |f_{j-1}^{(d)}|^4 dm \right)$$

and, for $d = 0$, we use (1.18) to get

$$(1.28) \quad \int |\check{P}_j^{(0)}|^4 dm \lesssim |\Delta_j|^4 \int |g_{j-1}^{(0)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm,$$

$$(1.29) \quad \int |f_{j-1}^{(0)}\check{P}_j^{(0)}|^2 dm \lesssim |\Delta_j|^2 \left(\int |g_{j-1}^{(0)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm + \int |f_{j-1}^{(0)}|^4 dm \right),$$

$$(1.30) \quad \int |f_{j-1}^{(0)}| \cdot |\check{P}_j^{(0)}|^3 dm \lesssim |\Delta_j|^3 \left(\int |g_{j-1}^{(0)} + \dots + g_{j-1}^{(\mu-1)}|^4 dm + \int |f_{j-1}^{(0)}|^4 dm \right).$$

Analogous bounds hold for $\{g_j^{(d)}\}$. Then, summing up in d in (1.22), we have

$$\sum_{d=0}^{\mu-1} \int |f_j^{(d)}|^4 dm \leq \left(\sum_{d=0}^{\mu-1} \int |f_{j-1}^{(d)}|^4 dm \right) (1 + C\Delta_j^2) + C_\mu \Delta_j^2 \sum_{d=0}^{\mu-1} \int (|g_j^{(d)}|^4 + |g_{j-1}^{(d)}|^4) dm$$

and, similarly, one gets

$$\sum_{d=0}^{\mu-1} \int |g_j^{(d)}|^4 dm \leq \left(\sum_{d=0}^{\mu-1} \int |g_{j-1}^{(d)}|^4 dm \right) (1 + C\Delta_j^2) + C_\mu \Delta_j^2 \sum_{d=0}^{\mu-1} \int (|f_j^{(d)}|^4 + |f_{j-1}^{(d)}|^4) dm.$$

We now add up these two bounds, use inequality $\sum_j \Delta_j^2 < \infty$ and Lemma 4.1 from [4] to get

$$(1.31) \quad \sup_n \left(\sum_{d=0}^{\mu-1} \int |f_n^{(d)}|^4 dm + \sum_{d=0}^{\mu-1} \int |g_n^{(d)}|^4 dm \right) < \infty$$

and, by (1.19), $\sup_n \|\tilde{\mathbf{a}}_n^{(*)}\|_{L^4(\mathbb{T}, m)} < \infty$ and $\sup_n \|\tilde{\mathbf{b}}_n\|_{L^4(\mathbb{T}, m)} < \infty$. If we denote $\tilde{\mathbf{a}}^{(*)} = \tilde{x}$ and $\tilde{\mathbf{b}} = \tilde{y}$, the convergence $\|\tilde{\mathbf{a}}_n^{(*)} - \tilde{\mathbf{a}}^{(*)}\|_{L^4(\mathbb{T}, m)} \rightarrow 0$ and $\|\tilde{\mathbf{b}}_n - \tilde{\mathbf{b}}\|_{L^4(\mathbb{T}, m)} \rightarrow 0$ can be proved similarly using the Cauchy criterion and estimates on the tails of the series $\sum_j \check{P}_j$ and $\sum_j \check{Q}_j$.

For $\ell > 2$, the argument is the same. Instead of (1.20), we get

$$(1.32) \quad (f_j^{(d)})^\ell = (f_{j-1}^{(d)})^\ell + \dots + (\check{P}_j^{(d)})^\ell, \quad (g_j^{(d)})^\ell = (g_{j-1}^{(d)})^\ell + \dots + (\check{P}_j^{(d)})^\ell$$

by Newton's binomial formula. Choosing μ large enough, we can again make sure that the first terms in both sums are orthogonal to all others. Then, taking the square of the absolute values and using Young's inequality, one can obtain the required bounds just like in (1.22)-(1.31). $\square$

Theorem 1.7. *Under the assumptions of Corollary 1.5, it holds that the orthogonality measure σ satisfies $d\sigma = w dm$ and $w \in L^p(\mathbb{T})$ with an arbitrary $p \in [2, \infty)$. Moreover, the sequences $\{\varphi_n^*\}$ and $\{\Phi_n^*\}$ converge in $L^p(\mathbb{T}, m)$.*

Proof. Assume without loss of generality that $\|\{\delta_j\}\|_{\ell^2} \leq \frac{1}{2}$. Following [5], we define the Wall polynomials associated with σ by A_j and B_j and consider Bernstein-Szegő approximations

$$d\sigma_j = \sigma'_j dm = \frac{1}{|\varphi_j|^2} dm, \quad r_j := \frac{A_{j-1}}{B_{j-1}}.$$

Then, by the formula (6.3) in [5], we get

$$\sigma'_j = \frac{1 - |r_j|^2}{|1 - z r_j|^2} \leq \frac{1 + |r_j|}{1 - |r_j|} = \frac{(1 + |r_j|)^2}{1 - |r_j|^2} \stackrel{(4.14) \text{ in [5]}}{\lesssim} |B_{j-1}|^2 (1 + |r_j|)^2 \lesssim |B_{j-1}|^2.$$

The connection between $\mathbf{a}_j, \mathbf{b}_j$ and φ_j^*, A_j, B_j , already used in [4] in Section 3, shows the $L^p(\mathbb{T}, m)$ estimates hold for A_j, B_j as well, i.e., $\sup_j \|A_j\|_{L^p(\mathbb{T}, m)} < \infty$ and $\sup_j \|B_j\|_{L^p(\mathbb{T}, m)} < \infty$ for all $p \in [2, \infty)$.

Hence, $\sup_j \|\sigma'_j\|_{L^p(\mathbb{T}, m)} < \infty$ for every $p \in [1, \infty)$. Since $\sigma_j \xrightarrow{(*)} \sigma$ (see, e.g., Lemma 5.4 in [5]), we get our first statement by using the weak compactness of balls in $L^p(\mathbb{T}, m)$. The second claim follows from the previous theorem and the relation between the $SU(1, 1)$ NLFT and the OPUC. $\square$

REFERENCES

- [1] Michel Alexis, Gevorg Mnatsakanyan, and Christoph Thiele. Quantum signal processing and nonlinear Fourier analysis. *Rev. Mat. Complut.*, 37(3):655–694, 2024. 4
- [2] Michel Alexis, Gevorg Mnatsakanyan, and Christoph Thiele. One sided orthogonal polynomials and a pointwise convergence result for $SU(2)$ -valued nonlinear Fourier series. *arXiv preprint*, 2025. <https://arxiv.org/abs/2507.05124>. 4
- [3] Lennart Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966. 3
- [4] Sergey Denisov. Pointwise behavior of $SU(1, 1)$ nonlinear Fourier transform. *arXiv preprint*, 2026. <https://arxiv.org/abs/2605.25108>. 1, 3, 6
- [5] Sergei Khrushchev. Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$. *J. Approx. Theory*, 108(2):161–248, 2001. 6
- [6] Vjekoslav Kovač, Diogo Oliveira e Silva, and Jelena Rupčić. A sharp nonlinear Hausdorff-Young inequality for small potentials. *Proc. Amer. Math. Soc.*, 147(1):239–253, 2019. 1
- [7] Vjekoslav Kovač, Diogo Oliveira e Silva, and Jelena Rupčić. Asymptotically sharp discrete nonlinear Hausdorff-Young inequalities for the $SU(1, 1)$ -valued Fourier products. *Q. J. Math.*, 73(3):1179–1188, 2022. 1
- [8] Gevorg Mnatsakanyan. ℓ^1 mapping properties, smoothness and decay for $SU(2)$ -valued nonlinear Fourier transform. *arXiv preprint*, 2026. <https://arxiv.org/abs/2603.02021>. 4
- [9] Camil Muscalu, Terence Tao, and Christoph Thiele. A counterexample to a multilinear endpoint question of Christ and Kiselev. *Math. Res. Lett.*, 10(2-3):237–246, 2003. 1
- [10] Richard Oberlin, Andreas Seeger, Terence Tao, Christoph Thiele, and James Wright. A variation norm Carleson theorem. *J. Eur. Math. Soc. (JEMS)*, 14(2):421–464, 2012. 1
- [11] Jelena Rupčić. Convergence of lacunary $SU(1, 1)$ -valued trigonometric product. *Commun. Pure Appl. Anal.*, 19(3):1275–1289, 2020. 1, 2, 3

- [12] Barry Simon. *Orthogonal polynomials on the unit circle. Part 1*, volume 54, Part 1 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory. [3](#)
- [13] Terence Tao and Christoph Thiele. Nonlinear Fourier Analysis. *arXiv preprint*, 2012. <https://arxiv.org/abs/1201.5129>. [1](#), [2](#), [3](#)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DR., MADISON, WI 53706, USA

Email address: denissov@wisc.edu