

# GENERALIZATIONS OF MENCHOV-RADEMACHER THEOREM AND EXISTENCE OF WAVE OPERATORS IN SCHRÖDINGER EVOLUTION

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ABSTRACT. We obtain generalizations of the classical Menchov-Rademacher theorem to the case of continuous orthogonal systems. These results are applied to show the existence of Moller wave operators in Schrödinger evolution.

## 1. INTRODUCTION

The celebrated Menchov-Rademacher Theorem (see, e.g., [10]) gives a general condition for a.e. convergence of the orthogonal series:

**Theorem 1.1 (Menchov-Rademacher).** *Suppose  $\{\phi_n(x)\}, n \in \mathbb{N}$  is orthonormal system in  $L^2(0, 1)$  and the sequence  $\{a_n\}$  satisfies*

$$l \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n^2 \log^2(n+1) < \infty.$$

*Then, the series  $\sum_{n=1}^{\infty} a_n \phi_n(x)$  converges for a.e.  $x \in (0, 1)$ . Moreover, if*

$$m(x) \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \left| \sum_{j=1}^n a_j \phi_j(x) \right|$$

*defines a maximal function, then*

$$\|m\|_{L^2(0,1)} \leqslant Cl^{1/2}$$

*with some absolute constant  $C$ .*

This result can be easily modified to cover orthonormal systems in  $L^2_{\mu}(0, 1)$  where  $\mu$  is a measure on  $(0, 1)$ . In this paper, we prove an analog of this result for orthogonal systems with “continuous” parameters of orthogonality and apply it to show the existence of wave operators for Schrödinger evolution.

We start with the following definitions.

**Definition.** We say that  $f \in L^2_{\text{loc}}(\mathbb{R}^+)$  if

$$(1.1) \quad \int_0^a |f(r)|^2 dr < \infty$$

for all  $a > 0$ .

**Definition.** Let a pair  $(P, \sigma)$  consist of a function  $P(r, k) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{C}$  and a measure  $\sigma$  on  $\mathbb{R}$ . We say that  $(P, \sigma)$  is a continuous orthonormal system if

- (a) for  $\sigma$ -a.e.  $k \in \mathbb{R}$ ,  $P(r, k) \in L^2_{\text{loc}}(\mathbb{R}^+)$ ,
- (b) for every  $f \in L^2(\mathbb{R}^+)$  and every  $a > 0$ , we have

$$\int_{\mathbb{R}} \left| \int_0^a f(r) P(r, k) dr \right|^2 d\sigma(k) = \int_0^a |f(r)|^2 dr.$$

Our first result is the following theorem.

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**Theorem 1.2.** *Suppose  $(P, \sigma)$  is continuous orthonormal system and*

$$L \stackrel{\text{def}}{=} \int_{\mathbb{R}^+} |f(r)|^2 \log^2(2+r) dr.$$

*Then, the sequence  $\left\{ \int_0^n f(r)P(r, k) dr \right\}$  converges for  $\sigma$ -a.e.  $k \in \mathbb{R}$ . Moreover, if*

$$M(k) \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \left| \int_0^n f(r)P(r, k) dr \right|,$$

*then  $\|M\|_{L^2_\sigma(\mathbb{R})} \leq CL^{1/2}$  with some absolute constant  $C$ .*

**Definition.** We will call a continuous orthonormal system  $(P, \sigma)$  normalized if there is a continuous positive function  $\kappa$  defined on  $\mathbb{R}$  such that

$$(1.2) \quad \kappa^{-1} \in L^\infty(\mathbb{R}), \quad K \stackrel{\text{def}}{=} \sup_{r \geq 0} \int_{\mathbb{R}} \frac{|P(r, k)|^2}{\kappa(k)} d\sigma < \infty.$$

For normalized systems, the previous theorem can be improved in the following way.

**Theorem 1.3.** *Consider a normalized continuous orthonormal system  $(P, \sigma, \kappa)$  and suppose that  $f \log(2+r) \in L^2(\mathbb{R}^+)$ , then*

$$(1.3) \quad \int_{\mathbb{R}} \sup_{t > 0} \left| \int_0^t f(r)P(r, k) dr \right|^2 \frac{d\sigma}{\kappa(k)} \lesssim (\|\kappa^{-1}\|_{L^\infty(\mathbb{R})} + K) \int_0^\infty |f(r)|^2 \log^2(2+r) dr.$$

*Moreover, as  $R \rightarrow \infty$ ,*

$$(1.4) \quad \int_0^R f(r)P(r, k) dr \rightarrow \int_0^\infty f(r)P(r, k) dr$$

*for a.e.  $k$  with respect to measure  $\sigma$ .*

One example of continuous orthonormal system is given by solutions  $\{P(r, k)\}$  to Krein systems [5, 12]. Krein systems are given by the following linear system of differential equations

$$(1.5) \quad \begin{cases} P'(r, k) = ikP(r, k) - \overline{A(r)}P_*(r, k), & P(0, k) = 1 \\ P'_*(r, k) = -A(r)P(r, k), & P_*(0, k) = 1 \end{cases}, \quad k \in \mathbb{C}, \quad r \geq 0.$$

In this paper, we will always assume that the coefficient  $A \in L^2_{\text{loc}}(\mathbb{R}^+)$ . The Cauchy problem (1.5) has the unique solution  $(P(r, k), P_*(r, k))$ . In [12] (see also, e.g., [4]), Krein showed that  $\{P(r, k)\}$  with  $r \geq 0$  and  $k \in \mathbb{R}$  can be viewed as continuous analogs of polynomials, orthogonal on the unit circle. In particular, there is a measure  $\sigma$  on  $\mathbb{R}$ , which satisfies

$$\int_{\mathbb{R}} \frac{d\sigma(k)}{1+k^2} < \infty,$$

and the following property

$$(1.6) \quad \int_{\mathbb{R}} \left| \int_0^a f(r)P(r, k) dr \right|^2 d\sigma = \int_0^a |f(r)|^2 dr$$

holds for every  $f \in L^2(\mathbb{R}^+)$ . In other words, any pair  $(P, \sigma)$  gives an example of a continuous orthonormal system. Notice that (1.6) allows us to define the generalized Fourier transform of  $f$

$$\int_0^\infty f(r)P(r, k) dr$$

as an element of  $L^2_\sigma(\mathbb{R})$ .

Under a mild extra assumption on coefficient  $A$ , the system  $(P, \sigma)$  becomes normalized and the previous theorem can be applied. More precisely, the following lemma holds.

**Lemma 1.4.** *Suppose the coefficient  $A$  in a Krein system belongs to the Stummel class, i.e.,*

$$(1.7) \quad \|A\|_{\text{St}} \stackrel{\text{def}}{=} \sup_{r \geq 0} \left( \int_r^{r+1} |A(\rho)|^2 d\rho \right)^{1/2} < \infty.$$

Then,

$$(1.8) \quad \sup_{r>0} \int_{\mathbb{R}} \frac{|P(r, k)|^2}{1+k^2} d\sigma \lesssim 1 + \|A\|_{\text{St}}^2.$$

Moreover, we have (1.3) and (1.4) with  $\kappa(k) = 1 + k^2$  and  $K \lesssim 1 + \|A\|_{\text{St}}^2$ .

The proof of this Lemma is given in the Appendix.

Another application of our general results to Krein systems is given in the following Lemma.

**Lemma 1.5.** *Suppose the coefficient in a Krein system satisfies  $A(r) \log(2+r) \in L^2(\mathbb{R}^+)$ , then*

$$(1.9) \quad \int_{\mathbb{R}} \left( \sup_{\rho < r_1 < r_2} \left| \int_{r_1}^{r_2} A(x) P(x, k) dx \right| \right)^2 \frac{d\sigma}{1+k^2} = \int_{\mathbb{R}} \left( \sup_{\rho < r_1 < r_2} |P_*(r_2, k) - P_*(r_1, k)| \right)^2 \frac{d\sigma}{1+k^2} \lesssim (1 + \|A\|_2^2) \int_{\rho}^{\infty} |A(r)|^2 \log^2(2+r) dr, \rho > 0.$$

Moreover, for Lebesgue a.e.  $k \in \mathbb{R}$ , there is a limit  $\Pi(k) = \lim_{r \rightarrow \infty} P_*(r, k)$ .

Theorem 1.2, Theorem 1.3 and Lemma 1.5 are proved in the second section. In section 3, we apply Lemma 1.5 to show existence of wave operators for Schrödinger evolution, which is our central result. Consider

$$H = -\partial_{xx}^2 + v$$

on  $\mathbb{R}^+$  with Dirichlet boundary condition at zero and denote by  $H_0 = -\partial_{xx}^2$  the free Schrödinger operator with the same Dirichlet condition at zero. The Moller wave operators (see, e.g., [15]) are defined by

$$W^{\pm}(H, H_0) \stackrel{\text{def}}{=} \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0},$$

where the limit is the strong limit in  $L^2(\mathbb{R}^+)$ . The main result of our paper is the following theorem.

**Theorem 1.6.** *Suppose  $v = a' + q$  where  $q \in L^1(\mathbb{R}^+)$ ,  $a$  is absolutely continuous on  $\mathbb{R}^+$ , and*

$$(1.10) \quad a' \in L^{\infty}(\mathbb{R}^+), \quad a \log(2+r) \in L^2(\mathbb{R}^+).$$

Then, the wave operators  $W^{\pm}(H, H_0)$  exist.

The existence of wave and modified wave operators for Schrödinger and Dirac equations was extensively studied in the scattering theory of wave propagation, see, e.g., the classical papers by Agmon [1], Hörmander [9], and a book by T. Kato [11] on the subject. The case  $v \in L^p(\mathbb{R}^+)$ ,  $1 \leq p < 2$  was considered in [3] where the existence of modified wave operators was proved. See [6] for later developments. In [4], the presence of wave operators was established for Dirac equations with potential in  $L^2(\mathbb{R}^+)$ . This result is optimal on  $L^p(\mathbb{R}^+)$  scale. For more general potentials in Dirac equations and connection to the Szegő condition on measure  $\sigma$ , see [2]. Some related recent results, including the multidimensional setting, can be found in, e.g., [7, 8, 13].

### Notation

- (1) If  $f$  is defined on  $\mathbb{R}$ ,  $\hat{f}$  denotes its Fourier transform:

$$\hat{f}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx.$$

The inverse Fourier transform is defined as

$$\check{f}(k) = f^{\vee}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ikx} dx.$$

- (2)  $C^{\infty}(\mathbb{R})$  stands for infinitely smooth functions defined on the real line and  $C_c^{\infty}(\mathbb{R})$  denotes the space of smooth functions with compact support.
- (3) We will use the symbol  $C_{(a_1, \dots, a_k)}$  to indicate a nonnegative function which depends on parameters  $(a_1, \dots, a_k)$ . The actual value of  $C$  can change from one formula to another.
- (4) If  $E$  is a set on the real line,  $E^c$  denotes its complement.

- (5) For two non-negative functions  $f_{1(2)}$ , we write  $f_1 \lesssim f_2$  if there is an absolute constant  $C$  such that

$$f_1 \leq C f_2$$

for all values of the arguments of  $f_{1(2)}$ . We define  $\gtrsim$  similarly and say that  $f_1 \sim f_2$  if  $f_1 \lesssim f_2$  and  $f_2 \lesssim f_1$  simultaneously.

- (6) If  $f_2$  is a non-negative function and  $|f_1| \lesssim f_2$ , we write  $f_1 = O(f_2)$ .

## 2. MENCHOV-RADEMACHER THEOREM FOR CONTINUOUS ORTHOGONAL SYSTEMS

We start by giving a proof of Theorem 1.2. It is a direct adaptation of the proof of Menchov-Rademacher Theorem in [10] but we present it here for the reader's convenience.

**Proof of Theorem 1.2.** For  $j \in \mathbb{N}$ , let  $P_j(k) = \int_{2^{j-1}}^{2^j} f(r)P(r, k)dr$  and

$$S'_j(k) = \sum_{l=1}^j P_l(k) = \int_1^{2^j} f(r)P(r, k)dr.$$

Now,

$$\|P_j\|_{L^2_\sigma(\mathbb{R})}^2 = \int_{\mathbb{R}} \left| \int_{2^{j-1}}^{2^j} f(r)P(r, k)dr \right|^2 d\sigma(k) = \int_{2^{j-1}}^{2^j} |f(r)|^2 dr$$

and so

$$(2.1) \quad \sum_{j \in \mathbb{N}} j^2 \|P_j\|_{L^2_\sigma(\mathbb{R})}^2 \sim \int_1^\infty |f(r)|^2 \log^2(2+r) dr.$$

For any  $a > 0$ , we have

$$\begin{aligned} \sum_{j \in \mathbb{N}} \int_{-a}^a |P_j(k)| d\sigma(k) &\leq \sum_{j \in \mathbb{N}} \left( \int_{-a}^a |P_j(k)|^2 d\sigma(k) \right)^{1/2} \left( \int_{-a}^a d\sigma(k) \right)^{1/2} \leq \\ &\sqrt{\sigma([-a, a])} \sum_{j \in \mathbb{N}} \|P_j\|_{L^2_\sigma(\mathbb{R})} j j^{-1} \leq \sqrt{\sigma([-a, a])} \left( \sum_{j \in \mathbb{N}} j^2 \|P_j\|_{L^2_\sigma(\mathbb{R})}^2 \right)^{1/2} \left( \sum_{j \in \mathbb{N}} j^{-2} \right)^{1/2} \\ &\lesssim \sqrt{\sigma([-a, a])} \left( \int_{\mathbb{R}_+} |f(r)|^2 \log^2(2+r) dr \right)^{1/2} = \sqrt{\sigma([-a, a])} L^{1/2}. \end{aligned}$$

Since  $a$  is arbitrary large, by the theorem of Beppo Levi,  $\sum_{j \in \mathbb{N}} |P_j(k)|$  converges for  $\sigma$ -a.e.  $k$ , as does  $\{S'_j(k)\}$ .

Let  $S'(k) \stackrel{\text{def}}{=} \sup_{j \in \mathbb{N}} |S'_j(k)|$  be the maximal function over dyadic partial sums. Since  $S'(k) \leq \sum_{j \in \mathbb{N}} |P_j(k)|$ , we have

$$(2.2) \quad \|S'\|_{L^2_\sigma(\mathbb{R})} \leq \left\| \sum_{j \in \mathbb{N}} |P_j| \right\|_{L^2_\sigma(\mathbb{R})} \leq \sum_{j \in \mathbb{N}} \|P_j\|_{L^2_\sigma(\mathbb{R})} = \sum_{j \in \mathbb{N}} j^{-1} j \|P_j\|_{L^2_\sigma(\mathbb{R})} \lesssim L^{1/2}$$

after applying the Cauchy-Schwarz inequality and (2.1).

For  $n \in \{0, 1, 2, \dots, 2^N\}$ , we can write  $n = \sum_{m=0}^N \epsilon_m(n) 2^{N-m}$  with  $\epsilon_m(n) \in \{0, 1\}$ . For  $j \in \{0, 1, \dots, N\}$ , let  $n_j = \sum_{m=0}^j \epsilon_m(n) 2^{N-m}$ .

Noting that  $\left| \sum_{j=1}^N x_j \right|^2 \leq N \sum_{j=1}^N |x_j|^2$ , we have:

$$\begin{aligned} \left| \int_{2^N}^{2^{N+n}} f(r)P(r, k)dr \right|^2 &= \left| \sum_{j=1}^N \int_{2^{N+n_{j-1}}}^{2^{N+n_j}} f(r)P(r, k)dr \right|^2 \leq \\ N \sum_{j=1}^N \left| \int_{2^{N+n_{j-1}}}^{2^{N+n_j}} f(r)P(r, k)dr \right|^2 &\leq N \sum_{j=1}^N \sum_{p=0}^{2^j-1} \left| \int_{2^{N+p2^{N-j}}}^{2^{N+(p+1)2^{N-j}}} f(r)P(r, k)dr \right|^2 \end{aligned}$$

and the last expression does not depend on  $n$ . Let

$$S_j''(k) \stackrel{\text{def}}{=} \sup_{0 \leq n \leq 2^j} \left| \int_{2^j}^{2^{j+n}} f(r)P(r, k) dr \right|$$

denote the maximal function over dyadic interval  $[2^j, 2^{j+1}]$ . We apply the above estimate to get

$$\begin{aligned} (2.3) \quad \|S_N''\|_{L_\sigma^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \sup_{0 \leq n \leq 2^N} \left| \int_{2^N}^{2^{N+n}} f(r)P(r, k) dr \right|^2 d\sigma(k) \\ &\leq \int_{\mathbb{R}} N \sum_{j=1}^N \sum_{p=0}^{2^j-1} \left| \int_{2^{N+p}2^{N-j}}^{2^{N+(p+1)2^{N-j}}} f(r)P(r, k) dr \right|^2 d\sigma(k) \\ &= N \sum_{j=1}^N \sum_{p=0}^{2^j-1} \int_{\mathbb{R}} \left| \int_{2^{N+p}2^{N-j}}^{2^{N+(p+1)2^{N-j}}} f(r)P(r, k) dr \right|^2 d\sigma(k) \\ &= N \sum_{j=1}^N \sum_{p=0}^{2^j-1} \int_{2^{N+p}2^{N-j}}^{2^{N+(p+1)2^{N-j}}} |f(r)|^2 dr = N^2 \int_{2^N}^{2^{N+1}} |f(r)|^2 dr. \end{aligned}$$

Taking  $S'' = \sup_{j \in \mathbb{N}} S_j''$ , we note that  $S'' \leq \left( \sum_{j \in \mathbb{N}} |S_j''|^2 \right)^{1/2}$  so

$$\|S''\|_{L_\sigma^2(\mathbb{R})} \lesssim \left( \sum_{j \in \mathbb{N}} j^2 \int_{2^j}^{2^{j+1}} |f(r)|^2 dr \right)^{1/2} \lesssim L^{1/2}.$$

Finally, we have

$$\begin{aligned} \|M\|_{L_\sigma^2(\mathbb{R})}^2 &\lesssim \int_0^1 |f(r)|^2 dr + \int_{\mathbb{R}} \sup_{j \in \mathbb{N}} \left| \int_1^{2^j} f(r)P(r, k) dr \right|^2 d\sigma(k) + \\ &\int_{\mathbb{R}} \sup_{j \in \mathbb{N}} \sup_{2^j \leq n \leq 2^{j+1}} \left| \int_{2^j}^n f(r)P(r, k) dr \right|^2 d\sigma(k) = \int_0^1 |f(r)|^2 dr + \|S'\|^2 + \|S''\|^2 \lesssim L. \end{aligned}$$

Convergence of the sequence  $\left\{ \int_0^n f(r)P(r, k) dr \right\}$  for  $\sigma$ -a.e.  $k$  follows from the convergence of  $\{S_j'(k)\}$  established above and the estimate  $\int_{\mathbb{R}} \sum_{j \in \mathbb{N}} |S_j''|^2 d\sigma \lesssim L$  which yields convergence of  $\sum_{j \in \mathbb{N}} |S_j''|^2$  for  $\sigma$ -a.e.  $k$ . □

**Proof of Theorem 1.3.** We have

$$\begin{aligned} \int_{\mathbb{R}} \sup_{t \in \mathbb{R}^+} \left| \int_0^t f(r)P(r, k) dr \right|^2 \frac{d\sigma(k)}{\kappa(k)} &= \int_{\mathbb{R}} \sup_{t \in \mathbb{R}^+} \left| \int_0^{[t]} f(r)P(r, k) dr + \int_{[t]}^t f(r)P(r, k) dr \right|^2 \frac{d\sigma(k)}{\kappa(k)} \\ &\lesssim \|\kappa^{-1}\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \sup_{n \in \mathbb{N}} \left| \int_0^n f(r)P(r, k) dr \right|^2 d\sigma(k) + \int_{\mathbb{R}} \sup_{t \in \mathbb{R}^+} \left| \int_{[t]}^t f(r)P(r, k) dr \right|^2 \frac{d\sigma(k)}{\kappa(k)}. \end{aligned}$$

The first integral was controlled in Theorem 1.2. The second one can be estimated as follows

$$\begin{aligned} (2.4) \quad \int_{\mathbb{R}} \sup_{t \in \mathbb{R}^+} \left| \int_{[t]}^t f(r)P(r, k) dr \right|^2 \frac{d\sigma(k)}{\kappa(k)} &\leq \int_{\mathbb{R}} \sup_{n \in \mathbb{Z}^+} \left( \int_n^{n+1} |f(r)P(r, k)| dr \right)^2 \frac{d\sigma(k)}{\kappa(k)} \leq \\ &\int_{\mathbb{R}} \sup_{n \in \mathbb{Z}^+} \left( \left( \int_n^{n+1} |f|^2 dr \right) \left( \int_n^{n+1} |P(r, k)|^2 dr \right) \right) \frac{d\sigma(k)}{\kappa(k)} \leq \\ &\int_{\mathbb{R}} \sum_{n=0}^{\infty} \left( \left( \int_n^{n+1} |f|^2 dr \right) \left( \int_n^{n+1} |P(r, k)|^2 dr \right) \right) \frac{d\sigma(k)}{\kappa(k)} \leq \\ &\sum_{n=0}^{\infty} \left( \int_n^{n+1} |f|^2 dr \right) \left( \int_n^{n+1} \left( \int_{\mathbb{R}} \frac{|P(r, k)|^2}{\kappa(k)} d\sigma(k) \right) dr \right) \stackrel{(1.2)}{\leq} K \|f\|_2^2, \end{aligned}$$

which proves (1.3).

To establish (1.4), we notice that

$$\int_0^r f(\rho)P(\rho, k)d\rho = \int_0^{[r]} f(\rho)P(\rho, k)d\rho + \int_{[r]}^r f(\rho)P(\rho, k)d\rho.$$

The first term has a limit as  $r \rightarrow \infty$  for  $\sigma$ -a.e.  $k$  as follows from Theorem 1.2. For the second one, we can write

$$\left| \int_{[r]}^r f(\rho)P(\rho, k)d\rho \right| \leq \int_{[r]}^{[r]+1} |f(\rho)P(\rho, k)|d\rho$$

and the last expression goes to 0 for  $\sigma$ -a.e.  $k$  since the series

$$\sum_{n \in \mathbb{N}} \left( \int_n^{n+1} |f(r)P(r, k)|dr \right)^2$$

converges for  $\sigma$ -a.e.  $k$ . This convergence follows from the following bound

$$\begin{aligned} \int_{\mathbb{R}} \sum_{n \in \mathbb{N}} \left( \int_n^{n+1} |f(r)P(r, k)|dr \right)^2 \frac{d\sigma}{\kappa} &\leq \int_{\mathbb{R}} \sum_{n \in \mathbb{N}} \left( \left( \int_n^{n+1} |f(r)|^2 dr \right) \left( \int_n^{n+1} |P(r, k)|^2 dr \right) \right) \frac{d\sigma}{\kappa} \leq \\ &\left( \sup_{r \geq 0} \int_{\mathbb{R}} \frac{|P(r, k)|^2}{\kappa} d\sigma \right) \sum_{n \in \mathbb{N}} \int_n^{n+1} |f(r)|^2 dr \stackrel{(1.2)}{<} \infty. \end{aligned}$$

□

Before giving our proof of Lemma 1.5, we list some basic properties of Krein systems which will be needed later in the text. We start by remarking that

$$(2.5) \quad P(r, k) = e^{irk} \overline{P_*(r, k)},$$

provided that  $k \in \mathbb{R}$ . This identity follows directly from (1.5) and can be found in, e.g., [5].

Next, we consider an important case when  $A \in L^2(\mathbb{R}^+)$ . In [4] (see also original Krein's paper [12]), it was shown that the following properties hold under this condition:

- There is a function  $\Pi(k), k \in \mathbb{C}^+$  such that

$$(2.6) \quad \lim_{r \rightarrow \infty} P_*(r, k) = \Pi(k)$$

uniformly over compact sets in  $\mathbb{C}^+$ . This  $\Pi$  is outer and the orthogonality measure  $\sigma$  can be written as follows

$$(2.7) \quad d\sigma = \frac{dk}{2\pi|\Pi(k)|^2} + d\sigma_s,$$

where  $\sigma_s$  is its singular part.

- Integrating the second equation in (1.5), we have

$$(2.8) \quad P_*(r, k) = 1 - \int_0^r A(\rho)P(\rho, k)d\rho.$$

Therefore

$$1 - P_*(r, k) = \int_0^r A(\rho)P(\rho, k)d\rho \rightarrow \tilde{A}(k) \stackrel{\text{def}}{=} \int_0^\infty A(\rho)P(\rho, k)d\rho,$$

when  $r \rightarrow \infty$  and convergence is in  $L^2(\mathbb{R}, \sigma)$  norm. On the other hand, the formula (12.37) in [4] gives

$$\tilde{A}(k) = 1 - \Pi(k) \cdot \chi_{E_s^c},$$

where  $E_s^c$  denotes the complement to  $E_s$ , the support of  $\sigma_s$ . Therefore,

$$(2.9) \quad \lim_{r \rightarrow \infty} \|P_*(r, k) - \Pi(k) \cdot \chi_{E_s^c}\|_{2, \sigma} = 0.$$

- From (2.8) and orthogonality, we get

$$\int_{\mathbb{R}} |P_*(r, k) - 1|^2 d\sigma = \int_0^r |A(\rho)|^2 d\rho.$$

**Proof of Lemma 1.5.** The second equation in (1.5) gives

$$(2.10) \quad P_*(r_2, k) - P_*(r_1, k) = - \int_{r_1}^{r_2} A(r)P(r, k)dr.$$

Theorem 1.3 yields necessary estimate on the maximal function and convergence of  $P_*(r, k)$   $\sigma$ -a.e. The limit is equal to  $\Pi$  from (2.6) due to (2.9).  $\square$

### 3. WAVE OPERATORS FOR SCHRÖDINGER EVOLUTION: PROOF OF THEOREM 1.6

We start this section by describing a connection between Krein systems and Dirac and Schrödinger operators on  $\mathbb{R}^+$ . Consider a Krein system with coefficient  $A \in L^2_{\text{loc}}(\mathbb{R}^+)$ . It corresponds to a Dirac operator

$$(3.1) \quad \mathcal{D} = \begin{pmatrix} -b & \partial_x - a \\ -\partial_x - a & b \end{pmatrix}$$

defined on the Hilbert space  $(f_1, f_2) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ , where  $a(x) = 2 \operatorname{Re} A(2x)$ ,  $b(x) = 2 \operatorname{Im} A(2x)$  with the boundary condition  $f_2(0) = 0$ . Indeed, define real-valued functions  $\phi$  and  $\psi$  by writing  $\phi(x, k) + i\psi(x, k) \stackrel{\text{def}}{=} P(2x, k)e^{-ikx}$ . It can be checked [5, 12] that  $(\phi, \psi)$  are generalized eigenfunctions for Dirac operator (3.1) and that  $2\sigma$  is its spectral measure. Define  $\{\mathcal{E}(x, k)\}$ ,  $x \geq 0$  by

$$(3.2) \quad \mathcal{E}(x, k) \stackrel{\text{def}}{=} P(2x, k)e^{-ikx}.$$

This is also a continuous orthonormal system with respect to  $\sigma$ , i.e.,

$$(3.3) \quad \int_{\mathbb{R}} \left| \int_0^\infty f(x)\mathcal{E}(x, k)dx \right|^2 d\sigma = \|f\|_2^2,$$

for every  $f \in L^2(\mathbb{R}^+)$  (see [4, 12]). Making the extra assumption that  $A$  is real-valued, i.e., that  $b = 0$ , and absolutely continuous on  $\mathbb{R}^+$  and taking the square of  $\mathcal{D}$  reveals the connections between Dirac and Schrödinger operators. Indeed,

$$(3.4) \quad \mathcal{D}^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

where  $H_1 f = -\partial_{xx}^2 f + q_1 f$ ,  $f'(0) + a(0)f(0) = 0$ ,  $H_2 f = -\partial_{xx}^2 f + q_2 f$ ,  $f(0) = 0$ ,

$$q_1 = a^2 - a', q_2 = a^2 + a'.$$

Later in the proof, we will use the spectral decomposition for Dirac  $\mathcal{D}$  and the formula (3.4) to write a suitable expression for  $e^{itH_2}$ .

The following result implies Theorem 1.6 thanks to Lemma 1.5.

**Theorem 3.1.** *Suppose the coefficient  $A$  in a Krein system is real and absolutely continuous,  $A \in L^2(\mathbb{R}^+)$ ,  $A' \in L^\infty(\mathbb{R}^+)$ , and*

$$(3.5) \quad \lim_{\rho \rightarrow \infty} \int_{\mathbb{R}} \left( \sup_{\rho < r_1 < r_2} \left| \int_{r_1}^{r_2} A(r)P(r, k)dr \right| \right)^2 \frac{d\sigma}{1+k^2} = 0.$$

Let  $a(x) = 2A(2x)$  and let  $q$  be a real-valued function on  $\mathbb{R}^+$  satisfying  $q \in L^1(\mathbb{R}^+)$ . Then, taking two operators  $H = -\partial_{xx}^2 + a' + q$  and  $H_0 = -\partial_{xx}^2$  both with Dirichlet boundary condition at zero, we get the existence of wave operators  $W^\pm(H, H_0)$ .

This Theorem is the central technical result of our paper. Before giving its proof, we state the following Lemma.

**Lemma 3.2.** *Suppose  $t \geq 0$ ,  $\mu$  is a measure on  $\mathbb{R}$ , and  $p(k), p_t(k) \in L^2_\mu(\mathbb{R})$ . Let  $\|p\|_{2,\mu} = 1$  and*

$$(3.6) \quad \lim_{t \rightarrow \infty} \|p_t\|_{2,\mu} = 1, \quad \lim_{t \rightarrow \infty} \int_\Delta |p - p_t|^2 d\mu = 0$$

for every interval  $\Delta \subset \mathbb{R}$ . Then,  $\lim_{t \rightarrow \infty} \|p - p_t\|_{2,\mu} = 0$ .

*Proof.* The proof is based on a standard exhaustion principle. For every  $\epsilon \in (0, 1)$ , we can choose  $L > 0$  such that  $\int_{\Delta^c} |p|^2 d\mu \leq \epsilon$  where  $\Delta \stackrel{\text{def}}{=} [-L, L]$ . By (3.6), there is a  $T$  so that

$$|1 - \|p_t\|_{2,\mu}^2| < \epsilon, \quad \int_{\Delta} |p - p_t|^2 d\mu < \epsilon$$

for  $t > T$ . Thus, for  $t > T$ , we also have

$$\begin{aligned} \int_{\Delta^c} |p_t|^2 d\mu &= \|p_t\|_{2,\mu}^2 - \int_{\Delta} |p_t|^2 d\mu = \\ \|p_t\|_{2,\mu}^2 - \left(1 - \int_{\Delta^c} |p|^2 d\mu - \int_{\Delta} (|p|^2 - |p_t|^2) d\mu\right) &\leq \\ \|p_t\|_{2,\mu}^2 - 1 + \int_{\Delta^c} |p|^2 d\mu + \left| \int_{\Delta} (|p|^2 - |p_t|^2) d\mu \right| &\lesssim \\ &\epsilon + \sqrt{\epsilon}, \end{aligned}$$

where we used the triangle inequality to estimate

$$\begin{aligned} \left| \int_{\Delta} (|p|^2 - |p_t|^2) d\mu \right| &= \left| \|p\|_{L_\mu^2(\Delta)}^2 - \|p_t\|_{L_\mu^2(\Delta)}^2 \right| = \\ \left( \|p\|_{L_\mu^2(\Delta)} + \|p_t\|_{L_\mu^2(\Delta)} \right) \cdot \left| \|p\|_{L_\mu^2(\Delta)} - \|p_t\|_{L_\mu^2(\Delta)} \right| &\lesssim \|p - p_t\|_{L_\mu^2(\Delta)} \leq \sqrt{\epsilon}. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}} |p - p_t|^2 d\mu = \int_{\Delta} |p - p_t|^2 d\mu + \int_{\Delta^c} |p - p_t|^2 d\mu \leq \epsilon + 2 \int_{\Delta^c} |p|^2 d\mu + 2 \int_{\Delta^c} |p_t|^2 d\mu \lesssim \sqrt{\epsilon}$$

for  $t > T$  and the proof is finished.  $\square$

**Proof of Theorem 3.1.** Since  $a^2, q \in L^1(\mathbb{R}^+)$  and relative trace class perturbations do not change the existence of wave operators (Birman-Kuroda Theorem, [14], p. 27), it is enough to consider  $H = H_2 = a' + a^2$ . Take  $f \in L^2(\mathbb{R}^+)$ . We need to prove existence of

$$(3.7) \quad \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} f,$$

where the limit is understood in  $L^2(\mathbb{R}^+)$  topology. Notice that, since both groups  $e^{itH}$  and  $e^{-itH_0}$  preserve  $L^2(\mathbb{R}^+)$  norm, it is enough to prove existence of the limit for every  $f \in \mathcal{T}$  where  $\mathcal{T}$  is any dense subset in  $L^2(\mathbb{R}^+)$ . We define  $\mathcal{T}$  as follows:  $\mathcal{T} \stackrel{\text{def}}{=} \{f : \hat{f}_o \in C_c^\infty(\mathbb{R}), 0 \notin \text{supp} \hat{f}_o\}$ , where  $f_o$  denotes the odd extension of  $f$  to  $\mathbb{R}$ . From now on, we assume that  $f \in \mathcal{T}$ ,  $\|f\|_2 = 1$  and that  $t \rightarrow +\infty$  in (3.7) (the case  $t \rightarrow -\infty$  can be handled similarly). Denote  $f_+ \stackrel{\text{def}}{=} (\hat{f}_o \cdot \chi_{\xi > 0})^\vee$ ,  $f_- \stackrel{\text{def}}{=} (\hat{f}_o \cdot \chi_{\xi < 0})^\vee$ . Working on the Fourier side, we get

$$e^{-itH_0} f = \frac{1}{\pi} \int_{\mathbb{R}} e^{-it\xi^2} \left( \int_{\mathbb{R}^+} f(u) \sin(\xi u) du \right) \sin(\xi x) d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi^2} \left( \int_{\mathbb{R}} f_o(u) e^{-i\xi u} du \right) e^{i\xi x} d\xi.$$

The last expression is equal to the restriction of  $e^{it\partial_{xx}^2} f_o$  to  $\mathbb{R}^+$ , where  $\partial_{xx}^2$  is considered on all of  $\mathbb{R}$ . The large time asymptotic behavior of  $e^{it\partial_{xx}^2} h$  for  $h \in L^2(\mathbb{R})$  is known and given in Lemma 4.1 from Appendix. Since  $\hat{f}_o(\xi) = \hat{f}_+(\xi)$  for  $\xi > 0$ , it is enough to show that

$$(3.8) \quad I \stackrel{\text{def}}{=} \frac{e^{itk^2}}{1+i} \int_0^\infty \frac{e^{ix^2/(4t)}}{\sqrt{t}} \hat{f}_+(x/(2t)) \psi(x, k) dx$$

has a limit in  $L^2(\mathbb{R}, 2\sigma)$  when  $t \rightarrow +\infty$ . Indeed, the spectral measure for Dirac operator  $\mathcal{D}$  is equal to  $2\sigma$ , the generalized eigenfunctions are  $(\phi, \psi)$ , and the Schrödinger operator is related to the Dirac operator by (3.4) so we can use spectral decomposition for the Dirac operator to compute  $e^{itH}$  where  $H = H_2$ . To this end, we will use the following generalized Fourier transform

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rightarrow \mathcal{F} = \int_0^\infty f_1(x) \phi(x, k) dx + \int_0^\infty f_2(x) \psi(x, k) dx$$

and the analog of Plancherel's Theorem

$$\|f_1\|_2^2 + \|f_2\|_2^2 = \|\mathcal{F}\|_{2,2\sigma}^2,$$

which holds since  $f \in \mathcal{T}$ ,  $\widehat{f}_+$  is supported on some interval  $[a, b]$  and  $a > 0$ . Use (2.5) and substitute

$$\psi(x, k) = \frac{\overline{P}_*(2x, k)e^{ikx} - P_*(2x, k)e^{-ikx}}{2i}$$

into (3.8) to get

$$(3.9) \quad I = I_1 - I_2,$$

where

$$I_1 = \frac{e^{itk^2}}{2i(1+i)} \int_{2at}^{2bt} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{f}_+(x/(2t)) \overline{P}_*(2x, k) e^{ikx} dx,$$

$$I_2 = \frac{e^{itk^2}}{2i(1+i)} \int_{2at}^{2bt} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{f}_+(x/(2t)) P_*(2x, k) e^{-ikx} dx.$$

Consider  $I_2$  - the analysis of  $I_1$  is similar. Integrating by parts, we get

$$\int_{2at}^{2bt} P_*(2x, k) \left( \int_{2at}^x \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right)' dx =$$

$$= P_*(4bt, k) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du - J_2,$$

where, thanks to the second equation in (1.5),

$$J_2 = \int_{2at}^{2bt} 2A(2x)P(2x, k) \left( \int_{2at}^x \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right) dx.$$

For the first term, we can write

$$P_*(4bt, k) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du =$$

$$(P_*(4bt, k) - \Pi(k) \cdot \chi_{E_s^c}) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du$$

$$+ \Pi(k) \cdot \chi_{E_s^c} \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du.$$

From (4.12), we get

$$\sup_{t>1} \left\| \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right\|_{L^\infty(\mathbb{R})} < C(f)$$

and (2.9) implies

$$\lim_{t \rightarrow +\infty} \left\| (P_*(4bt, k) - \Pi(k) \cdot \chi_{E_s^c}) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right\|_{2, \sigma} = 0.$$

From (2.7) and (4.11), we obtain

$$(3.10) \quad \lim_{t \rightarrow \infty} \left\| \frac{e^{itk^2} \Pi(k)}{2i(1+i)} \cdot \chi_{E_s^c} \cdot \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du - \frac{\sqrt{2\pi} \Pi(k)}{2i} \cdot \chi_{E_s^c} \widehat{f}_+(k) \right\|_{2, \sigma} = 0.$$

The analysis for  $I_1$  is analogous and it also gives a main term converging to

$$\frac{\sqrt{2\pi} \cdot \overline{\Pi(k)}}{2i} \cdot \chi_{E_s^c} \widehat{f}_+(-k)$$

and a correction which we call  $J_1$ . Consider  $J_1$  and  $J_2$ . We claim that if we show that

$$(3.11) \quad \lim_{t \rightarrow \infty} \int_{\Delta} |J_1|^2 d\sigma = 0, \quad \lim_{t \rightarrow \infty} \int_{\Delta} |J_2|^2 d\sigma = 0$$

for every interval  $\Delta \subset \mathbb{R}$ , then the proof of Theorem 3.1 will be finished after application of Lemma 3.2.

Indeed, in this lemma, we set  $\mu = 2\sigma$ ,  $p_t = I$  and the limiting function  $p$  is

$$p = \chi_{E_s^c} \cdot \sqrt{2\pi} \frac{\overline{\Pi(k)} \widehat{f}_+(-k) - \Pi(k) \widehat{f}_+(k)}{2i}.$$

To apply Lemma 3.2, we notice that  $\|I\|_{2,2\sigma} \rightarrow 1$  by Lemma 4.1. Moreover, (2.7) gives  $\|p\|_{2,2\sigma} = \|f\|_2 = 1$ .

We will prove the second identity in (3.11); the first one can be obtained similarly. For  $J_2$ , we have

$$J_2 = 2 \int_{2at}^{2bt} A(2x)P(2x, k) \left( \int_{2at}^x \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du \right) dx.$$

One can write

$$\int_{2at}^x \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du = \int_{2at}^0 \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du + \int_0^x \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du.$$

The first term does not depend on  $x$  and we can use (4.12) and (1.6) to write

$$(3.12) \quad \left\| \int_{2at}^{2bt} A(2x)P(2x, k) \left( \int_{2at}^0 \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du \right) dx \right\|_{2,\sigma} \leq C_{(f)} \int_{at}^{bt} |A(x)|^2 dx,$$

where the last expression converges to zero as  $t \rightarrow \infty$ . For the other term, we have

$$\int_0^x \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \hat{f}_+(u/(2t)) du = e^{-itk^2} \int_0^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \hat{f}_+(u/(2t)) du.$$

The integral can be rewritten as

$$\begin{aligned} & \int_0^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \hat{f}_+(u/(2t)) du = \\ & \int_{-\infty}^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \hat{f}_+(u/(2t)) du - \int_{-\infty}^0 \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \hat{f}_+(u/(2t)) du. \end{aligned}$$

The second term is  $x$ -independent so its contribution is negligible by an argument identical to (3.12). For the first one, we change variables and write, using the same variable  $u$ ,

$$(3.13) \quad \begin{aligned} & \int_{-\infty}^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \hat{f}_+(u/(2t)) du = 2 \int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2} \hat{f}_+(k + u/\sqrt{t}) du \\ & = 2 \int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2} \left( \hat{f}_+(k + u/\sqrt{t}) - \hat{f}_+(k) \right) du + 2\hat{f}_+(k) \int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2} du. \end{aligned}$$

We can continue as follows

$$\begin{aligned} & \int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2} \left( \hat{f}_+(k + u/\sqrt{t}) - \hat{f}_+(k) \right) du = \int_{-\infty}^0 e^{iu^2} \left( \hat{f}_+(k + u/\sqrt{t}) - \hat{f}_+(k) \right) du + \\ & \int_0^{(x-2kt)/2\sqrt{t}} e^{iu^2} \left( \hat{f}_+(k + u/\sqrt{t}) - \hat{f}_+(k) \right) du. \end{aligned}$$

The first term in the right-hand side does not depend on  $x$  and it is uniformly bounded in  $k \in \mathbb{R}$  and  $t \geq 1$  as can be seen by integrating by parts. Thus, its contribution to  $\|J_2\|_{L^2_\sigma(\Delta)}$  is also negligible.

We want to apply Lemma 4.2 from Appendix to the second term. Since we are interested in  $k \in \Delta$  and  $x \in [at, bt]$ , then  $|(x-2kt)/2t| < C_{(a,b,\Delta)}$ . Hence, the Lemma is applicable with  $\epsilon = 1/\sqrt{t}$ ,  $g(u) = \hat{f}_+(k+u) - \hat{f}_+(k)$  which gives

$$\left| \int_0^{(x-2kt)/2\sqrt{t}} e^{iu^2} \left( \hat{f}_+(k + u/\sqrt{t}) - \hat{f}_+(k) \right) du \right| \leq C_{a,b,\Delta,f} / \sqrt{t}.$$

The proof of Lemma 4.2 shows that this bound is uniform in  $k \in \Delta$ . We substitute it and apply (1.8) along with the generalized Minkowski inequality to get

$$\begin{aligned} & \left( \int_{\Delta} \left| \frac{1}{\sqrt{t}} \int_{2at}^{2bt} |A(2x)P(2x, k)| \right|^2 d\sigma \right)^{1/2} \\ & \frac{1}{\sqrt{t}} \int_{2at}^{2bt} |A(2x)| \cdot \left( \int_{\Delta} |P(2x, k)|^2 d\sigma \right)^{1/2} dx \stackrel{(1.8)}{\lesssim} \\ & \frac{C_{(\Delta, \|A\|_{St})}}{\sqrt{t}} \int_{2at}^{2bt} |A(2x)| dx \leq C_{(\Delta, a, b, \|A\|_{St})} \left( \int_{at}^{bt} |A(x)|^2 dx \right)^{1/2} \end{aligned}$$

and the last expression converges to zero when  $t \rightarrow +\infty$ . We are only left with controlling the contribution from the last term in (3.13), i.e.,

$$\hat{f}_+(k) \int_{2at}^{2bt} A(2x)P(2x, k) \left( \int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) dx.$$

Let us write a partition of unity

$$(3.14) \quad 1 = \mu_- + \mu_0 + \mu_+,$$

where  $\mu_0$  is even, smooth, supported in  $(-2, 2)$  and

$$0 \leq \mu_0 \leq 1, \quad \mu_0 = 1 \text{ if } |x| < 1.$$

The function  $\mu_+$  is supported on  $(1, \infty)$  and is non-decreasing,  $\mu_-(x) \stackrel{\text{def}}{=} \mu_+(-x)$ . Then,

$$\int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du = \left( \int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) \left( \mu_-((x-2kt)/(2\sqrt{t})) + \mu_0(\cdot) + \mu_+(\cdot) \right).$$

We will apply the following trick several times. Notice that the function  $F(x) \stackrel{\text{def}}{=} \left( \int_0^x e^{iu^2} du \right) \mu_0(x) \in C_c^\infty(\mathbb{R})$  thus  $\hat{F} \in L^1(\mathbb{R})$  and we can write

$$F((x-2kt)/(2\sqrt{t})) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{F}(\xi) \exp(i\xi(x-2kt)/(2\sqrt{t})) d\xi.$$

Then,

$$\begin{aligned} & \hat{f}_+(k) \int_{2at}^{2bt} A(2x)P(2x, k) \left( \mu_0((x-2kt)/(2\sqrt{t})) \int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) dx = \\ & \hat{f}_+(k) \int_{2at}^{2bt} A(2x)P(2x, k) F((x-2kt)/(2\sqrt{t})) dx = \\ & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{F}(\xi) \left( \hat{f}_+(k) e^{-i\xi k\sqrt{t}} \int_{2at}^{2bt} A(2x)P(2x, k) \exp(i\xi x/(2\sqrt{t})) dx \right) d\xi. \end{aligned}$$

We use the generalized Minkowski inequality and (1.6) to estimate the last quantity as follows

$$\begin{aligned} & \left\| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{F}(\xi) \left( \hat{f}_+(k) e^{-i\xi k\sqrt{t}} \int_{2at}^{2bt} A(2x)P(2x, k) \exp(i\xi x/(2\sqrt{t})) dx \right) d\xi \right\|_{2, \sigma} \\ & \left( \int_{\mathbb{R}} |\hat{F}(\xi)| d\xi \right) \|\hat{f}_+\|_\infty \left( \int_{at}^{bt} |A(x)|^2 dx \right)^{1/2} \end{aligned}$$

and the last quantity converges to zero when  $t \rightarrow \infty$ . We apply a similar strategy to other terms.

$$\begin{aligned} & \left( \int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t})) = C \mu_+((x-2kt)/(2\sqrt{t})) \\ & - \left( \int_{(x-2kt)/(2\sqrt{t})}^\infty e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t})), \end{aligned}$$

where  $C \stackrel{\text{def}}{=} \int_0^\infty e^{iu^2} du$ . Consider

$$\begin{aligned} \int_{2at}^{2bt} A(2x)P(2x, k) \mu_+((x-2kt)/(2\sqrt{t})) dx &= \int_{2at}^{2bt} \left( \int_{2at}^x A(2u)P(2u, k) du \right)' \mu_+((x-2kt)/(2\sqrt{t})) dx \\ &= \left( \int_{2at}^{2bt} A(2u)P(2u, k) du \right) \mu_+((b-k)\sqrt{t}) - \int_{2at}^{2bt} \left( \int_{2at}^x A(2u)P(2u, k) du \right) \frac{\mu'_+((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} dx. \end{aligned}$$

The first term gives the contribution

$$\int_{\mathbb{R}} \left| \widehat{f}_+(k) \left( \int_{2at}^{2bt} A(2u)P(2u, k) du \right) \mu_+((b-k)\sqrt{t}) \right|^2 d\sigma \lesssim \|\widehat{f}_+\|_\infty^2 \int_{2at}^{2bt} |A(2u)|^2 du$$

and the last quantity converges to zero when  $t \rightarrow \infty$ . For the second one, we can write an estimate

$$(3.15) \quad \left| \int_{2at}^{2bt} \left( \int_{2at}^x A(2u)P(2u, k) du \right) \frac{\mu'_+((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} dx \right| \leq \left( \sup_{2at < r_1 < r_2} \left| \int_{r_1}^{r_2} A(2u)P(2u, k) du \right| \right) \cdot \int_{2at}^{2bt} \left| \frac{\mu'_+((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} \right| dx.$$

Since  $\mu_+$  was chosen to be non-decreasing, one obtains

$$\int_{2at}^{2bt} \left| \frac{\mu'_+((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} \right| dx \lesssim 1.$$

Under the assumptions of the theorem, we get

$$\left\| |\widehat{f}_+| \cdot \sup_{2at < r_1 < r_2} \left| \int_{r_1}^{r_2} A(2u)P(2u, k) du \right| \right\|_{L^2_\sigma(\Delta)} \rightarrow 0$$

when  $t \rightarrow \infty$ . Consider the expression

$$\left( \int_{(x-2kt)/(2\sqrt{t})}^\infty e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t}))$$

and apply Lemma 4.3 from Appendix to write it as

$$\begin{aligned} \left( \int_{(x-2kt)/(2\sqrt{t})}^\infty e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t})) &= \\ (2\pi)^{-1/2} e^{ix^2/(4t)} e^{-ixk} e^{ik^2t} \int_{\mathbb{R}} e^{i\xi(x-2kt)/(2\sqrt{t})} \Psi(\xi) d\xi, \end{aligned}$$

where  $\Psi \in L^1(\mathbb{R})$ . Then,

$$\begin{aligned} \int_{2at}^{2bt} A(2x)P(2x, k) e^{ix^2/(4t)} e^{-ixk} e^{ik^2t} \left( \int_{\mathbb{R}} e^{i\xi(x-2kt)/(2\sqrt{t})} \Psi(\xi) d\xi \right) dx \\ = e^{ik^2t} \int_{\mathbb{R}} \Psi(\xi) e^{-i\xi k\sqrt{t}} \left( \int_{2at}^{2bt} A(2x) e^{ix^2/(4t)} e^{i\xi x/(2\sqrt{t})} \mathcal{E}(x, k) dx \right) d\xi, \end{aligned}$$

where  $\mathcal{E}(x, k) = P(2x, k) e^{-ikx}$  was introduced in (3.2). Using the generalized Minkowski inequality and (3.3), we get

$$\begin{aligned} \left\| \widehat{f}_+(k) \cdot e^{ik^2t} \int_{\mathbb{R}} \Psi(\xi) e^{-i\xi k\sqrt{t}} \left( \int_{2at}^{2bt} A(2x) e^{ix^2/(4t)} e^{i\xi x/(2\sqrt{t})} \mathcal{E}(x, k) dx \right) d\xi \right\|_{2, \sigma} \\ \lesssim \|\widehat{f}_+\|_\infty \cdot \left( \int_{\mathbb{R}} |\Psi(\xi)| d\xi \right) \cdot \left( \int_{2at}^{2bt} |A(2x)|^2 dx \right)^{1/2} \end{aligned}$$

and the last quantity converges to zero when  $t \rightarrow \infty$ .

The contribution from the term

$$\left( \int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) \mu_-((x-2kt)/(2\sqrt{t}))$$

can be handled in the same way. Thus,

$$\lim_{t \rightarrow \infty} \int_{\Delta} |J_2|^2 d\sigma = 0$$

and our Theorem is proved.  $\square$

**Remark.** Notice that we had to use our additional assumption about the maximal function (3.5) only when handling (3.15). It is an intriguing question whether this extra hypothesis can be dropped.

#### 4. APPENDIX

In this Appendix, we collect results that are used in the main text. Although some of them are standard, we provide their proofs for completeness.

**Proof of Lemma 1.4.** In Section 13 of [5], the following formula for the Green's function of an operator  $\mathcal{D}$  (i.e., the integral kernel of  $R_z = (\mathcal{D} - z)^{-1}$ ) was obtained

$$(4.1) \quad G(x, y, z) = \begin{pmatrix} G_{11}(x, y, z) & G_{12}(x, y, z) \\ G_{21}(x, y, z) & G_{22}(x, y, z) \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}} \frac{\phi(x, k)\phi(y, k)}{k - z} d\sigma_d(k) & \int_{\mathbb{R}} \frac{\phi(x, k)\psi(y, k)}{k - z} d\sigma_d(k) \\ \int_{\mathbb{R}} \frac{\psi(x, k)\phi(y, k)}{k - z} d\sigma_d(k) & \int_{\mathbb{R}} \frac{\psi(x, k)\psi(y, k)}{k - z} d\sigma_d(k) \end{pmatrix}$$

and  $\sigma_d = 2\sigma$ . We now introduce an auxiliary parameter  $\rho \in [1, \infty)$  to be chosen later as  $\rho \sim 1 + \|A\|_{\text{St}}^2$ . Since  $|P(2x, k)|^2 = \phi^2(x, k) + \psi^2(x, k)$  and  $\sup_{k \in \mathbb{R}} (k^2 + \rho^2)/(k^2 + 1) \lesssim \rho^2$ , then

$$(4.2) \quad \sup_{x \geq 0} \int_{\mathbb{R}} \frac{|P(x, k)|^2}{k^2 + 1} d\sigma = \sup_{x \geq 0} \int_{\mathbb{R}} \frac{(k^2 + \rho^2)|P(x, k)|^2}{(k^2 + \rho^2)(k^2 + 1)} d\sigma \lesssim \rho \sup_{x \geq 0} \int_{\mathbb{R}} \frac{\rho|P(x, k)|^2}{k^2 + \rho^2} d\sigma.$$

Hence, we only need to prove that

$$(4.3) \quad \sup_{x \geq 0} \text{Im}(G_{11}(x, x, i\rho) + G_{22}(x, x, i\rho)) \lesssim 1.$$

To control  $G(x, y, i\rho)$ , i.e., the integral kernel of the resolvent  $R_{i\rho}$ , we will use the standard perturbation series. If  $R_{i\rho}^0$  denotes the resolvent of free Dirac operator, we write the second resolvent identity:

$$R_{i\rho} = R_{i\rho}^0 - R_{i\rho}^0 V R_{i\rho}^0, \quad V \stackrel{\text{def}}{=} \begin{pmatrix} -b & -a \\ -a & b \end{pmatrix}$$

and iterate it to get the series

$$(4.4) \quad R_{i\rho} = R_{i\rho}^0 - R_{i\rho}^0 V R_{i\rho}^0 + R_{i\rho}^0 V R_{i\rho}^0 V R_{i\rho}^0 - \dots$$

In the series (4.4), each term starting from the second one takes the form  $(-1)^{j+1} (R_{i\rho}^0 V)^j (R_{i\rho}^0 V R_{i\rho}^0)$  and  $j = 0, 1, 2, \dots$ . If we denote its kernel by  $k_j(x, y)$ , then

$$(4.5) \quad G(x, y, i\rho) = G^0(x, y, i\rho) - k_0(x, y) + k_1(x, y) - \dots$$

and  $G^0(x, y, z)$  stands for the Green's function of the free Dirac operator. Next, we will show convergence of this series for suitable choice of parameter  $\rho$  and will provide an estimate for it.

First, we claim that for every  $j = 0, 1, \dots$ , we have

$$(4.6) \quad \|k_j(x, y)\| \leq C^{j+1} \frac{e^{-\rho|x-y|/2} \|A\|_{\text{St}}^{j+1}}{\rho^{(j+1)/2}},$$

where  $C$  is an absolute constant to be specified below. We will prove (4.6) by induction. To this end, we use formula (4.1) and residue calculus to obtain the bound

$$\|G^0(x, y, i\rho)\| \lesssim e^{-\rho|x-y|} + e^{-\rho(x+y)} \lesssim e^{-\rho|x-y|}.$$

Thus, for  $k_0(x, y)$ , we have

$$\|k_0(x, y)\| \lesssim \int_0^\infty e^{-\rho|x-\xi|} |\alpha(\xi)| e^{-\rho|y-\xi|} d\xi, \quad \alpha \stackrel{\text{def}}{=} |a| + |b|.$$

Continue  $\alpha(\xi)$  to negative  $\xi$  by zero. We write

$$(4.7) \quad \|k_0(x, 0)\| \lesssim \int_0^\infty e^{-\rho|x-\xi|} \alpha(\xi) e^{-\rho\xi} d\xi \leq e^{-\rho x} \int_0^x \alpha(\xi) d\xi + e^{\rho x} \int_x^\infty \alpha(\xi) e^{-2\rho\xi} d\xi.$$

Then, using the Cauchy-Schwarz inequality, one has  $\int_0^\infty \alpha(\xi) d\xi \lesssim (x + x^{1/2}) \|A\|_{\text{St}}$ . By a change of variable,

$$e^{\rho x} \int_x^\infty \alpha(\xi) e^{-2\rho\xi} d\xi = e^{-\rho x} \int_0^\infty e^{-2\rho\eta} \alpha(x + \eta) d\eta.$$

We have

$$\begin{aligned} \int_0^\infty e^{-2\rho\eta} \alpha(x + \eta) d\eta &= \int_0^1 e^{-2\rho\eta} \alpha(x + \eta) d\eta + \sum_{j=1}^\infty \int_j^{j+1} e^{-2\rho\eta} \alpha(x + \eta) d\eta \leq \\ &\left( \int_0^1 e^{-4\rho\eta} d\eta \right)^{1/2} \left( \int_0^1 \alpha^2(x + \eta) d\eta \right)^{1/2} + \sum_{j=1}^\infty e^{-2\rho j} \left( \int_j^{j+1} \alpha^2(x + \eta) d\eta \right)^{1/2} \lesssim \frac{\|A\|_{\text{St}}}{\rho^{1/2}} \end{aligned}$$

by virtue of the Cauchy-Schwarz inequality. Summing up, we get

$$\|k_0(x, 0)\| \lesssim (x + x^{1/2} + \rho^{-1/2}) e^{-\rho x} \|A\|_{\text{St}} \lesssim \frac{e^{-\rho x/2} \|A\|_{\text{St}}}{\rho^{1/2}}.$$

The Stummel condition is translation-invariant on the line, which implies (4.6) for  $j = 0$ :

$$(4.8) \quad \|k_0(x, y)\| \leq C \frac{e^{-\rho|x-y|/2}}{\rho^{1/2}} \|A\|_{\text{St}}.$$

We can write  $k_{j+1}(x, y) = \int_{\mathbb{R}^+} G^0(x, \xi, i\rho) V(\xi) k_j(\xi, y) d\xi$  and use the inductive assumption to conclude that

$$\|k_{j+1}(x, y)\| \leq C_1 \int_{\mathbb{R}^+} e^{-\rho|x-\xi|} \alpha(\xi) \cdot \|k_j(\xi, y)\| d\xi \leq \frac{C_1 C^{j+1} \|A\|_{\text{St}}^{j+1}}{\rho^{(j+1)/2}} \int_{\mathbb{R}^+} e^{-\rho|x-\xi|} \alpha(\xi) e^{-\rho|\xi-y|/2} d\xi.$$

For  $y = 0$ , we get

$$(4.9) \quad \int_{\mathbb{R}^+} e^{-\rho|x-\xi|} \alpha(\xi) e^{-\rho\xi/2} d\xi = e^{-\rho x/2} \cdot e^{-\rho x/2} \int_0^x e^{\rho\xi/2} \alpha(\xi) d\xi + e^{\rho x} \int_x^\infty \alpha(\xi) e^{-3\rho\xi/2} d\xi.$$

Then, we write

$$\begin{aligned} e^{-\rho x/2} \int_0^x e^{\rho\xi/2} \alpha(\xi) d\xi &= \int_0^x e^{-\rho\eta/2} \alpha(x - \eta) d\eta \leq \int_0^1 e^{-\rho\eta/2} \alpha(x - \eta) d\eta + \sum_{j=1}^\infty \int_j^{j+1} e^{-\rho\eta/2} \alpha(x - \eta) d\eta \leq \\ &\left( \int_0^1 e^{-\rho\eta} d\eta \right)^{1/2} \left( \int_0^1 \alpha^2(x - \eta) d\eta \right)^{1/2} + \sum_{j=1}^\infty e^{-\rho j/2} \left( \int_j^{j+1} \alpha^2(x - \eta) d\eta \right)^{1/2} \lesssim \frac{\|A\|_{\text{St}}}{\rho^{1/2}}. \end{aligned}$$

Estimating the second integral in (4.9) in a similar way, we have

$$\int_{\mathbb{R}^+} e^{-\rho|x-\xi|} \alpha(\xi) e^{-\rho\xi/2} d\xi \leq C_2 \frac{e^{-\rho x/2} \|A\|_{\text{St}}}{\rho^{1/2}}$$

and, using the translation invariance of the Stummel condition,

$$\int_{\mathbb{R}^+} e^{-\rho|x-\xi|} \alpha(\xi) e^{-\rho|\xi-y|/2} d\xi \leq C_2 \frac{e^{-\rho|x-y|/2} \|A\|_{\text{St}}}{\rho^{1/2}}.$$

Thus,

$$\|k_{j+1}(x, y)\| \leq \frac{C_1 C_2 C^{j+1} e^{-\rho|x-y|/2} \|A\|_{\text{St}}^{j+2}}{\rho^{(j+2)/2}}.$$

Choosing  $C$  sufficiently large, e.g., larger than  $C_1 C_2$ , we show (4.6) for  $j + 1$ . This proves the claim. Now, (4.5) implies  $\|G(x, y, i\rho)\| \lesssim e^{-\rho|x-y|/2}$  provided that  $\rho = 2C(1 + \|A\|_{\text{St}}^2)$ . Thus, (4.2) finishes the proof.  $\square$

**Lemma 4.1.** *Let  $h \in L^2(\mathbb{R})$ . Then,*

$$(4.10) \quad \lim_{t \rightarrow +\infty} \left\| e^{it\delta_{xx}^2} h - \frac{1}{1+i} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) \right\|_{L^2(\mathbb{R})} = 0,$$

and, taking the inverse Fourier transform,

$$(4.11) \quad \lim_{t \rightarrow +\infty} \left\| \frac{1}{1+i} \left( \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) \right)^\vee - e^{-it\xi^2} \check{h}(\xi) \right\|_{L^2(\mathbb{R})} = 0.$$

Suppose  $\widehat{h} \in C_c^\infty(\mathbb{R})$ , then

$$(4.12) \quad \sup_{t>1, \alpha, \beta \in \mathbb{R}} \left\| \int_{\alpha t}^{\beta t} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) e^{ixk} dx \right\|_{L^\infty(\mathbb{R})} < C_{(h)}.$$

*Proof.* Formula (4.10) can be found in [15] (see formulas (4.10) and (4.12) there). Then, (4.11) is a direct corollary. Proof of (4.12) follows from a direct calculation:

$$\begin{aligned} \int_{\alpha t}^{\beta t} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) e^{ixk} dx &= \frac{e^{-itk^2}}{\sqrt{t}} \int_{\alpha t}^{\beta t} \exp\left(i\left(\frac{x}{2\sqrt{t}} + k\sqrt{t}\right)^2\right) \widehat{h}(x/(2t)) dx \\ &= 2e^{-itk^2} \int_{\sqrt{t}(0.5\alpha+k)}^{\sqrt{t}(0.5\beta+k)} \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) d\xi. \end{aligned}$$

Now, consider the integral

$$\int_0^l \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) d\xi$$

for arbitrary  $l \in \mathbb{R}, k \in \mathbb{R}, t \geq 1$  and let  $\mu_0$  be a bump function introduced in (3.14). We have

$$\int_0^l \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) d\xi = \int_0^l \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) \mu_0 d\xi + \int_0^l \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) (1 - \mu_0) d\xi.$$

The first integral is bounded uniformly in all parameters since  $\widehat{h} \in C_c^\infty(\mathbb{R})$ . For the second one, we can write

$$\begin{aligned} \int_0^l \exp(i\xi^2) \widehat{h}(-k + \xi/\sqrt{t}) (1 - \mu_0) d\xi &= \int_0^l \left( \exp(i\xi^2) \right)' \frac{\widehat{h}(-k + \xi/\sqrt{t}) (1 - \mu_0)}{2i\xi} d\xi \\ &= \exp(il^2) \frac{\widehat{h}(-k + l/\sqrt{t}) (1 - \mu_0(l))}{2il} - \int_0^l \exp(i\xi^2) \left( \frac{\widehat{h}(-k + \xi/\sqrt{t}) (1 - \mu_0(\xi))}{2i\xi} \right)' d\xi. \end{aligned}$$

The first term is uniformly bounded because  $1 - \mu_0(0) = 0$ . For the second one, we can show that each resulting integral is uniformly bounded, e.g.,

$$\begin{aligned} \left| \frac{1}{\sqrt{t}} \int_0^l \exp(i\xi^2) \frac{\widehat{h}'(-k + \xi/\sqrt{t}) (1 - \mu_0)}{2i\xi} d\xi \right| &\lesssim \frac{1}{\sqrt{t}} \int_0^l |\widehat{h}'(-k + \xi/\sqrt{t})| d\xi \leq \|\widehat{h}'\|_1, \\ \left| \int_0^l \exp(i\xi^2) \frac{\widehat{h}(-k + \xi/\sqrt{t}) (1 - \mu_0(\xi))}{\xi^2} d\xi \right| &\leq \|\widehat{h}\|_\infty \int_0^l \frac{|1 - \mu_0(\xi)|}{\xi^2} d\xi \lesssim \|\widehat{h}\|_\infty, \\ \left| \int_0^l \exp(i\xi^2) \frac{\widehat{h}(-k + \xi/\sqrt{t}) \mu_0'(\xi)}{2i\xi} d\xi \right| &\lesssim \|\widehat{h}\|_\infty \end{aligned}$$

and (4.12) is proved.  $\square$

**Lemma 4.2.** *Let  $\epsilon \in (0, 1), \nu > 0, a > 0$ , and  $|a\epsilon| \leq \nu$ . We have*

$$\left| \int_0^a e^{iu^2} g(u\epsilon) du \right| \leq C_{(g, \nu)} \epsilon$$

provided that  $g \in C^\infty(\mathbb{R})$  and  $g(0) = 0$ .

*Proof.* We have

$$(4.13) \quad \int_0^a (e^{iu^2})' g(u\epsilon) u^{-1} du = e^{ia^2} \epsilon \left( \frac{g(a\epsilon)}{a\epsilon} \right) - \epsilon g'(0) - \epsilon \int_0^a e^{iu^2} \left( \frac{g(u\epsilon)}{\epsilon u} \right)' du.$$

We can write  $|g(\xi)| \leq C_{(g,\nu)} |\xi|$  for  $\xi \in [-\nu, \nu]$  and the first term is controlled by  $C_{(g,\nu)} \epsilon$  since  $|a\epsilon| \leq \nu$ . For the third one, we introduce  $G(u) \stackrel{\text{def}}{=} (g(u)/u)' \in C^\infty(\mathbb{R})$  and write

$$G_1(u) \stackrel{\text{def}}{=} G(u) - G(0), \quad G(u) = G(0) + G_1(u)$$

so that

$$\int_0^a e^{iu^2} G(u\epsilon) du = G(0) \int_0^a e^{iu^2} du + \int_0^a e^{iu^2} G_1(\epsilon u) du.$$

The absolute value of the first term is bounded by  $C_{(g)}$  uniformly in  $a$ . For the second one, we can iterate the argument since  $G_1 \in C^\infty(\mathbb{R})$  and  $G_1(0) = 0$ . We get

$$(4.14) \quad \begin{aligned} & \int_0^a e^{iu^2} G_1(\epsilon u) du = -0.5i\epsilon \int_0^a (e^{iu^2})' \frac{G_1(\epsilon u)}{\epsilon u} du \\ & = -0.5i\epsilon \left( e^{ia^2} \frac{G_1(\epsilon a)}{\epsilon a} - G_1'(0) - \int_0^a e^{iu^2} \left( \frac{G_1(\epsilon u)}{\epsilon u} \right)' du \right). \end{aligned}$$

Writing a rough estimate

$$\left| \int_0^a e^{iu^2} \left( \frac{G_1(\epsilon u)}{\epsilon u} \right)' du \right| \leq C_{(g)} |a|$$

and substituting it into (4.14) gives

$$\left| \int_0^a e^{iu^2} G_1(\epsilon u) du \right| \leq C_{(g)} (\epsilon + \epsilon |a|) = C_{(g)} (\epsilon + \nu).$$

We bring it to (4.13) to finish the proof of the Lemma.  $\square$

Consider  $H$  defined as

$$H(x) = \int_x^\infty e^{it^2} dt.$$

This integral can be related to the so-called erf-function whose properties are well-known. However, our purpose is to obtain a specific representation for  $H$  for  $x \in [1, \infty)$  and we proceed directly as follows. We change variables and iteratively integrate by parts  $n$  times to get

$$H(x) = i \frac{e^{ix^2}}{2x} - \frac{i}{2} \int_{x^2}^\infty \frac{e^{iu}}{u^{3/2}} du = e^{ix^2} \left( \sum_{j=0}^{n-1} \frac{c_j}{x^{1+2j}} + c'_n e^{-ix^2} \int_{x^2}^\infty \frac{e^{iu}}{u^{n+1/2}} du \right) \stackrel{\text{def}}{=} e^{ix^2} (H_{1,n} + H_{2,n})$$

where  $\{c_j\}$  and  $c'_n$  are some constants. Let  $\mu_+$  be a cutoff function that satisfies the following conditions:  $\mu_+$  is supported on  $(1, \infty)$ ,  $\mu_+(x) = 1$  for  $x > 2$ ,  $\mu_+ \in C^\infty(\mathbb{R})$ . Define

$$H_{1,n}^{(m)} \stackrel{\text{def}}{=} H_{1,n} \mu_+, \quad H_{2,n}^{(m)} \stackrel{\text{def}}{=} H_{2,n} \mu_+.$$

**Lemma 4.3.** *Let  $n > 1$ . We have  $\widehat{H_{1,n}^{(m)}} \in L^1(\mathbb{R})$ ,  $\widehat{H_{2,n}^{(m)}} \in L^1(\mathbb{R})$ .*

*Proof.* Consider  $H_{2,n}^{(m)}$  first. We have

$$|H_{2,n}^{(m)}| \leq C_n (1 + |x|)^{-(2n+1)}, \quad |\partial_x H_{2,n}^{(m)}| \leq C_n (1 + |x|)^{-(2n)}, \quad |\partial_{xx}^2 H_{2,n}^{(m)}| \leq C_n (1 + |x|)^{-(2n-1)}.$$

Therefore,

$$|\widehat{H_{2,n}^{(m)}}(\xi)| < C_n (1 + |\xi|)^{-2}$$

and hence  $\widehat{H_{2,n}^{(m)}} \in L^1(\mathbb{R})$ . For  $H_{1,n}^{(m)}$ , consider the first term,  $x^{-1} \mu_+$ . Other terms can be handled similarly. We have  $x^{-1} \mu_+ \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$  and all of its derivatives are in  $L^2(\mathbb{R})$ . Thus,  $\xi^j \widehat{(x^{-1} \mu_+)} \in L^2(\mathbb{R})$  for all  $j \in \mathbb{Z}^+$ . Therefore,  $\widehat{(x^{-1} \mu_+)}(\xi) \in L^1(|\xi| > 1)$ . For  $|\xi| < 1$ , we can write an estimate

$$|\widehat{(x^{-1} \mu_+)}| < C |\log \xi|,$$

which can be verified directly:

$$\int_1^{\infty} \frac{\mu_+(x)}{x} e^{-i\xi x} dx = \int_2^{\infty} \frac{e^{-i\xi x}}{x} dx + O(1).$$

For  $\xi \in (0, 1)$ ,

$$\int_2^{\infty} \frac{e^{-i\xi x}}{x} dx = \int_{2\xi}^{\infty} \frac{e^{-iu}}{u} du = \int_{2\xi}^1 \frac{e^{-iu}}{u} du + \int_1^{\infty} \frac{e^{-iu}}{u} du = O(|\log \xi| + 1).$$

For  $\xi \in (-1, 0)$ , the argument is analogous and we get the statement of the Lemma.  $\square$

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