ZERO SETS, ENTROPY, AND POINTWISE ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. Let μ be a measure from Szegő class on the unit circle $\mathbb{T}$ and let $\{f_n\}$ be the family of Schur functions generated by μ. In this paper, we prove a version of the classical Szegő's formula, which controls the oscillation of f_n on $\mathbb{T}$ for all $n \geqslant 0$. Then, we focus on an analog of Lusin's conjecture for polynomials $\{\varphi_n\}$ orthogonal with respect to measure μ and prove that pointwise convergence of $\{|\varphi_n|\}$ almost everywhere on $\mathbb{T}$ is equivalent to a certain condition on zeroes of φ_n.

1. INTRODUCTION

Consider a probability measure μ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane $\mathbb{C}$. The Schur function of μ is the analytic function f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ defined by the relation

$$\frac{1 + zf(z)}{1 - zf(z)} = \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z}\, d\mu(\xi), \qquad z \in \mathbb{D}. \tag{1}$$

Taking the real part of both sides of (1) and using the Schwarz lemma, it is not difficult to see that $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$. In particular, the function f has non-tangential boundary values (to be denoted by the same letter f) almost everywhere on the unit circle $\mathbb{T}$. Set $f_0 = f$ and denote the Schur iterates of f by f_n:

$$zf_{n+1}(z) = \frac{f_n(z) - f_n(0)}{1 - \overline{f_n(0)}f_n(z)}, \qquad z \in \mathbb{D}, \qquad n \geqslant 0. \tag{2}$$

Schur's algorithm (2) produces an infinite family $\{f_n\}_{n\geqslant 0}$ of analytic contractions unless μ is supported on a finite subset of $\mathbb{T}$, or, equivalently, f is a finite Blaschke product. Knowing coefficients $f_k(0)$ for $0 \leqslant k \leqslant n$, one can set $f_{n+1} = 0$ and reverse the recursion in (2) to obtain an efficient approximation to f in $\mathbb{D}$ by a rational contraction of degree n, see Corollary 4.7 in [10].

Let m be the Lebesgue measure on the unit circle $\mathbb{T}$ normalized by $m(\mathbb{T}) = 1$, and let $\mu = w\, dm + \mu_{\mathbf{s}}$ be the decomposition of μ into the absolutely continuous and singular parts. The measure μ is said to belong to the Szegő class $\mathrm{Sz}(\mathbb{T})$ if $\log w \in L^1(\mathbb{T})$. To every measure $\mu \in \mathrm{Sz}(\mathbb{T})$, we associate the entropy function

$$\mathcal{K}(\mu, z) = \log \mathcal{P}(\mu, z) - \mathcal{P}(\log w, z), \qquad z \in \mathbb{D}, \tag{3}$$

where $\mathcal{P}$ stands for the harmonic extension to $\mathbb{D}$:

$$\mathcal{P}(\mu, z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\xi}z|^2}\, d\mu(\xi),$$

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and we set $\mathcal{P}(v,z)=\mathcal{P}(v\,dm,z)$ for $v\in L^1(\mathbb{T})$. Roughly speaking, $\mathcal{K}(\mu,z)$ measures a "size of oscillation" of μ on the arc $\{\xi\in\mathbb{T}:|\xi-a_z|\leqslant 1-|z|\}$, $a_z=z/|z|$. By Jensen's inequality, we have $\mathcal{K}(\mu,z)\geqslant 0$ for every $z\in\mathbb{D}$ and $\mathcal{K}(\mu,z)=0$ if and only if $\mu=m$. Notice also that $\mathcal{K}(\mu,\cdot)$ is superharmonic in $\mathbb{D}$ and its nontangential boundary value is zero almost everywhere on $\mathbb{T}$.

The celebrated Szegő theorem says that a probability measure μ on the unit circle $\mathbb{T}$ belongs to the Szegő class $\mathrm{Sz}(\mathbb{T})$ if and only if $\sum_{n\geqslant 0}|f_n(z)|^2<\infty$ for some (and then for every) $z\in\mathbb{D}$. Moreover, in the latter case we have

$$\mathcal{K}(\mu,0)=-\int_{\mathbb{T}}\log w\,dm=-\log\prod_{n\geqslant 0}(1-|f_n(0)|^2). \tag{4}$$

This result has many equivalent reformulations, see, e.g., Section 2.7.8 in [17]. Our first aim is to extend formula (4) to the whole unit disk $\mathbb{D}$.

Theorem 1. *Let $\mu\in\mathrm{Sz}(\mathbb{T})$ and let $\{f_n\}$ be the Schur family of μ. Then*

$$\mathcal{K}(\mu,z)=\log\prod_{n\geqslant 0}\frac{1-|zf_n(z)|^2}{1-|f_n(z)|^2},\qquad z\in\mathbb{D}. \tag{5}$$

Substituting $z=0$ into (5), we get (4). As an immediate consequence of (5), we see that $\sup_{n\geqslant 0}|f_n(z)|$ cannot be close to 1 if $\mathcal{K}(\mu,z)$ is small.

Given a measure $\mu\in\mathrm{Sz}(\mathbb{T})$ and its Schur family $\{f_n\}$, we let μ_n denote the probability measure on $\mathbb{T}$ whose Schur function f in (1) equals f_n. A standard problem in the field is to relate properties of μ_n to those of μ when n is large. The following inequality is another immediate consequence of Theorem 1.

Corollary 1. *We have $\mathcal{K}(\mu_n,z)\leqslant\mathcal{K}(\mu,z)$ for all $n\geqslant 0$ and all $z\in\mathbb{D}$.*

Indeed, due to Theorem 1 and Schur's algorithm, we have

$$\mathcal{K}(\mu_n,z)=\log\prod_{k\geqslant n}\frac{1-|zf_k(z)|^2}{1-|f_k(z)|^2}.$$

Since the terms in the product above are greater than 1, we have $\mathcal{K}(\mu_n,z)\leqslant\mathcal{K}(\mu,z)$.

Theorem 1 implies a uniform bound for oscillation of Schur family generated by a Szegő measure.

Theorem 2. *Suppose $\mu\in\mathrm{Sz}(\mathbb{T})$ and let $\{f_n\}$ be the family of Schur functions of μ. Then, we have*

$$\mathcal{P}(|f_n-f_n(z)|,z)\leqslant c\sqrt{\mathcal{K}(\mu,z)},\qquad z\in\mathbb{D},$$

with an absolute constant c and all $n\geqslant 0$.

Let us now turn to an application of these results to study asymptotic behavior of orthogonal polynomials. To every measure $\mu\in\mathrm{Sz}(\mathbb{T})$ we associate the Szegő function D_μ. This is the outer function in $\mathbb{D}$ with modulus $\sqrt{w}$ on $\mathbb{T}$:

$$D_\mu(z)=\exp\left(\int_{\mathbb{T}}\log\sqrt{w(\xi)}\cdot\frac{1+\bar\xi z}{1-\bar\xi z}\,dm(\xi)\right),\qquad z\in\mathbb{D}.$$

The family $\{\varphi_n\}_{n\geqslant 0}$ of orthonormal polynomials in $L^2(\mu)$ is defined by

$$\deg\varphi_n=n,\qquad k_n=\mathrm{coeff}_n\varphi_n>0,\qquad(\varphi_n,\varphi_k)_{L^2(\mu)}=\delta_{n,k}, \tag{6}$$

where $\delta_{n,k}$ is the Kronecker symbol and $\mathrm{coeff}_j Q$ denotes the coefficient at the power z^j in polynomial Q. Let also $\varphi_n^*(z)=z^n\overline{\varphi_n(1/\bar z)}$ denote the reversed orthogonal polynomial. Due to a version of Szegő theorem, we have $\mu \in \mathrm{Sz}(\mathbb{T})$ if and only if for some (and then for every) $z \in \mathbb{D}$ we have

$$\lim_{n\to+\infty} \varphi_n^*(z) = D_\mu^{-1}(z). \tag{7}$$

A well-known conjecture in the theory of orthogonal polynomials on the unit circle (an analog of Lusin's conjecture for trigonometric series [9, 12], see p. 135 in [3] for its positive solution) asks whether (7) holds for almost every $z \in \mathbb{T}$. As usual, for $z \in \mathbb{T}$ we understand $D_\mu^{-1}(z)$ as non-tangential boundary value. While not stated explicitly, the conjecture goes back to works of Bernstein, Szegő, and Steklov who studied asymptotics of orthogonal polynomials. Recently, it attracted more attention due to its connection to "nonlinear Carleson problem" in the scattering theory, see, e.g., [4], [5], [14], [15]. In the theorem below, we relate pointwise asymptotics of $\{\varphi_n(z)\}$, $z \in \mathbb{T}$, to the distribution of their zeroes near the unit circle. Our analysis is based on controlling oscillation of Schur functions $\{f_n\}$ in terms of the entropy function $\mathcal{K}$ in (3). The introduction of $\mathcal{K}$ was inspired by recent analysis of Szegő condition for canonical systems [1], [2].

Given a parameter $\rho \in (0,1)$ and a point $\xi \in \mathbb{T}$, define the Stolz angle $S_\rho^*(\xi)$ to be the convex hull of $\rho\mathbb{D}$ and ξ. Here is our main result.

Theorem 3. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $Z(\varphi_n) = \{z \in \mathbb{D} : \varphi_n(z) = 0\}$. Take any $a > 0$ and denote $r_{a,n} = 1 - a/n$. Then, for almost every $\xi \in \mathbb{T}$, the following assertions are equivalent:*

- *(a)* $\lim_{n\to\infty} |\varphi_n^*(\xi)|^2 = |D_\mu^{-1}(\xi)|^2$,
- *(b)* $\lim_{n\to\infty} \mathrm{dist}(Z(\varphi_n), \xi)\, n = +\infty$,
- *(c)* $\lim_{n\to\infty} f_n(r_{a,n}\xi) = 0$,
- *(d)* $\lim_{n\to\infty} \sup_{z\in S_\rho^*(\xi)} |f_n(z)| = 0$ *for every* $\rho \in (0,1)$.

The paper is organized as follows. In Section 2, we prove Theorem 1 and discuss its corollaries. Theorem 2 is proved in Section 3. In Section 4, we collect some facts about finite sums of Poisson kernels that will be used in Section 5 to prove Theorem 3.

2. Proof of Theorem 1 and some corollaries

We start by giving an expression for $\mathcal{K}(\mu, z)$ in terms of f, the Schur function of measure μ.

Lemma 1. *If $\mu \in \mathrm{Sz}(\mathbb{T})$ and f is its Schur function, then*

$$\mathcal{K}(\mu, z) = \int_{\mathbb{T}} \log\left(\frac{1-|zf(z)|^2}{1-|f(\xi)|^2}\right) \frac{1-|z|^2}{|1-\bar{\xi}z|^2} dm(\xi), \tag{8}$$

for every $z \in \mathbb{D}$.

Proof. Let w be the density of μ with respect to m. Taking the real part of both sides of (1), we obtain

$$\frac{1-|zf(z)|^2}{|1-zf(z)|^2} = \mathcal{P}(\mu, z), \qquad z \in \mathbb{D}.$$

Hence, $w = \frac{1-|f|^2}{|1-\xi f|^2}$ almost everywhere on $\mathbb{T}$. Then, the mean value formula for harmonic function $\log|1-zf|^2$ implies

$$\begin{aligned}\mathcal{K}(\mu, z) &= \log\frac{1-|zf(z)|^2}{|1-zf(z)|^2} - \int_{\mathbb{R}} \log\frac{1-|f(\xi)|^2}{|1-\xi f(\xi)|^2}\frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,dm(\xi)\\ &= \log(1-|zf(z)|^2) - \int_{\mathbb{T}} \log(1-|f(\xi)|^2)\,\frac{1-|z|^2}{|1-\bar{\xi}z|^2}dm(\xi)\\ &= \int_{\mathbb{T}} \log\left(\frac{1-|zf(z)|^2}{1-|f(\xi)|^2}\right)\,\frac{1-|z|^2}{|1-\bar{\xi}z|^2}dm(\xi).\end{aligned}$$

The lemma follows. □

Now, let $\mu \in \mathrm{Sz}(\mathbb{T})$ and sequence $\{f_n\}_{n\geqslant 0}$ be the family of Schur functions generated by μ via the Schur's algorithm (2). Denote by μ_n the probability measure on $\mathbb{T}$ whose Schur function coincides with f_n. Its existence follows if we notice that the function defined for $z \in \mathbb{D}$ by

$$\frac{1-|zf_n(z)|^2}{|1-zf_n(z)|^2} = \mathrm{Re}\left(\frac{1+zf_n(z)}{1-zf_n(z)}\right)$$

is a nonnegative harmonic function in $\mathbb{D}$ and therefore it is a Poisson integral of a unique nonnegative measure on $\mathbb{T}$. This is our μ_n. Taking $z = 0$ in the formula $\mathcal{P}(\mu_n, z) = \frac{1-|zf_n(z)|^2}{|1-zf_n(z)|^2}$, we get $\mu_n(\mathbb{T}) = 1$ so μ_n is a probability measure.

It is clear from construction that the Schur family of μ_n is $\{f_{n+k}\}_{k\geqslant 0}$. After making these observations, we proceed with the proof of Theorem 1.

Proof of Theorem 1. For a measure $\mu \in \mathrm{Sz}(\mathbb{T})$, consider the family of Schur functions $\{f_n\}_{n\geqslant 0}$ and associated probability measures $\{\mu_n\}_{n\geqslant 0}$. By Szegő theorem (see, e.g., p. 4 in [17]), we have $\sum_{k\geqslant 0}|f_k(0)|^2 < \infty$. It follows (again from the Szegő theorem) that $\mu_n \in \mathrm{Sz}(\mathbb{T})$ and

$$\int_{\mathbb{T}} \log(1-|f_n(\xi)|^2)\,dm = \log\prod_{k\geqslant n}(1-|f_k(0)|^2) \to 0, \qquad n \to +\infty. \tag{9}$$

In particular, functions f_n tend to zero in Lebesgue measure on $\mathbb{T}$ and, since they are uniformly bounded, we have $\lim_{n\to\infty} f_n(z) = 0$ for every $z \in \mathbb{D}$. From (9) and Lemma 1, we get

$$\mathcal{K}(\mu_n, z) = \int_{\mathbb{T}} \log\left(\frac{1-|zf_n(z)|^2}{1-|f_n(\xi)|^2}\right)\frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,dm(\xi) \to 0, \qquad n \to +\infty,$$

for every $z \in \mathbb{D}$. Thus, to prove Theorem 1, we only need to check that

$$\mathcal{K}(\mu, z) = \mathcal{K}(\mu_1, z) + \log\frac{1-|zf(z)|^2}{1-|f(z)|^2} \tag{10}$$

and then iterate this formula. From (8), we have

$$\begin{aligned}\mathcal{K}(\mu, z) &= \log(1-|zf(z)|^2) - \mathcal{P}(\log(1-|f(\xi)|^2), z),\\ \mathcal{K}(\mu_1, z) &= \log(1-|zf_1(z)|^2) - \mathcal{P}(\log(1-|f_1(\xi)|^2), z),\end{aligned}$$

for every $z \in \mathbb{D}$. Due to Schur's algorithm (2), one can write

$$zf_1(z) = \frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}, \qquad 1-|zf_1(z)|^2 = \frac{(1-|f(0)|^2)(1-|f(z)|^2)}{|1-\overline{f(0)}f(z)|^2}.$$

Using this computation, the mean value formula, and identity $|\xi| = 1, \xi \in \mathbb{T}$, we get

$$\begin{aligned}\mathcal{K}(\mu_1, z) &= \log \frac{(1-|f(0)|^2)(1-|f(z)|^2)}{|1-\overline{f(0)}f(z)|^2} - \mathcal{P}\left(\log \frac{(1-|f(0)|^2)(1-|f(\xi)|^2)}{|1-\overline{f(0)}f(\xi)|^2}, z\right)\\ &= \log(1-|f(z)|^2) - \mathcal{P}\left(\log(1-|f(\xi)|^2), z\right)\\ &= \log \frac{1-|f(z)|^2}{1-|zf(z)|^2} + \mathcal{K}(\mu, z),\end{aligned}$$

as required. □

Corollary 2. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and $\{f_n\}_{n\geqslant 0}$ be the Schur family of μ. Then,*

$$\mathcal{K}(1-|f_n(\xi)|^2, z) \leqslant \mathcal{K}(\mu, z)$$

for every $z \in \mathbb{D}$ and $n \geqslant 0$.

Proof. Since $|zf_n(z)|^2$ is subharmonic in $\mathbb{D}$, we get

$$\mathcal{P}(|\xi f_n(\xi)|^2, z) \geqslant |zf_n(z)|^2 .$$

Therefore,

$$\log(1-|zf_n(z)|^2) \geqslant \log \mathcal{P}(1-|f_n(\xi)|^2, z) .$$

So, applying Lemma 1 to measure μ_n, we have

$$\begin{aligned}\mathcal{K}(\mu_n, z) &= \log(1-|zf_n(z)|^2) - \int_{\mathbb{T}} \log(1-|f_n(\xi)|^2)\,\frac{1-|z|^2}{|1-\bar{\xi}z|^2} dm(\xi)\\ &\geqslant \log \mathcal{P}(1-|f_n(\xi)|^2, z) - \mathcal{P}(\log(1-|f_n(\xi)|^2), z)\\ &= \mathcal{K}(1-|f_n(\xi)|^2, z).\end{aligned}$$

It remains to use Corollary 1. □

Let $\alpha \in \mathbb{T}$, and let f be the Schur function of a measure $\mu \in \mathrm{Sz}(\mathbb{T})$. Then, the family of measures μ_α defined by

$$\mathcal{P}(\mu_\alpha, z) = \mathrm{Re}\left(\frac{1+\alpha z f(z)}{1-\alpha z f(z)}\right), \qquad z \in \mathbb{D},$$

is called the Aleksandrov-Clark family of μ. From (1), we see that αf is the Schur function of μ_α.

Corollary 3. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$ and let $\{f_n\}_{n\geqslant 0}$ be the Schur family of μ. Then, for every $z \in \mathbb{D}$, the entropy $\mathcal{K}(\mu, z)$ depends only on absolute value of $f(z)$. In particular, we have $\mathcal{K}(\mu, z) = \mathcal{K}(\mu_\alpha, z)$ for every $\alpha \in \mathbb{T}$.*

Proof. This follows from (8). □

The case $\alpha = -1$ in Corollary 3 corresponds to the "dual measure" μ_{dual}, playing an important role in the theory of orthogonal polynomials on the unit circle. The measure μ_{dual} is defined by

$$\int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\, d\mu_{\mathrm{dual}}(\xi) = \left(\int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\, d\mu(\xi)\right)^{-1}, \qquad z \in \mathbb{D}.$$

From (1), we infer that the Schur function of μ_{dual} equals $-f$. In particular, the last corollary yields

$$\mathcal{K}(\mu, z) = \mathcal{K}(\mu_{\mathrm{dual}}, z)\,, \qquad z \in \mathbb{D}. \tag{11}$$

It is well-known (see, e.g, Section 5 in [10]) that orthonormal polynomials φ_n defined in (6) satisfy recurrence relations

$$\sqrt{1-|a_n|^2}\cdot \varphi^*_{n+1} = \varphi^*_n - z a_n \varphi_n, \qquad \varphi_0 = \varphi^*_0 = 1, \qquad n \geqslant 0, \tag{12}$$

for coefficients $a_n = f_n(0)$ in $\mathbb{D}$, $\{f_n\}$ being the Schur family of μ. Conversely, each sequence $\{a_k\}_{k\geqslant 0} \subset \mathbb{D}$ gives rise to a unique probability measure μ on $\mathbb{T}$ with infinite number of points in $\operatorname{supp}\mu$ such that its orthonormal polynomials satisfy relations (12). In the next result we determine $\widehat{\mu}_{n,z}$, a variant of Bernstein-Szegő approximation to μ such that $\mathcal{K}(\mu, z) = \mathcal{K}(\widehat{\mu}_{n,z}, z) + \mathcal{K}(\mu_{n+1}, z)$.

Corollary 4. *Let $n \geqslant 0$ and $z^* \in \mathbb{D}$. Consider the measure $\widehat{\mu}_{n,z^*} = w_{n,z^*}\,dm$, where*

$$w_{n,z^*}(\xi) = \frac{1 - |f_n(z^*)|^2}{|\varphi_n^*(\xi) - \xi\overline{f_n(z^*)}\varphi_n(\xi)|^2}, \qquad \xi \in \mathbb{T}. \tag{13}$$

Then, $\widehat{\mu}_{n,z^*}$ is a probability measure whose Schur functions $\{\widehat{f}_k\}$ satisfy

$$\widehat{f}_k(z^*) = \begin{cases} f_k(z^*), & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases} \tag{14}$$

at the point z^*. Moreover, we have $\mathcal{K}(\mu, z^*) = \mathcal{K}(\widehat{\mu}_{n,z^*}, z^*) + \mathcal{K}(\mu_{n+1}, z^*)$.

Proof. Consider the family of orthonormal polynomials $\{\widehat{\varphi}_j\}$ whose recurrence coefficients are given by $\widehat{a}_k = f_k(0)$ for $0 \leqslant k \leqslant n-1$, $\widehat{a}_n = f_n(z^*)$, and $\widehat{a}_k = 0$ for $k > n$. It is well-known that the measure $\nu = |\widehat{\varphi}_{n+1}^*|^{-2}\,dm$ is a probability measure on $\mathbb{T}$ and its Schur functions $\{f_{\nu,k}\}_{k\geqslant 0}$ satisfy $f_{\nu,k}(0) = \widehat{a}_k$ for all $k \geqslant 0$. To see this, combine formulas (4.17) and (5.11) in [10]. It follows that for all $w \in \mathbb{D}$ we have $f_{\nu,n+1}(w) = 0$. Therefore, from the definition of Schur's algorithm (2), we have

$$0 = \frac{f_{\nu,n}(w) - f_{\nu,n}(0)}{1 - \overline{f_{\nu,n}(0)}f_{\nu,n}(w)}$$

for all $w \in \mathbb{D}$ and so

$$f_{\nu,n}(w) = f_{\nu,n}(0) = \widehat{a}_n = f_n(z^*).$$

Then, since $\widehat{a}_k = f_k(0)$ for all $0 \leqslant k \leqslant n-1$, we have

$$f_{\nu,k}(z^*) = f_k(z^*), \qquad 0 \leqslant k \leqslant n-1$$

by Schur's algorithm (2) since $\{f_{\nu,k}\}$ and $\{f_k\}$ satisfy the same recursion at point z^* when $k = 0, 1, \ldots, n-1$. We take $\widehat{\mu}_{n,z^*} = \nu$, and, to finish the proof, it remains to check that $|\widehat{\varphi}_{n+1}^*(\xi)|^{-2} = w_{n,z^*}(\xi)$ for $\xi \in \mathbb{T}$. To this end, observe that polynomials $\widehat{\varphi}_n^*$ and φ_n^* are identical since the recurrence coefficients defining them are the same. Then, from (12) we get

$$\sqrt{1 - |\widehat{a}_n|^2} \cdot \widehat{\varphi}_{n+1}^* = \widehat{\varphi}_n^* - \xi\overline{\widehat{a}_n}\widehat{\varphi}_n, \qquad \widehat{a}_n = f_n(z^*)\,.$$

Hence, (13) follows due to

$$|\widehat{\varphi}_{n+1}^*(\xi)|^{-2} = \frac{1 - |f_n(z^*)|^2}{|\widehat{\varphi}_n^* - \xi\overline{f_n(z^*)}\widehat{\varphi}_n|^2} = \frac{1 - |f_n(z^*)|^2}{|\varphi_n^*(\xi) - \xi\overline{f_n(z^*)}\varphi_n(\xi)|^2}\,,$$

where we used $\widehat{\varphi}_n^* = \varphi_n^*$ and $\widehat{\varphi}_n = \varphi_n$. □

According to a theorem by Khrushchev (Theorem 3 in [10]), the Schur function of the probability measure $|\varphi_n^*|^2\,d\mu$ is equal to $b_n f_n$, where $b_n = \varphi_n/\varphi_n^*$ is the Blaschke product of order n. In other words, we have (formula (2.14) in [10])

$$\int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z}|\varphi_n^*(\xi)|^2\,d\mu(\xi) = \frac{1 + zb_n(z)f_n(z)}{1 - zb_n(z)f_n(z)}, \qquad z \in \mathbb{D}, \tag{15}$$

and hence (formula (1.18) in [10])

$$|\varphi_n^*(\xi)|^2 w(\xi) = \frac{1 - |f_n(\xi)|^2}{|1 - \xi b_n(\xi)f_n(\xi)|^2}, \qquad \xi \in \mathbb{T}. \tag{16}$$

Identity $|b_n(\xi)| = 1, \xi \in \mathbb{T}$ implies the following corollary.

Corollary 5. *We have*

$$\mathcal{K}(|\varphi_n^*(\xi)|^2\,d\mu,z)=\mathcal{K}(\mu_n,z)+\log\left(\frac{1-|zb_n(z)f_n(z)|^2}{1-|zf_n(z)|^2}\right)$$

for every $n\geqslant 0$ and $z\in\mathbb{D}$.

Proof. Fix $n\geqslant 0$ and $z\in\mathbb{D}$. It follows from (8) that

$$\begin{aligned}\mathcal{K}(|\varphi_n^*|^2\,d\mu,z)&=\log(1-|zb_n(z)f_n(z)|^2)-\mathcal{P}(\log(1-|b_nf_n|^2),z)\\&=\log(1-|zb_n(z)f_n(z)|^2)-\mathcal{P}(\log(1-|f_n|^2),z)\\&=\mathcal{K}(\mu_n,z)+\log\left(\frac{1-|zb_n(z)f_n(z)|^2}{1-|zf_n(z)|^2}\right),\end{aligned}$$

as required. □

Let us now consider the case when μ is absolute continuous and its density does not oscillate too much. We say that $w\in A_\infty^P(\mathbb{T})$ if

$$[w]_{\infty,P}=\sup_{z\in\mathbb{D}}\mathcal{P}(w,z)\exp\Big(-\mathcal{P}(\log w,z)\Big)<\infty\,. \tag{17}$$

It is known that $A_\infty^P(\mathbb{T})\subsetneq A_\infty(\mathbb{T})$, where $A_\infty(\mathbb{T})$ is the usual Muckenhoupt class (see p. 212 in [18] for its definition).

Lemma 2. *We have $w\in A_\infty^P(\mathbb{T})$ if and only if $\sup_{z\in\mathbb{D}}\mathcal{K}(w\,dm,z)<\infty$. Moreover, the dual measure of wdm is absolutely continuous – i.e., $(w\,dm)_{\rm dual}=w_{\rm dual}dm$, and its density satisfies $w_{\rm dual}\in A_\infty^P(\mathbb{T})$.*

Proof. The first statement is immediate from the definition. To prove the second one, we use (11) and notice that $\mu\in{\rm Sz}(\mathbb{T})$ and $\mathcal{K}(\mu,z)\in L^\infty(\mathbb{D})$ imply that μ has no singular part and $\mu=wdm$ with $w\in A_\infty^P(\mathbb{T})$. Indeed, if $\mu=w\,dm+\mu_{\mathbf{s}}$ where $\mu_{\mathbf{s}}$ is the singular measure, then

$$\log\left(\mathcal{P}(\mu_{\mathbf{s}},z)+\mathcal{P}(w,z)\right)-\mathcal{P}(\log w,z)\leqslant C,\quad z\in\mathbb{D},$$

by our assumptions. This implies

$$\mathcal{P}(\mu_{\mathbf{s}},z)\leqslant\mathcal{P}(\mu_{\mathbf{s}},z)+\mathcal{P}(w,z)\leqslant C\exp\left(\mathcal{P}(\log w,z)\right)\leqslant C\mathcal{P}(w,z),$$

by Jensen inequality, hence, $\mu_{\mathbf{s}}=0$. □

Corollary 6. *Let the probability measure μ be defined by $\mu=wdm$ and w satisfy $w\in A_\infty^P(\mathbb{T})$. If $\{f_n\}$ denotes the Schur family of μ, then $1-|f_n|^2\in A_\infty^P(\mathbb{T})$ and $[1-|f_n|^2]_{\infty,P}\leqslant[w]_{\infty,P}$, $n\geqslant 0$.*

Proof. By Corollary 2, for each $n\geqslant 0$ and $z\in\mathbb{D}$, we have

$$\log\mathcal{P}(1-|f_n|^2,z)-\mathcal{P}(\log(1-|f_n|^2),z)\leqslant\mathcal{K}(\mu,z)\leqslant\log[w]_{\infty,P}.$$

It follows that $\log[1-|f_n|^2]_{\infty,P}\leqslant\log[w]_{\infty,P}$. □

3. The space BMO_η and proof of Theorem 2

Given a function $\eta:\mathbb{D}\to[0,+\infty]$, we define the space BMO_η to be the set of functions $v\in L^1(\mathbb{T})$ such that the following characteristic

$$\|v\|_\eta^*=\inf\{c\geqslant 0:\mathcal{P}(|v-\mathcal{P}(v,z)|,z)\leqslant c\eta(z),\ z\in\mathbb{D}\}$$

is finite. The next result is a direct analogue of an estimate by M. Korey (see Section 3.2 in [11]).

Lemma 3. *Suppose that $v, e^v \in L^1(\mathbb{T})$ and let $\mathcal{P}(e^v, z)/e^{\mathcal{P}(v,z)} = 1 + \gamma$ for some $\gamma \geqslant 0$ and $z \in \mathbb{D}$. Then,*

$$\mathcal{P}(|v - \mathcal{P}(v, z)|, z) \leqslant c \begin{cases} \sqrt{\gamma}, & \gamma < 1, \\ \log(1+\gamma), & \gamma \geqslant 1, \end{cases}$$

for an absolute constant c.

Proof. The proof is an adaptation of the original argument in [11]. For the reader's convenience, we reproduce it here. It suffices to prove the inequality

$$\mathcal{P}(|v - m_z(v)|, z) \leqslant c \begin{cases} \sqrt{\gamma}, & \gamma < 1, \\ \log(1+\gamma), & \gamma \geqslant 1, \end{cases}$$

where $m_z(v)$ is the median value of v on $\mathbb{T}$ with respect to the probability measure $\nu = (1-|z|^2)/|1-\bar{\xi}z|^2\, dm$. Adding a constant to v if needed, one can assume that $m_z(v) = 0$. Then, there are two disjoint measurable subsets $E \subseteq \{\xi : v(\xi) \geqslant 0\}$ and $F \subseteq \{\xi : v(\xi) \leqslant 0\}$ of $\mathbb{T}$ such that $\nu(E) = \nu(F) = 1/2$. Set

$$a = 2\mathcal{P}(\chi_E e^v, z), \quad b = 2\mathcal{P}(\chi_F e^v, z), \qquad a' = e^{2\mathcal{P}(\chi_E v, z)}, \quad b' = e^{2\mathcal{P}(\chi_F v, z)}.$$

By construction and by Jensen's inequality, one gets

$$1 + \gamma = \frac{\mathcal{P}(e^v, z)}{e^{\mathcal{P}(v,z)}} = \frac{a+b}{2\sqrt{a'b'}} \geqslant \frac{a'+b'}{2\sqrt{a'b'}}\,,$$

which implies $a'/b' \leqslant 1 + \widetilde{c}\max(\sqrt{\gamma}, \gamma^2)$ with an absolute constant $\widetilde{c}$. On the other hand, we have $a'/b' = e^{2\mathcal{P}(\chi_E v, z) - 2\mathcal{P}(\chi_F v, z)}$. It follows that

$$\mathcal{P}(|v|, z) = \mathcal{P}(\chi_E v, z) - \mathcal{P}(\chi_F v, z) \leqslant c\log(1 + \max(\sqrt{\gamma}, \gamma)),$$

for another absolute constant c, as claimed. $\square$

Given a measure $\mu \in \mathrm{Sz}(\mathbb{T})$, we introduce the function

$$\eta(z) = \max\left(\sqrt{\mathcal{K}(\mu, z)}, \mathcal{K}(\mu, z)e^{\mathcal{K}(\mu,z)/2}\right), \tag{18}$$

on the unit disk $\mathbb{D}$. The next lemma is crucial for later analysis.

Lemma 4. *Consider $\mu \in \mathrm{Sz}(\mathbb{T})$. Let $\{f_n\}$ be the Schur family of μ and $\{\varphi_n\}$ be orthogonal polynomials generated by μ. Then the functions $\log|\varphi_n^* - \xi f_n \varphi_n|^2$ and f_n belong to BMO_η for all $n \geqslant 0$ and*

$$\left\|\log|\varphi_n^* - \xi f_n\varphi_n|^2\right\|_\eta^* \leqslant c, \quad \|f_n\|_\eta^* \leqslant c, \qquad n \geqslant 0,$$

with an absolute constant c.

Proof. Consider the weight $v_n = 1 - |f_n|^2$ on the unit circle $\mathbb{T}$. By Corollary 2, we have $\mathcal{K}(v_n, z) \leqslant \mathcal{K}(\mu, z)$. Hence, applying Lemma 3 to $v = \log v_n$ one has

$$\mathcal{P}(|\log v_n - \mathcal{P}(\log v_n, z)|, z) \leqslant c\max(\sqrt{\mathcal{K}(\mu, z)}, \mathcal{K}(\mu, z)) \leqslant c\eta(z), \qquad z \in \mathbb{D}.$$

It follows that $\|\log v_n\|_\eta^* \leqslant c$ for all $n \geqslant 0$. In a similar way, we get $\log w \in \mathrm{BMO}_\eta$. We now use (16) to write

$$\log w = \log v_n - \log|\varphi_n^* - \xi f_n\varphi_n|^2,$$

hence $\log|\varphi_n^* - \xi f_n\varphi_n|^2 \in \mathrm{BMO}_\eta$ with the characteristic $\|.\|_\eta^*$ at most $2c$. Next, we use Jensen's inequality to write

$$\mathcal{P}(\log(1 - |f_n|^2), z) \leqslant \log\mathcal{P}(1 - |f_n|^2, z) = \log(1 - \mathcal{P}(|f_n|^2, z)).$$

Therefore, applying Lemma 1 to measure μ_n, one has

$$\mathcal{K}(\mu_n,z)=\log(1-|zf_n(z)|^2)-\mathcal{P}(\log(1-|f_n|^2),z)\geqslant\log\frac{1-|f_n(z)|^2}{1-\mathcal{P}(|f_n|^2,z)}.$$

Since $\mathcal{K}(\mu_n,z)\leqslant\mathcal{K}(\mu,z)$ by Corollary 1, we have

$$1-|f_n(z)|^2\leqslant e^{\mathcal{K}(\mu,z)}(1-\mathcal{P}(|f_n|^2,z)),$$

which can be rewritten as

$$e^{\mathcal{K}(\mu,z)}\mathcal{P}(|f_n|^2,z)-|f_n(z)|^2\leqslant e^{\mathcal{K}(\mu,z)}-1\,.$$

Since $\mathcal{K}\geqslant 0$, the following inequality holds

$$\mathcal{P}(|f_n|^2,z)-|f_n(z)|^2\leqslant e^{\mathcal{K}(\mu,z)}\mathcal{P}(|f_n|^2,z)-|f_n(z)|^2\leqslant e^{\mathcal{K}(\mu,z)}-1\,.$$

The last bound along with mean value formula for harmonic functions imply

$$\begin{aligned}\mathcal{P}(|f_n-\mathcal{P}(f_n,z)|^2,z)&=\mathcal{P}(|f_n-f_n(z)|^2,z),\\&=\mathcal{P}(|f_n|^2,z)+|f_n(z)|^2-2\mathcal{P}(\mathrm{Re}(f_n\overline{f_n(z)}),z),\\&=\mathcal{P}(|f_n|^2,z)-|f_n(z)|^2\leqslant e^{\mathcal{K}(\mu,z)}-1.\end{aligned}$$

By Cauchy-Schwarz inequality, we get

$$\mathcal{P}(|f_n-\mathcal{P}(f_n,z)|,z)\leqslant\sqrt{e^{\mathcal{K}(\mu,z)}-1}\leqslant c\eta(z)\,.$$

That finishes the proof. □

Proof of Theorem 2. By Lemma 4, for every $n\geqslant 0$ we have

$$\mathcal{P}(|f_n-f_n(z)|,z)\leqslant c\eta(z),\qquad z\in\mathbb{D}.$$

On the other hand, $\mathcal{P}(|f_n-f_n(z)|,z)\leqslant 2$ since $|f_n|\leqslant 1$ on $\mathbb{D}\cup\mathbb{T}$. This yields the statement of the theorem. □

Next, we will estimate the harmonic conjugates of functions in BMO_η. Some notation is needed first. We denote $|E|=m(E)$ for Borel subsets of $\mathbb{T}$. If $I\subset\mathbb{T}$ is an arc with center at ξ, set $z_I=\xi(1-|I|)$ and denote by $2I$ the arc with center at ξ such that $|2I|=2|I|$. We also let $\langle f\rangle_{I,P}=\mathcal{P}(f,z_I)$. For $u\in L^1(\mathbb{T})$, we define the harmonic conjugate function v by the formula

$$v(\xi)=(Qu)(\xi)=\lim_{r\to 1}\int_{\mathbb{T}}u(\zeta)Q_r(\zeta,\xi)\,dm(\zeta),\quad Q_r(\zeta,\xi)=\mathrm{Im}\,\frac{1+r\bar\zeta\xi}{1-r\bar\zeta\xi},\quad \xi\in\mathbb{T}\,.$$

From the standard estimates for singular integrals, one knows that the limit exists almost everywhere on $\mathbb{T}$ and defines the function $Qu\in L^{1,\infty}(\mathbb{T})$. Notice that the harmonic conjugate of a constant function is identically zero. Finally, given real-valued $u\in L^1(\mathbb{T})$, the function $u+i(Qu)$ is the nontangential boundary value of the function

$$\mathcal{F}(z)=\int_{\mathbb{T}}u(\zeta)\frac{1+\bar\zeta z}{1-\bar\zeta z}\,dm(\zeta)$$

analytic in $\mathbb{D}$. Function $\mathrm{Re}\,\mathcal{F}$ is Poisson extension of u and harmonic conjugate of u is the boundary value of $\mathrm{Im}\,\mathcal{F}$. Next, we recall that, given a parameter $\rho\in(0,1)$, the symbol $S^*_\rho(\xi)$ denotes the convex hull of $\rho\mathbb{D}$ and a point $\xi\in\mathbb{T}$.

Below we write $A\lesssim B$ for quantities A, B if there is an absolute constant c such that $A\leqslant cB$. Notation $A\sim B$ is used when $A\lesssim B$ and $B\lesssim A$.

Lemma 5. *Let $u\in\mathrm{BMO}_\eta$ and let v be the harmonic conjugate of u. Let I be an arc with center at $\xi_0\in\mathbb{T}$. Then, there is a constant c_I such that*

$$\frac{|\{\xi\in I:|v(\xi)-c_I|>t\}|}{|I|}\lesssim t^{-1}\|u\|^*_\eta\sum\nolimits_{j\geqslant 0}2^{-j}\eta(z_j),$$

for some $z_j\in S^*_{0.9}(\xi_0)$ such that $|z_j-\xi_0|\sim 2^j|I|$, $j\geqslant 0$.

Proof. Write $u = u_1 + u_2 + \langle u\rangle_{2I,P}$ for $u_1 = \chi_{2I}(u - \langle u\rangle_{2I,P})$, $u_2 = \chi_{\mathbb{T}\setminus 2I}(u - \langle u\rangle_{2I,P})$, and denote by v_1, v_2 the harmonic conjugates of u_1, u_2, respectively. Since Q is the continuous operator from $L^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$, we have

$$\begin{aligned} m(\{\xi \in I : |v_1(\xi)| > t\}) &\lesssim t^{-1}\|u_1\|_{L^1(\mathbb{T})}, \\ &\lesssim t^{-1}\langle |u - \langle u\rangle_{2I,P}|\rangle_{2I,P}|I|, \\ &\lesssim t^{-1}\|u\|_\eta^* \cdot \eta(z_0)\cdot |I|, \end{aligned}$$

for $z_0 = z_{2I} = (1-2|I|)\xi_0$. Next, we estimate the distribution function of v_2. Put $c_I = \int_{\mathbb{T}\setminus 2I} u_2(\zeta)Q(\zeta,\xi_0)\,dm(\zeta)$ and write for $\xi \in I$:

$$v_2(\xi) - c_I = (Qu_2)(\xi) - c_I = \int_{\mathbb{T}\setminus 2I} u_2(\zeta)\Big(Q(\zeta,\xi) - Q(\zeta,\xi_0)\Big)\,dm(\zeta).$$

Let us estimate the norm of $v_2 - c_I$ in $L^1(I)$ to later use Chebyshev inequality. For $k \geqslant 1$, denote by I_k the arcs of $\mathbb{T}$ of size $2^k|I|$ with center at ξ_0. Notice that we take only those k for which $|I_k| \leqslant 2\pi$. Then, for $\xi \in I$, $\zeta \in I_{k+1}\setminus I_k$, we have

$$|Q(\zeta,\xi) - Q(\zeta,\xi_0)| \lesssim \frac{|\xi-\xi_0|}{|(\zeta-\xi)(\zeta-\xi_0)|} \lesssim \frac{|I|}{|I_k|^2} = \frac{1}{2^{2k}|I|}.$$

Using this relation, we get

$$\begin{aligned} \int_I |v_2 - c_I|\,dm &\lesssim \sum_{k\geqslant 1}\frac{1}{2^{2k}|I|}\int_I\int_{I_{k+1}\setminus I_k}|u_2(\zeta)|\,dm(\zeta)\,dm(\xi) \\ &\lesssim \sum_{k\geqslant 1}\frac{1}{2^{2k}}\int_{I_{k+1}\setminus I_k}|u_2|\,dm. \end{aligned}$$

Set $J_0 = 2I$ and let J_k, $k \geqslant 1$, be one of two arcs of $I_{k+1}\setminus I_k$ such that

$$\int_{I_{k+1}\setminus I_k}|u_2|\,dm \leqslant 2\int_{J_k}|u_2|\,dm.$$

We have

$$\begin{aligned} \int_{J_k}|u_2|\,dm &\leqslant |J_k|\cdot\Big(\langle|u - \langle u\rangle_{J_k,P}|\rangle_{J_k,P} + |\langle u\rangle_{J_k,P} - \langle u\rangle_{2I,P}|\Big) \\ &\lesssim 2^k|I|\cdot\Big(\|u\|_\eta^*\eta(z_k) + \sum_{j=1}^k |\langle u\rangle_{J_j,P} - \langle u\rangle_{J_{j-1},P}|\Big), \end{aligned}$$

where $z_k = (1-|J_k|)\xi_k$ and ξ_k denotes the center of J_k. Since $|\zeta - z_j| \sim |\zeta - z_{j+1}|$ for $\zeta \in \mathbb{T}$, we can write

$$|\langle u\rangle_{J_j,P} - \langle u\rangle_{J_{j-1},P}| = |\mathcal{P}(u - \langle u\rangle_{J_{j-1},P}, z_j)| \lesssim \mathcal{P}(|u - \langle u\rangle_{J_{j-1},P}|, z_{j-1}) \lesssim \|u\|_\eta^*\eta(z_{j-1})\,.$$

Hence,

$$\int_{J_k}|u_2|\,dm \lesssim 2^k\cdot|I|\cdot\|u\|_\eta^*\sum_{j=0}^k\eta(z_j).$$

It follows that

$$\frac{1}{|I|}\int_I|v_2 - c_I|\,dm \lesssim \|u\|_\eta^*\cdot\sum_{k\geqslant 1}2^{-k}\sum_{j=0}^k\eta(z_j) \lesssim \|u\|_\eta^*\cdot\sum_{j\geqslant 0}2^{-j}\eta(z_j).$$

Now we collect estimates to get the bound

$$\begin{aligned}|\{\xi \in I : |v(\xi) - c_I| > 2t\}| &\leqslant |\{\xi \in I : |v_1(\xi)| > t\}| + |\{\xi \in I : |v_2(\xi) - c_I| > t\}| \\ &\lesssim t^{-1}\|u\|_\eta^* \cdot \eta(z_0) \cdot |I| + t^{-1}\int_I |v_2 - c_I|\, dm \\ &\lesssim t^{-1}\|u\|_\eta^* \cdot |I| \sum\nolimits_{j\geqslant 0} 2^{-j}\eta(z_j).\end{aligned}$$

The simple geometric considerations (see Figure 1 below) yield $z_j \in S^*_{0.9}(\xi_0)$ and the lemma is proved.

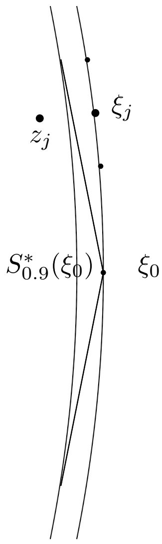

Figure 1

□

4. Sums of Poisson kernels

In this section, we study the properties of finite sums of Poisson kernels. They will be used in the proof of Theorem 3.

We denote by $C[a,b]$ the space of functions continuous on $[a,b]$. The following elementary result is well-known (see problem 13(b), p. 167 in [16]).

Lemma 6. *Suppose the sequence $\{g_n\}$ of non-decreasing functions converges to a function $g \in C[a,b]$ on a dense subset of $[a,b]$. Then, $\{g_n\}$ converges to g uniformly on $[a,b]$.*

We start with the calculation, which reveals the connection between the zeroes of the polynomial φ_n and the sum of Poisson kernels. Consider $b_n = \varphi_n/\varphi_n^*$. We can write it as

$$b_n(z) = \alpha_n z^{l_n} \prod_{j=1}^{m_n} \frac{z - z_{j,n}}{z - \bar{z}_{j,n}^{-1}}, \quad \alpha_n > 0, \quad l_n + m_n = n\,,$$

where $\{z_{j,n}\}$ are zeroes of φ_n different from 0. That is the product of Möbius transforms each of which has an argument which is increasing monotonically on $\mathbb{T}$

since this transform is a conformal map of $\mathbb{D}$ onto $\mathbb{D}$. Calculating the derivative of its argument

$$\partial_\theta \arg b_n(e^{i\theta}) = l_n + \operatorname{Im} \partial_\theta \left(\sum_{j=1}^{m_n} \log \left(\frac{e^{i\theta} - z_{j,n}}{e^{i\theta} - \bar{z}_{j,n}^{-1}} \right) \right) = l_n + \sum_{j=1}^{m_n} \frac{1 - |z_{j,n}|^2}{|e^{i\theta} - z_{j,n}|^2} \tag{19}$$

one can recognize the Poisson kernel as terms in the last sum.

Lemma 7. *Assume that h_n are smooth functions on $(-\pi n, \pi n)$ with derivatives h_n' given by*

$$h_n'(t) = \frac{1}{n} \sum_{k=1}^{n} \frac{1 - |z_{k,n}|^2}{|e^{it/n} - z_{k,n}|^2}, \qquad z_{k,n} \in \mathbb{D}. \tag{20}$$

If $\{h_n\}$ converges to a smooth function h uniformly on compact subsets of $\mathbb{R}$, then $\{h_n'\}$ converges to h' uniformly on compact subsets of $\mathbb{R}$.

Proof. We will assume that the points $z_{k,n}$ are enumerated so that

$$|1 - z_{k,n}| \leqslant |1 - z_{k+1,n}|, \qquad 1 \leqslant k \leqslant n.$$

Take an arbitrary $b > 0$. It suffices to show that $\{h_n'\}$ converges to h' uniformly over $[-b/2, b/2]$. We write h_n' as $h_n' = G_n + H_n$, where G_n is the sum which corresponds to all terms (if any) for which $n|1 - z_{k,n}| > 1.9b$ and, respectively, terms in H_n satisfy $n|1 - z_{k,n}| \leqslant 1.9b$. For G_n, we have

$$|G_n'(t)| \lesssim b^{-1} G_n(t) \tag{21}$$

when $t \in [-b, b]$. Indeed, if $t \in [-b, b]$ and $n|1 - z| \geqslant 1.9b$, we have

$$\partial_t \left(\frac{1}{|e^{it/n} - z|^2} \right) = \frac{2}{n} \frac{\operatorname{Re}(i\bar{z}e^{it/n})}{|e^{it/n} - z|^4} \leqslant \frac{1}{n|e^{it/n} - z|^3} \lesssim \frac{1}{b|e^{it/n} - z|^2}, \tag{22}$$

which yields the required estimate. It follows that

$$\limsup_{n\to\infty} \int_{-b}^{b} |G_n'| \, dt \lesssim \limsup_{n\to\infty} \int_{-2b}^{2b} b^{-1} G_n \, dt \leqslant b^{-1}(h(2b) - h(-2b)).$$

Thus, functions $\{G_n\}$ are uniformly bounded. The estimate (21) then implies that the set $\{G_n\}$ is also equicontinuous on $[-b, b]$. Choose a subsequence $\{G_{n_j}\}$ which converges to some continuous function G uniformly over $[-b, b]$. Then, $\{\int_{-b}^{x} G_{n_j} \, dt\}$ converges to $\int_{-b}^{x} G \, dt$ uniformly over $[-b, b]$ as well. Since we know by conditions of the lemma that $\{\int_{-b}^{x} h_n' \, dt\}$ converges uniformly to a smooth function, the sequence $\{\int_{-b}^{x} H_{n_j} \, dt\}$ also converges uniformly to a function continuous on $[-b, b]$.

Now let z_{k,n_j}, $k = 1, \ldots, c(n_j)$ be all points that satisfy $n_j|1 - z_{k,n_j}| \leqslant 1.9b$. For every $z \in \mathbb{D}$ such that $n_j|1 - z| \leqslant 1.9b$, we have

$$\frac{1}{n_j} \int_{-2b}^{2b} \frac{1 - |z|^2}{|e^{it/n_j} - z|^2} dt = \int_{-2b/n_j}^{2b/n_j} \frac{1 - |z|^2}{|e^{i\tau} - z|^2} d\tau \geqslant c_b > 0,$$

where the constant c_b depends only on b. It follows that

$$\limsup_{j} c(n_j) \leqslant c_b^{-1} \limsup_{j} \int_{-2b}^{2b} H_{n_j} \, dt \leqslant (h(2b) - h(-2b))/c_b.$$

Hence, $\limsup_j c(n_j) = N_b$ for some $N_b \geqslant 0$ (we set $N_b = 0$ if there are no zeroes z_{k,n_j} such that $n_j|1 - z_{k,n_j}| \leqslant 1.9b$ for all j large enough). Choosing a subsequence, one can assume that $c(n_j) = N_b$ for all j. If $N_b > 0$, we set $\xi_{k,n_j} = in_j(1 - z_{k,n_j})$ for every $k = 1, \ldots, N_b$. Note that ξ_{k,n_j} belong to $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and, moreover, $|\xi_{k,n_j}| < 1.9b$. Choosing again a subsequence, we may assume that $\{\xi_{k,n_j}\}$ converges to $\xi_k \in \overline{\mathbb{C}^+}$, $k = 1, \ldots, N_b$. We claim that none of these limiting points belongs to the segment $[-b/2,b/2]$ on the real line. Indeed, if $\xi_k\in[-b/2,b/2]$, then the sequence of functions

$$\frac{1}{n_j}\frac{1-|z_{k,n_j}|^2}{|e^{it/n_j}-z_{k,n_j}|^2},\qquad j=1,2,\ldots \tag{23}$$

converges to $2\pi\delta_{\xi_k}$ in the weak-$*$ sense because substituting the Taylor expansion of e^{it/n_j} gives the Poisson kernel as the main term. That contradicts the fact that $\int_{-b}^x H_{n_j}dt$ converges uniformly to a continuous function on $[-b,b]$. Knowing that all limiting points ξ_k are separated from the real line, it is easy to see that $\{H_{n_j}\}$ converges uniformly over $[-b/2,b/2]$. Thus, we can guarantee that some subsequence $\{h'_{n_j}\}$ converges uniformly on $[-b/2,b/2]$. Denote its limit by F. Since

$$\lim_{n\to\infty}\int_{-b/2}^x h'_n(t)\,dt=\lim_{n\to\infty}\Big(h_n(x)-h_n(-b/2)\Big)=h(x)-h(-b/2)$$

by assumption of the lemma, we get $F=h'$. In the standard way, we now get $\lim_{n\to+\infty}h'_n=h'$ uniformly on compacts in $\mathbb{R}$ over the whole original sequence. Indeed, if this is not true, then there is $\varepsilon>0$, $b>0$, $t_n\in[-b/2,b/2]$ and a sequence $\{m_n\}$ such that $|h'_{m_n}(t_n)-h'(t_n)|>\varepsilon$. However, by the argument above we can take a subsequence of $\{m_n\}$, call it $\{m'_n\}$, so that $\{h'_{m'_n}\}$ converges to h' uniformly on $[-b/2,b/2]$. That gives a contradiction. The lemma is proved. $\square$

Lemma 8. *If, under the conditions of Lemma 7, we also assume that $h(t)=t+c$ for all $t\in\mathbb{R}$ and some constant c, then*

$$\lim_{n\to\infty}\left(n\min_{1\leqslant k\leqslant n}|1-z_{k,n}|\right)=\infty.$$

Proof. Given $b\in\mathbb{R}^+$, let $\{z_{j,n}\}$, $j=1,\ldots,c(n,b)$, be all zeroes of φ_n, counting multiplicity, that satisfy $n|1-z_{j,n}|<1.9b$, and set $c(n,b)=0$ if there are no such zeroes. From the previous proof, we know that $\limsup_n c(n,b)<\infty$. We need to show that $\limsup_{n\to\infty}c(n,b)=0$ for every b. Suppose this is not the case and there is some $\widehat{b}$ such that $c_1\stackrel{\text{def}}{=}\limsup_{n\to\infty}c(n,\widehat{b})\geqslant 1$. Then, there is a subsequence $\{n_k^{(1)}\}$ such that each $\varphi_{n_k^{(1)}}(z)$ has exactly c_1 zeroes, counting multiplicity, at points $\{z_{j,n_k^{(1)}}\}, j=1,\ldots,c_1$, and all of these zeroes are inside the open disc of radius $1.9\widehat{b}/n$ centered at 1. Using compactness argument, we can find $\{\widehat{n}_k^{(1)}\}$, a subsequence of $\{n_k^{(1)}\}$, such that

$$\lim_{k\to\infty}\xi_{j,\widehat{n}_k^{(1)}}=\xi_j,\quad \xi_{j,n}\stackrel{\text{def}}{=}in(1-z_{j,n}),\qquad j=1,\ldots,c_1$$

and c_1 points $\{\xi_j\}$, counting multiplicity, all belong to the set $\{\xi\in\mathbb{C}^+:|\xi|\leqslant 1.9\widehat{b}\}$. Notice that none of these points can be on the real line, that follows from the proof of the previous lemma.

Next, we look at zeroes of a polynomial $\varphi_{\widehat{n}_k^{(1)}}(z)$ that belong to the annulus

$$1.9\widehat{b}\leqslant\widehat{n}_k^{(1)}|1-z|<1.9(\widehat{b}+1).$$

Applying the same argument, we can find $\{\widehat{n}_k^{(2)}\}$, a subsequence of $\{\widehat{n}_k^{(1)}\}$, for which each $\varphi_{\widehat{n}_k^{(2)}}(z)$ has exactly c_2 zeroes $\{z_{j,n_k^{(2)}}\}, j\in\{c_1,\ldots,c_1+c_2\}$ in that annulus and they all satisfy

$$\lim_{k\to\infty}\xi_{j,\widehat{n}_k^{(2)}}=\xi_j,\qquad j=c_1,\ldots,c_1+c_2\,.$$

Notice that it might be that $c_2=0$ but we always have $c_2<\infty$.

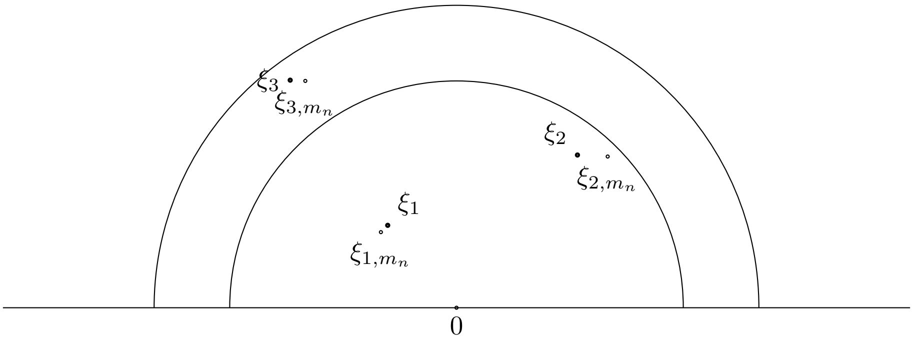

Figure 2

At the next step, we consider zeroes of $\varphi_{\widehat{n}_k^{(2)}}$ that satisfy

$$1.9(\widehat{b}+1) \leqslant \widehat{n}_k^{(2)}|1-z| < 1.9(\widehat{b}+2)$$

and select a subsequence $\{\widehat{n}_k^{(3)}\}$ out of $\{\widehat{n}_k^{(2)}\}$ over which these zeroes have limiting values. We continue this process and find a subsequence $\{m_n\}$ of the original sequence such that all zeroes $\{z_{j,m_n}\}$ satisfy conditions:

$$\lim_{n\to\infty} \xi_{j,m_n} = \xi_j, \qquad \xi_j \in \mathbb{C}^+,$$

for every $j = 1, 2, \ldots, N$, where $N = \sum_{l\geqslant 1} c_l \in [1, \infty]$. Moreover, if $N = \infty$, then $\lim_{j\to\infty} |\xi_j| = \infty$ by our construction. Figure 2 illustrates the case when $c_1 = 2$ and $c_2 = 1$.

For $z \in \mathbb{D}$ and $\xi = in(1-z)$, we have

$$\frac{1}{n}\frac{1-|z|^2}{|e^{it/n}-z|^2} = \frac{n(1-|z|^2)}{|in(e^{it/n}-1)+in(1-z)|^2} = \frac{2\,\mathrm{Im}\,\xi - |\xi|^2/n}{|in(e^{it/n}-1)+\xi|^2}.$$

That gives

$$\lim_{n\to\infty} \frac{1}{m_n}\frac{1-|z_{j,m_n}|^2}{|e^{it/m_n}-z_{j,m_n}|^2} = \frac{2\,\mathrm{Im}\,\xi_j}{|t-\xi_j|^2}, \tag{24}$$

and the convergence is uniform on compact subsets of $\mathbb{R}$. Since all terms in (20) are non-negative, we can define U as

$$U(t) = \sum_{j=1}^{N} \frac{2\,\mathrm{Im}\,\xi_j}{|t-\xi_j|^2}\,.$$

Next, we will study the properties of $U(t)$ and, in particular, show that it is finite. To this end, we fix arbitrary $l \in \mathbb{N}$ and consider the partial sum

$$\sum_{j=1}^{l} \frac{2\,\mathrm{Im}\,\xi_j}{|t-\xi_j|^2}\,.$$

From Lemma 7 and our additional assumption $h(t) = t + c$, we know that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1-|z_{k,n}|^2}{|e^{it/n}-z_{k,n}|^2} \to 1, \quad n \to \infty$$

and this convergence is uniform in t over compacts in $\mathbb{R}$. That implies

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{l} \frac{1-|z_{k,n}|^2}{|e^{it/n}-z_{k,n}|^2} \leqslant 1\,.$$

Now, we use (24) in the previous bound to get

$$\sum_{j=1}^{l}\frac{2\operatorname{Im}\xi_j}{|t-\xi_j|^2}\leqslant 1$$

for every $l\in\mathbb{N}$. Thus, $U(t)$ is finite for every $t\in\mathbb{R}$. Moreover, $U(t)\leqslant 1$, $t\in\mathbb{R}$. Substituting $t=0$, we see that $\{\xi_j\}$ satisfies Blaschke condition in $\mathbb{C}^+$. Consider the Blaschke product with zeroes at $\{\xi_j\}$, i.e.,

$$B(\xi)=\prod_{j=1}^{N}\left(e^{i\alpha_j}\frac{\xi-\xi_j}{\xi-\bar{\xi}_j}\right),\qquad \xi\in\mathbb{C}^+,\ N\geqslant 1\,, \tag{25}$$

where α_j are chosen such that

$$e^{i\alpha_j}\frac{i-\xi_j}{i-\bar{\xi}_j}>0 \quad \text{if} \quad \xi_j\neq i \quad \text{and} \quad \alpha_j=0 \quad \text{otherwise.}$$

We will show that $B(\xi)=e^{i(\beta_1\xi+\beta_2)}$ with some $\beta_1,\beta_2\in\mathbb{R}$ thus getting the contradiction with (25). To this end, write

$$h'_{m_n}(t)=\frac{1}{m_n}\sum_{k=1}^{m_n}\frac{1-|z_{k,m_n}|^2}{|e^{it/m_n}-z_{k,m_n}|^2}=\Psi_{1,m_n,L}(t)+\Psi_{2,m_n,L}(t),$$

where we define

$$\Psi_{1,m_n,L}(t)=\frac{1}{m_n}\sum_{k\in\Omega}\frac{1-|z_{k,m_n}|^2}{|e^{it/m_n}-z_{k,m_n}|^2}\,,\quad \Omega=\{k:\ |\xi_k|<L,\,|\xi_{k,m_n}-\xi_k|<0.1\},$$

and $\Psi_{2,m_n,L}=h'_{m_n}-\Psi_{1,m_n,L}$. We know from the previous lemma that $\lim_{n\to\infty}h'_n=1$ uniformly over compacts in $\mathbb{R}$. When L is fixed and $n\to\infty$, we have

$$U_L(t)\stackrel{\text{def}}{=}\sum_{j:|\xi_j|<L}\frac{2\operatorname{Im}\xi_j}{|t-\xi_j|^2}=\lim_{n\to\infty}\Psi_{1,m_n,L}(t)\leqslant h'(t)=1,\qquad t\in\mathbb{R}.$$

Moreover, from (22) we get $|\Psi'_{2,m_n,L}(t)|\lesssim L^{-1}\Psi_{2,m_n,L}(t)$ uniformly with respect to $t\in[-L/2,L/2]$. Since $\Psi_{1,m_n,L}$ and $\Psi_{2,m_n,L}$ are both nonnegative, $\Psi_{1,m_n,L}+\Psi_{2,m_n,L}=h'_{m_n}$, and $h'_{m_n}\to 1$ uniformly over compacts, we have $|\Psi'_{2,m_n,L}(t)|\lesssim L^{-1}$ for $t\in[-L/2,L/2]$ if n is large enough. Clearly, $\{\Psi_{2,m_n,L}\}$ converges uniformly on $[-L/2,L/2]$ as a difference of two uniformly convergent sequences. Therefore, if $\Psi_{2,L}$ denotes its limit, then

$$\|\Psi_{2,L}\|_{L^\infty[-L/2,L/2]}\leqslant 1,\quad \|\Psi_{2,L}\|_{\mathrm{Lip}[-L/2,L/2]}\lesssim L^{-1}\,,$$

where

$$\|f\|_{\mathrm{Lip}[a,b]}\stackrel{\text{def}}{=}\sup_{x,y\in[a,b],x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\,.$$

Notice that $U_L+\Psi_{2,L}=1$. Therefore, $\|U_L\|_{\mathrm{Lip}[-L/2,L/2]}\lesssim L^{-1}$, that is

$$|U_L(\xi_2)-U_L(\xi_1)|\lesssim|\xi_1-\xi_2|/L$$

for $\xi_1,\xi_2\in[-L/2,L/2]$. Taking the limit as $L\to\infty$ and recalling that $U=\lim_{L\to\infty}U_L$, we see that U is constant on $\mathbb{R}$. A direct calculation shows that $(\arg B)'=cU$ on $\mathbb{R}$ for some positive constant c. Thus, we have $\arg B(t)=\beta_1 t+\beta_2$, $t\in\mathbb{R}$, $\beta_1\geqslant 0$. The function $Be^{-i(\beta_1 z+\beta_2)}$ is unimodular on $\mathbb{R}$ and has zero argument there, so it is equal to 1 on $\mathbb{R}$ and, by uniqueness of holomorphic functions, $B(z)=e^{i(\beta_1 z+\beta_2)}$, $z\in\mathbb{C}^+$, yielding a contradiction. $\square$

Lemma 9. *Assume that smooth functions h_n defined on $(-\pi n, \pi n)$ have derivatives given by* (20) *and the sequence $\{h_n\}$ converges almost everywhere to some nondecreasing function h defined on $\mathbb{R}$. If*

$$\lim_{n\to\infty}\left(n \min_{1\leqslant j\leqslant n}|1-z_{j,n}|\right)=\infty\,, \tag{26}$$

then $h = c_1 t + c_2$ and $\{h_n\}$ converges uniformly over compacts in $\mathbb{R}$.

Proof. For arbitrary $b > 0$, we have

$$\int_{-b}^{b} h_n'(t)dt = h_n(b) - h_n(-b)$$

and, since $\lim_{n\to\infty} h_n(t) = h(t)$ a.e. and $h_n' \geqslant 0$, one gets

$$\sup_n \int_{-b}^{b} h_n'(t)dt < \infty\,. \tag{27}$$

Moreover, condition (26) and an estimate (22) give

$$|h_n''(t)| \leqslant \varepsilon_n |h_n'(t)|, \quad \lim_{n\to\infty}\varepsilon_n = 0\,,$$

for $t \in [-b, b]$. Thus, (27) implies

$$\lim_{n\to\infty}\int_{-b}^{b}|h_n''(t)|dt = 0\,.$$

From the relation

$$h_n'(t)(t_2-t_1) = h_n(t_2) - h_n(t_1) + \int_{t_1}^{t_2}\int_{\tau}^{t} h_n''(\tau_1)d\tau_1 d\tau\,,$$

we obtain

$$\lim_{n\to\infty} h_n'(t) = \frac{h(t_2)-h(t_1)}{t_2-t_1}\,.$$

In particular, the right hand side does not depend on t_1 and t_2. That implies that h is a linear function, i.e., $h = c_1 t + c_2$. Lemma 6 gives uniform convergence. □

5. Proof of Theorem 3

The proof of Theorem 3 is based on careful study of the arguments of orthogonal polynomials φ_n and Schur functions f_n. We proceed as follows. Theorem 1, Lemma 4, and Lemma 5 show that these arguments, after rescaling and taking a limit, satisfy equation (41) below. Since the derivative of the argument of a polynomial with zeroes in $\mathbb{D}$ is a finite sum of Poisson kernels (see formula (19)), equation (41) allows us to recover local asymptotics of all objects in Theorem 3 and prove that assertions (a)–(d) are equivalent to an identity $d = 0$ in (41).

In this section, we always assume $\mu \in \mathrm{Sz}(\mathbb{T})$. We start with several auxiliary results. Recall that $\{f_n\}$ denotes the family of Schur functions for a measure μ and that, given an arc $I \subset \mathbb{T}$ with the center at $\xi_I \in \mathbb{T}$, we let $z_I = (1-|I|)\xi_I$.

Lemma 10. *Let $I \subset \mathbb{T}$ be an arc and $|I| \leqslant 1/4$. Then,*

$$|\{\xi \in I : |f_n(\xi) - f_n(z_I)| > t\}|/|I| \lesssim \eta(z_I)/t,$$

where the function η is defined in (18).

Proof. By Lemma 4, we have $\|f_n\|_\eta^* \leqslant C$ for some constant C. Since

$$\frac{1-|z_I|^2}{|1-\bar{\xi}z_I|^2} \gtrsim \frac{1-(1-|I|)^2}{|I|^2} \gtrsim \frac{1}{|I|}, \qquad \xi \in I,$$

one has

$$\frac{1}{|I|}\int_I |f_n(\xi)-f_n(z_I)|\,dm(\xi) \lesssim \mathcal{P}(|f_n-f_n(z_I)|, z_I) \leqslant C\eta(z_I).$$

It remains to use Chebyshev inequality. □

Given $\xi \in \mathbb{T}$, $\rho \in (0,1)$, $\delta \in (0,1)$ and α, β: $0<\alpha<\beta$, set

$$\Upsilon_{\delta,\rho,\alpha,\beta}(\xi) = \{z \in S_\rho^*(\xi),\ \alpha\delta < |z-\xi| < \beta\delta\}, \tag{28}$$

where, as before, $S_\rho^*(\xi)$ is the convex hull of $\rho\mathbb{D}$ and point ξ. For a complex-valued function h defined on a domain $\Omega \subset \mathbb{C}$, we introduce its oscillation as

$$\operatorname{osc}_\Omega(h) = \sup_{z_1,z_2\in\Omega} |h(z_2)-h(z_1)|.$$

In the next lemma, we show that Schur family $\{f_k\}$ has small oscillation near the boundary of $\mathbb{D}$ uniformly in $k \geqslant 0$.

Lemma 11. *Suppose $\xi \in \mathbb{T}$ is such that $\lim_{r\to 1}\mathcal{K}(\mu, r\xi) = 0$. Then, for every ρ,α,β and $\{\delta_n\}$ such that $\lim_{n\to\infty}\delta_n = 0$, we have*

$$\lim_{n\to+\infty}\sup_k \operatorname{osc}_{\Upsilon_n}(f_k) = 0, \qquad \Upsilon_n = \Upsilon_{\delta_n,\rho,\alpha,\beta}(\xi).$$

Proof. Take an arc $I_n \subset \mathbb{T}$ centered at ξ so that $|I_n| = c_n\delta_n$ for some $c_n > 0$ such that $c_n \to \infty$, $c_n\delta_n \to 0$, and $c_n\sqrt{\rho_n} \to 0$, where $\rho_n = \eta(z_n)$, $z_n = \xi(1-|I_n|)$. For example, one can take $c_n = 1/(\sqrt{\delta_n}+\sqrt[4]{\widetilde{\rho}_n})$, $\widetilde{\rho}_n = \sup_{r\geqslant 1-\sqrt{\delta_n}}\eta(r\xi)$. By Lemma 10, we have

$$|\{\xi \in I_n : |f_k(\xi)-f_k(z_n)| > t\}|/|I_n| \lesssim \eta(z_n)/t, \quad t>0. \tag{29}$$

For every g that satisfies $\|g\|_{L^\infty(\mathbb{T})} \leqslant 2$ and every $z \in \Upsilon_n$, we have

$$\mathcal{P}(g,z) = \mathcal{P}(\chi_{I_n}g,z) + \mathcal{P}(\chi_{\mathbb{T}\setminus I_n}g,z) = \mathcal{P}(\chi_{I_n}g,z) + o(1), \quad n\to\infty,$$

since $\lim_{n\to\infty}c_n = \infty$, and this bound holds uniformly in g and z. Thus, having defined $\widetilde{f}_k = (f_k - f_k(z_n))\chi_{I_n}$, we get

$$f_k(z) - f_k(z_n) = \mathcal{P}(f_k - f_k(z_n), z) = \mathcal{P}(\widetilde{f}_k, z) + o(1). \tag{30}$$

Recall that $\lim_{n\to\infty}c_n\sqrt{\rho_n} = 0$. Thus,

$$\left|\mathcal{P}(\widetilde{f}_k,z)\right| \leqslant \mathcal{P}(\chi_{|\widetilde{f}_k|<\sqrt{\rho_n}}|\widetilde{f}_k|,z) + \mathcal{P}(\chi_{|\widetilde{f}_k|\geqslant\sqrt{\rho_n}}|\widetilde{f}_k|,z)\,.$$

The first term is bounded by $\sqrt{\rho_n}$. Consider the second one. Since $z \in \Upsilon_n$, we can estimate the Poisson kernel by $C\delta_n^{-1}$, bound $|\widetilde{f}_k|$ by 2, and apply (29) to write

$$\mathcal{P}(\chi_{|\widetilde{f}_k|\geqslant\sqrt{\rho_n}}|\widetilde{f}_k|,z) \lesssim \frac{|I_n|\rho_n}{\sqrt{\rho_n}\,\delta_n} = c_n\sqrt{\rho_n} \to 0.$$

From (30), we get

$$\lim_{n\to\infty}\sup_k\sup_{z\in\Upsilon_n}|f_k(z)-f_k(z_n)| = 0\,.$$

Since $|f_k(\xi_1)-f_k(\xi_2)| \leqslant |f_k(\xi_1)-f_k(z_n)| + |f_k(\xi_2)-f_k(z_n)|$, we get the statement of the lemma. □

Denote the argument of φ_n^* on $\mathbb{T}$ by ζ_n. Since φ_n^* has no zeroes in $\overline{\mathbb{D}}$, $\zeta_n = \operatorname{Im}\log\varphi_n^*(e^{it})$ is a continuous function and it coincides with the harmonic conjugate of $\log|\varphi_n^*(e^{it})|$ since $\varphi_n^*(0)$ is real. Moreover, $\zeta_n(e^{it}) = (nt-\gamma_n(t))/2$ where γ_n denotes an argument of the Blaschke product $b_n = \varphi_n/\varphi_n^*$. As was discussed previously, $\gamma_n(t)$ is increasing in $t \in [-\pi,\pi)$, see (19).

Lemma 12. *The function $\varphi_n^*(1-zb_nf_n)$ is outer in $\mathbb{D}$. For almost every $t \in (-\pi,\pi)$, the harmonic conjugate of the function $\log|\varphi_n^*(1-\xi b_nf_n)|^2$, $\xi \in \mathbb{T}$, at point e^{it} is given by*

$$v_n(t) = nt - \gamma_n(t) + 2\arctan\left(\frac{|f_n(e^{it})|\sin(\gamma_n(t)+t+\kappa_n(t))}{1+|f_n(e^{it})|\cos(\gamma_n(t)+t+\kappa_n(t))}\right). \tag{31}$$

In this formula, the function $\kappa_n(t)$ is uniquely defined by conditions: $\kappa_n(t) \in [-\pi,\pi)$ and $e^{i\kappa_n(t)} = -f_n(e^{it})/|f_n(e^{it})|$ in the case when f_n is not identically zero. If $f_n = 0$ identically, then the third term in (31) can be dropped.

Proof. Since the polynomial φ_n^* has no zeroes in $\mathbb{D}$ it is an outer function. We also have $\mathrm{Re}(1-zb_nf_n) \geqslant 0$ in $\mathbb{D}$, hence $1-zb_nf_n$ is an outer function as well, see Corollary 4.8 on page 74 in [7]. The harmonic conjugate of $\log|\varphi_n^*(1-zb_nf_n)|$ is the sum of harmonic conjugates of $\log|\varphi_n^*|$ and of $\log|1-zb_nf_n|$. The harmonic conjugate of $\log|\varphi_n^*|$ is ζ_n. The harmonic conjugate of $g = \log|1-\xi b_nf_n|$ is equal to $\mathrm{Im}\log(1-\xi b_nf_n)$ which is the boundary value of the argument of function $1-zb_nf_n$. The latter function has positive real part and its absolute value is bounded by 2. Therefore, $\widetilde{g}$ is well defined a.e. on $\mathbb{T}$ and $\widetilde{g} \in [-\pi/2,\pi/2]$. As we have seen in (9), for $\mu \in \mathrm{Sz}(\mathbb{R})$ we have

$$\int_{\mathbb{T}} \log(1-|f_n|^2)\,dm < +\infty,$$

in particular, $|f_n| < 1$ almost everywhere on $\mathbb{T}$ for each n. Suppose that $\xi = e^{it}$ is such that $f_n(\xi)$, the boundary value of f_n, satisfies $0 < |f_n(\xi)| < 1$. We know that this holds for almost every $\xi = e^{it} \in \mathbb{T}$ (if $|f_n| = 0$ on a set of positive Lebesgue measure, then $f_n = 0$ identically and the lemma holds trivially). Take $\kappa_n(t) \in [-\pi,\pi)$ such that $-f_n(e^{it})/|f_n(e^{it})| = e^{i\kappa_n(t)}$. Then, we have

$$\widetilde{g}(\xi) = \arctan\left(\frac{|b_n(\xi)f_n(\xi)|\sin(\gamma_n(t)+t+\kappa_n(t))}{1+|b_n(\xi)f_n(\xi)|\cos(\gamma_n(t)+t+\kappa_n(t))}\right),$$

due to the formula

$$\frac{1+ae^{i\psi}}{|1+ae^{i\psi}|} = \exp\left(i\arctan\left(\frac{a\sin\psi}{1+a\cos\psi}\right)\right), \tag{32}$$

for $a \in [0,1]$, $\psi \in \mathbb{R}$: $1+a\cos\psi \neq 0$, when we notice that $\mathrm{Re}(1+\xi b_nf_n) \neq 0$ almost everywhere on $\mathbb{T}$. Since $|b_n(\xi)| = 1$, the lemma is proved. $\square$

Now, we can control the oscillation of v_n.

Lemma 13. *Let v_n be defined by (31) and let $I \subset \mathbb{T}$ be an arc with center at $\xi_0 \in \mathbb{T}$. Then, there exist numbers $c_{I,n}$ such that*

$$|\{\xi \in \mathbb{T} : |v_n(\xi) - c_{I,n}| > t\}|/|I| \lesssim t^{-1}\sum_{j\geqslant 0} 2^{-j}\eta(z_j),$$

where the function η is defined in (18) and $\{z_j\}$ is the set of points constructed in Lemma 5.

Proof. Since v_n is the harmonic conjugate of $u_n = \log|\varphi_n^*(1-zb_nf_n)|^2$, we obtain

$$|\{\xi \in \mathbb{T} : |v_n(\xi) - c_{I,n}| > t\}|/|I| \lesssim t^{-1}\|u_n\|_\eta^* \textstyle\sum_{j\geqslant 0} 2^{-j}\eta(z_j)$$

from Lemma 5. It remains to note that $\{\|u_n\|_\eta^*\}$ is uniformly bounded due to Lemma 4. $\square$

We recall that Christoffel-Darboux kernel is defined by

$$k_{\xi,\mu,n}(z) = \sum_{j=0}^{n-1} \varphi_j(z)\overline{\varphi_j(\xi)},$$

where $\{\varphi_j\}$ are polynomials orthonormal with respect to measure μ.

Lemma 14. *If $\xi = e^{it}$ and $t \in \mathbb{R}$, then $\|k_{\xi,\mu,n}\|^2_{L^2(\mu)} = |\varphi_n^*(\xi)|^2 \gamma_n'(t)$.*

Proof. For $\xi \in \mathbb{T}$ and $z \neq \xi$, we have (see [8], Section 1)

$$k_{\xi,\mu,n}(z) = \frac{\varphi_n^*(z)\overline{\varphi_n^*(\xi)} - \varphi_n(z)\overline{\varphi_n(\xi)}}{1 - z\bar{\xi}} = \varphi_n^*(z)\overline{\varphi_n^*(\xi)}\frac{1 - b_n(z)\overline{b_n(\xi)}}{1 - z\bar{\xi}}.$$

Noting that $b_n(e^{is}) = e^{i\gamma_n(s)}$ for $s \in [-\pi, \pi)$, we get

$$\begin{aligned}\|k_{\xi,\mu,n}\|^2_{L^2(\mu)} &= k_{\xi,\mu,n}(\xi) = \lim_{z\to\xi} k_{\xi,\mu,n}(z),\\ &= |\varphi_n^*(\xi)|^2 \lim_{s\to t} \frac{1 - e^{i(\gamma_n(s)-\gamma_n(t))}}{1 - e^{i(s-t)}},\\ &= |\varphi_n^*(\xi)|^2 \gamma_n'(t).\end{aligned}$$

The lemma follows. □

Lemma 15. *Assume that $\lim_{n\to\infty} |\varphi_n^*(\xi)|^{-2} = |D_\mu(\xi)|^2$ for almost every $\xi \in \mathbb{T}$. Let $r_n = 1 - 1/n$ for $n \geqslant 1$. Then, $\lim_{n\to\infty} f_n(r_n\xi) = 0$ for almost every $\xi \in \mathbb{T}$.*

Proof. We claim that for each $\delta \in (0,1)$, there exists a subset $G_\delta(\mu) \subset \mathbb{T}$ with the properties:

$$m(G_\delta(\mu)) \geqslant 1 - \delta, \tag{33}$$

$$\text{each point of } G_\delta(\mu) \text{ is a Lebesgue point,} \tag{34}$$

$$\lim_{\varepsilon\to 0} \sup\{\mathcal{K}(\mu, z),\ z \in S_\rho^*(\xi),\ |z - \xi| < \varepsilon\} = 0 \text{ for } \xi \in G_\delta(\mu) \text{ and } \rho \in (0,1), \tag{35}$$

$$\lim_{n\to\infty} n^{-1} \cdot \|k_{\xi,\mu,n}\|^2_{L^2(\mu)} = |D_\mu(\xi)|^{-2} \text{ for } \xi \in G_\delta(\mu), \tag{36}$$

$$\lim_{n\to\infty} \gamma_n'(t)/n = 1 \text{ uniformly with respect to } t:\ e^{it} \in G_\delta(\mu). \tag{37}$$

Indeed, for every $\mu \in \mathrm{Sz}(\mathbb{T})$ we have (35) almost everywhere on $\mathbb{T}$ since the Poisson kernel is an approximate identity. Theorem 1 in [13] says that the limit relation in (36) holds almost everywhere on $\mathbb{T}$. By Lemma 14, this implies that the limit relation in (37) holds almost everywhere on $[-\pi, \pi]$. So, there is a set E of full Lebesgue measure on $\mathbb{T}$ such that limit relations in (36), (35), (37) holds for $\xi = e^{it}$ in E. Using Egorov's theorem, one can find a subset $\widetilde{E}$ of E of length $2\pi(1-\delta)$ such that the limit relation in (37) is uniform with respect to $t \in [-\pi, \pi]$ provided $e^{it} \in \widetilde{E}$. Then, we can denote by $G_\delta(\mu)$ the set of the Lebesgue points of $\widetilde{E}$.

Now, it suffices to prove that for every fixed $\delta > 0$ we have $\lim_{n\to\infty} f_n(r_n\xi) = 0$ for every $\xi \in G_\delta(\mu)$. Without loss of generality, we can assume that $\xi = 1$. Consider any convergent subsequence $\{f_{n_k}(r_{n_k})\}$ and let $\lim_{k\to\infty} |f_{n_k}(r_{n_k})| = d$. Let us show that $d = 0$. In this proof, we will several times choose subsequences of $\{f_{n_k}(r_{n_k})\}$. To simplify notation, we will assume that sequences under consideration converge without extracting subsequences. In particular, we let $\lim_{n\to\infty} |f_n(r_n)| = d$. By Lemma 11, we have $\lim_{n\to\infty} |f_n(1 - a/n)| = d$ for every $a > 0$. Let, as before, $\gamma_n : [-\pi, \pi) \to \mathbb{R}$ be a continuous branch of the argument of the function $b_n(e^{it})$, where $b_n = \varphi_n/\varphi_n^*$. Denote $I_n(a) = (-a/n, a/n)$ for all $n \geqslant 1$ and a constant $a \geqslant 10$. Consider $n \geqslant a/\pi$. It follows from Lemma 12, Lemma 13, and condition (35) that there are sets $E_n(a) \subset I_n(a)$ and numbers c_n such that functions

$$v_n(t) = nt - \gamma_n(t) + 2\arctan\left(\frac{|f_n(e^{it})|\sin(\gamma_n(t) + t + \kappa_n(t))}{1 + |f_n(e^{it})|\cos(\gamma_n(t) + t + \kappa_n(t))}\right) \tag{38}$$

satisfy the following relations:

(a) $|v_n(t) - c_n| \leqslant \varepsilon_n$ for all $t \in E_n(a)$,
(b) $|E_n(a)| \geqslant (1-\varepsilon_n)|I_n(a)|$,

for some positive sequence $\{\varepsilon_n\}_{n\geqslant 1}$ converging to zero. Next, we renormalize (38) as follows. For each n, take $\pi_n \in \{2\pi\mathbb{Z}\}$ such that $|c_n - \pi_n| \leqslant \pi$ so

$$|(v_n(t) - \pi_n) - (c_n - \pi_n)| \leqslant \varepsilon_n$$

for all $t \in E_n(a)$. We denote $\widehat{c}_n = c_n - \pi_n$, $\widehat{v}_n = v_n - \pi_n$ and $\widehat{\gamma}_n = \gamma_n + \pi_n$. Now, (38) can be rewritten as

$$\widehat{v}_n(t) = nt - \widehat{\gamma}_n(t) + 2\arctan\left(\frac{|f_n(e^{it})|\sin(\widehat{\gamma}_n(t) + t + \kappa_n(t))}{1 + |f_n(e^{it})|\cos(\widehat{\gamma}_n(t) + t + \kappa_n(t))}\right) \tag{39}$$

and the following relations hold:

(a') $|\widehat{c}_n| \leqslant \pi$ and $|\widehat{v}_n(t) - \widehat{c}_n| \leqslant \varepsilon_n$ for all $t \in E_n(a)$.

Since $\widehat{\gamma}_n$ is increasing on $(-\pi,\pi)$, relations (39) and (a') imply that there is a constant $c(a)$ depending only on a, such that $|\widehat{\gamma}_n(t)| \leqslant c(a)$, $t \in \operatorname{co}(E_n(a))$, where $\operatorname{co}(E_n(a))$ is the convex hull of the set $E_n(a) \subset I_n(a)$. Note that $\operatorname{co}(E_n(a))$ contains $I_n(a/2)$ for n such that $|\varepsilon_n| \leqslant 1/2$. Hence, for large enough n, the functions

$$h_n : s \mapsto \widehat{\gamma}_n(s/n)$$

are correctly defined on $[-a/2, a/2]$, increasing, and uniformly bounded by $c(a)$. Therefore, by Helly's selection theorem, one can choose a subsequence of $\{h_n\}$ that converges pointwise on $[-a/2, a/2]$ to a non-decreasing function h. We again will assume that the whole sequence converges to h. One can also assume that functions $\widehat{v}_n(s/n)$, $|f_n(e^{is/n})|$ on $[-a/2, a/2]$ converge in measure to constants $c \in [-\pi,\pi]$, d, respectively. Indeed, for $\widehat{v}_n(s/n)$ this follows from assertion (a'), while for $|f_n(e^{is/n})|$ – from Lemma 10. If $d \neq 0$, Lemma 10 implies also the converge of $\kappa_n(s/n)$ in measure on $[-a/2, a/2]$ to a constant $\kappa \in [-\pi,\pi]$. Choosing, if needed, a subsequence, one can assume (see [6], Theorem 2.30) that the convergence of $\widehat{v}_n(s/n)$, $|f_n(e^{is/n})|$ and $\kappa_n(s/n)$ is pointwise on a subset $E \subset [-a/2, a/2]$ of full Lebesgue measure. Since $\xi = 1$ is the Lebesgue point of the set $G_\delta(\mu)$ and $h_n'(s) = \gamma_n'(s/n)/n$, we use (37) to get

$$h(s_2) - h(s_1) = \lim_{n\to+\infty}(h_n(s_2) - h_n(s_1)) = \lim_{n\to+\infty}\int_{s_1}^{s_2} h_n'(s)\,ds \geqslant s_2 - s_1 \tag{40}$$

for every $s_1 \leqslant s_2$ in E. We consider two cases now.

Case 1. If $d \in [0,1)$, then relation (39) implies

$$c = s - h(s) + 2\arctan\left(\frac{d\cdot\sin(h(s)+\kappa)}{1 + d\cdot\cos(h(s)+\kappa)}\right), \qquad s \in E, \tag{41}$$

where we set $\kappa = 0$ if $d = 0$. The derivative

$$\partial_h\left(h - 2\arctan\left(\frac{d\sin(h+\kappa)}{1 + d\cos(h+\kappa)}\right)\right) = \frac{1-d^2}{1 + 2d\cos(h+\kappa) + d^2}$$

is within $[(1-d)/(1+d), (1+d)/(1-d)]$ so application of the inverse function theorem shows that (41) defines a smooth increasing function on $[-a/2, a/2]$. Since h is nondecreasing, we see that (41) holds for all $s \in [-a/2, a/2]$. Moreover, (40) gives $h'(s) \geqslant 1$ for such s. Differentiating (41), we obtain

$$h'(s) = \frac{1 + 2d\cos(h(s)+\kappa) + d^2}{1-d^2}, \qquad s \in [-a/2, a/2].$$

Since $h' \geqslant 1$ and a parameter a is large enough, there is $s^* \in [-a/2, a/2]$ so that $\cos(h(s^*)+\kappa) = -1$. Thus, $(1-d)/(1+d) \geqslant 1$ which implies $d = 0$ and we are done.

Case 2. Let $d = 1$ and rewrite (39) as

$$\widehat{v}_n(s/n) = s - \widehat{\gamma}_n(s/n) + 2\arctan\left(\frac{|f_n(e^{is/n})|\sin(\widehat{\gamma}_n(s/n) + s/n + \kappa_n(s/n))}{1 + |f_n(e^{is/n})|\cos(\widehat{\gamma}_n(s/n) + s/n + \kappa_n(s/n))}\right). \tag{42}$$

Taking the limit requires some care in this case. We have

$$\lim_{n\to\infty}\Big(1 - e^{is/n} f_n(e^{is/n}) b_n(e^{is/n})\Big) = 1 + e^{i(\kappa + h(s))}$$

for almost every $s \in [-a/2, a/2]$. Let $\widetilde{E}$ be a subset of E on which $1 + e^{i(\kappa+h(s))} \neq 0$. If the function H is defined by the formula

$$H(\alpha) = (\alpha - 2\pi j)/2 \qquad \text{if } \alpha \in (2\pi j - \pi, 2\pi j + \pi), \quad j \in \mathbb{Z},$$

then an identity

$$\arctan\left(\frac{\sin\alpha}{1 + \cos\alpha}\right) = H(\alpha), \quad \alpha:\ \cos\alpha \neq -1, \tag{43}$$

is immediate. Given (43), take a limit in (42) for every $s \in \widetilde{E}$ to get

$$c = s - h(s) + 2\arctan\left(\frac{\sin(h(s) + \kappa)}{1 + \cos(h(s) + \kappa)}\right) = s - h(s) + 2H(h(s) + \kappa)\,.$$

Thus, if $s_1 \neq s_2$ and $s_1, s_2 \in \widetilde{E}$, then $s_2 - s_1 \in \pi\mathbb{Z}$ and so $\widetilde{E}$ is either finite or empty. That implies $e^{i(\kappa+h)} = -1$ almost everywhere on $[-a/2, a/2]$ and h is a nondecreasing step function. That, however, contradicts (40) and we get $d \neq 1$ under assumptions of the lemma. □

We recall that the zeroes of φ_n were denoted by $\{z_{j,n}\}$ and they are all inside $\mathbb{D}$.

Lemma 16. *Suppose there is $a > 0$ such that*

$$\lim_{n\to\infty} f_n(r_{a,n}\xi) = 0$$

for almost every $\xi \in \mathbb{T}$. Then,

$$\lim_{n\to\infty} |\varphi_n^*(\xi)|^2 = |D(\xi)|^{-2}$$

and

$$\liminf_{n\to\infty}\Big(n \min_{1\leqslant j\leqslant n} |\xi - z_{j,n}|\Big) = +\infty \tag{44}$$

for almost every $\xi \in \mathbb{T}$.

Proof. Notice that we have

$$\lim_{n\to\infty}\ \sup_{z\in\Upsilon_{n^{-1},\rho,\alpha,\beta}(\xi)} |f_n(z)| = 0 \tag{45}$$

for all ρ, α, β and almost every $\xi \in \mathbb{T}$ by Lemma 11. Moreover, for almost every $\xi \in \mathbb{T}$, we have

$$\lim_{\epsilon\to 0}\ \sup_{z\in S_\rho^*(\xi), |z-\xi|<\epsilon} \mathcal{K}(\mu, z) = 0 \tag{46}$$

for any $\rho \in (0, 1)$, and

$$\lim_{n\to\infty} n^{-1}\cdot \|k_{\xi,\mu,n}\|^2_{L^2(\mu)} = |D_\mu(\xi)|^{-2}\,. \tag{47}$$

Without loss of generality, we assume that $\xi = 1$ is a point at which all these conditions are satisfied and let $I_n(a) = [-a/n, a/n]$. Like in the proof of previous lemma, we have

$$\widehat{v}_n(t) = nt - \widehat{\gamma}_n(t) + 2\arctan\left(\frac{|f_n(e^{it})|\sin(\widehat{\gamma}_n(t) + t + \kappa_n(t))}{1 + |f_n(e^{it})|\cos(\widehat{\gamma}_n(t) + t + \kappa_n(t))}\right) \tag{48}$$

and there is a set $E_n(a) \subseteq I_n(a)$ and a sequence $\widehat{c}_n \in [-\pi, \pi)$ such that

(a') $|\widehat{v}_n(t) - \widehat{c}_n| \leqslant \varepsilon_n$ for all $t \in E_n(a)$,
(b) $|E_n(a)| \geqslant (1-\varepsilon_n)|I_n(a)|$,

for some positive sequence $\{\varepsilon_n\}_{n\geqslant 1}$ converging to zero. Moreover, we can use (45), Lemma 10, and (46) to choose $E_n(a)$ such that an additional condition

(c) $|f_n(e^{it})| \leqslant \varepsilon_n$ for all $t \in E_n(a)$

is satisfied. Rescale $t \in I_n(a)$ as $t = s/n$, let $h_n(s) = \widehat{\gamma}_n(s/n)$ as in the previous proof, and write

$$\widehat{v}_n(s/n) = s - h_n(s) + 2\arctan\left(\frac{|f_n(e^{is/n})|\sin(h_n(s)+s/n+\kappa_n(s/n))}{1+|f_n(e^{is/n})|\cos(h_n(s)+\tau/n+\kappa_n(s/n))}\right).$$

Take a limit in measure on $[-a/2, a/2]$ in the above equation through some subsequence $\{n_j\}$, it exists thanks to $(a') - (c)$. It follows from (c) that the sequence $\{h_{n_j}\}$ converges to $s + c$ in measure, where c can depend on the choice of subsequence $\{n_j\}$. From each functional sequence converging in measure, we can choose a subsequence converging almost everywhere. We denote it by the same $\{n_j\}$. Since each function h_n is increasing, this convergence is in fact uniform over $[-a/2, a/2]$ due to Lemma 6. The parameter a was arbitrary so we can take an unbounded positive sequence $\{a_l\}$ and choose the subsequence of $\{h_n\}$ which converges to a linear function uniformly on all compacts in $\mathbb{R}$. We again denote it by $\{h_{n_j}\}$. Thus, in view of (19), we can apply Lemma 7 to show that $\{h'_{n_j}\}$ converges to 1 uniformly over compacts and this convergence holds at point $s = 0$, in particular. Arguing by contradiction, we can prove that in fact $\lim_{n\to\infty} h'_n(0) = 1$ through the whole sequence. By (47) and Lemma 14, we get an implication:

$$\lim_{n\to\infty} \frac{\gamma'_n(0)}{n} = 1 \Longrightarrow \lim_{n\to\infty} |\varphi^*_n(1)|^2 = |D_\mu(1)|^{-2}. \tag{49}$$

The property (44) of zeroes follows from Lemma 8. Indeed, $\lim_{n\to\infty} h'_n = 1$ uniformly over compacts in $\mathbb{R}$ so every subsequential limit of $\{h_n\}$ is a linear function of the form $h(t) = t + c$. So, if

$$\liminf_{n\to\infty}\Big(n \min_{1\leqslant j\leqslant n} |1 - z_{j,n}|\Big) < \infty\,, \tag{50}$$

we can choose a subsequence $\{k_n\}$ over which, first, $\{h_{k_n}\}$ converges uniformly to a linear function and, secondly,

$$\liminf_{n\to\infty}\Big(k_n \min_{1\leqslant j\leqslant n} |1 - z_{j,k_n}|\Big) < \infty\,. \tag{51}$$

That contradicts Lemma 8. $\square$

The next result shows that information about zeroes $\{z_{j,n}\}$ gives control of pointwise asymptotics of $\{|\varphi_n(\xi)|\}$ for $\xi \in \mathbb{T}$.

Lemma 17. *Suppose that*

$$\lim_{n\to\infty}\Big(n \min_{1\leqslant j\leqslant n} |\xi - z_{j,n}|\Big) = +\infty \tag{52}$$

holds for almost every $\xi \in \mathbb{T}$. Then,

$$\lim_{n\to\infty} |\varphi^*_n(\xi)|^2 = |D_\mu(\xi)|^{-2}$$

almost everywhere on $\mathbb{T}$.

Proof. We consider ξ in the full measure set of points on $\mathbb{T}$ where (39) and (52) hold. Assume again without loss of generality, that $\xi = 1$ and write renormalized equation (39) taking $s = tn$

$$\widehat{v}_n(s/n) = s - h_n(s) + 2\arctan\left(\frac{|f_n(e^{is/n})|\sin(h_n(s)+s/n+\kappa_n(s/n))}{1+|f_n(e^{is/n})|\cos(h_n(s)+s/n+\kappa_n(s/n))}\right) \tag{53}$$

and

(a') $|\widehat{v}_n(t)-\widehat{c}_n|\leqslant \varepsilon_n$ for all $t\in E_n$, and $|\widehat{c}_n|\leqslant \pi$.

(b) $|E_n(a)|\geqslant (1-\varepsilon_n)|I_n(a)|$,

for some positive sequence $\{\varepsilon_n\}_{n\geqslant 1}$ converging to zero. Therefore, since h_n is increasing,

$$\sup\{|h_n(s)|,\ n\geqslant 1,\ s\in[-a/2,a/2]\}<\infty,$$

and we can apply Helly's theorem on $[-a/2,a/2]$ to find a subsequence $\{h_{k_n}\}$ which converges to a limit h almost everywhere on $[-a/2,a/2]$. The parameter a is arbitrary so, going to subsequences, we can find a non-decreasing function h defined on $\mathbb{R}$ such that a subsequence of $\{h_n\}$ (call it $\{h_{k_n}\}$ also) converges to h almost everywhere on $\mathbb{R}$. From Lemma 9, we know that $h=c_1t+c_2$ and convergence $\lim_{n\to\infty}h_{k_n}=h$ is in fact uniform on compact subsets of $\mathbb{R}$. Formula (53) gives $c_1=1$ if we compare the variations of both sides on $[-a,a]$ when $a\to\infty$. Now, by Lemma 7, we get $\lim_{n\to\infty}h'_{k_n}=1$ uniformly over compacts in $\mathbb{R}$. In particular, $\lim_{n\to\infty}h'_{k_n}(0)=1$. Arguing by contradiction, we again can show that $\lim_{n\to\infty}h'_n(0)=1$ over the whole sequence. By the same reasoning we used in (49), one gets the statement of the lemma. □

What we proved so far implies that assertions (a), (b), (c) of Theorem 3 are equivalent on a subset of $\mathbb{T}$ of full Lebesgue measure. Let us proceed with item (d). The paper [10] will be the main reference in many arguments given below.

Lemma 18. *Suppose* $\lim_{n\to\infty}f_n=0$ *almost everywhere on* $\mathbb{T}$. *Then,*

$$\lim_{n\to\infty}|\varphi_n^*|^2=|D_\mu|^{-2}$$

almost everywhere on $\mathbb{T}$.

Proof. That immediately follows from Khrushchev's formula

$$|\varphi_n^*|^2|D_\mu|^2=\frac{1-|f_n|^2}{|1-\xi b_nf_n|^2},$$

see identity (1.18) in [10]. □

Given μ, we recall that the dual measure $\mu_{\rm dual}$ corresponds to the Schur function which is equal to $-f$. The associated orthonormal polynomials are called the polynomials of the second kind and they are denoted $\{\psi_n\}$. The Wall polynomials $\{A_n\},\{B_n\}$ are connected to orthogonal polynomials by (see formula (5.5) in [10])

$$\varphi_{n+1}=k_{n+1}(zB_n^*-A_n^*),\quad \varphi_{n+1}^*=k_{n+1}(B_n-zA_n)\,, \tag{54}$$

$$\psi_{n+1}=k_{n+1}(zB_n^*+A_n^*),\quad \psi_{n+1}^*=k_{n+1}(B_n+zA_n)\,, \tag{55}$$

where k_n is the leading coefficient of φ_n. For $n\geqslant 1$, let $\widehat{f}_n$ be the Schur function of the probability measure $|\varphi_n^*|^{-2}\,dm$. In fact, we have $\widehat{f}_n=A_{n-1}/B_{n-1}$, see formula (5.11) in [10]. Define

$$F=\frac{1+zf}{1-zf},\qquad \widehat{F}_n=\frac{1+z\widehat{f}_n}{1-z\widehat{f}_n}.$$

Then, from (54), (55) we get identities (see also formulas (5.10) and (5.11) in [10]):

$$\widehat{F}_n=\frac{1+zA_{n-1}/B_{n-1}}{1-zA_{n-1}/B_{n-1}}=\frac{\psi_n^*}{\varphi_n^*}.$$

To show that (d) in Theorem 3 follows from the other conditions, we proceed as follows. Since the real part of $\widehat{F}_n$ is nonnegative, it is an outer function in $\mathbb{D}$ and its behavior can be controlled by the argument. We have $\arg\widehat{F}_n=\arg\psi_n^*-\arg\varphi_n^*$ and this identity will give $\lim_{n\to\infty}\widehat{F}_n(\xi)=F(\xi)$. The latter condition implies

$\lim_{n\to\infty} \widehat{f}_n(\xi) = f(\xi)$, which yields $\lim_{n\to\infty} f_n(\xi) = 0$ by lemma 4.8 in [10]. The estimate on the nontangential maximal function will easily follow.

Lemma 19. *Suppose $Z_n \subset \overline{\mathbb{D}}$ and $\lim_{n\to\infty} \sup_{z\in Z_n} |f_n(z)| = 0$, then*

$$\lim_{n\to\infty} \sup_{z\in Z_n} |f(z) - \widehat{f}_n(z)| = 0.$$

Proof. Formula (4.19) in [10] reads

$$f = \frac{A_n + zB_n^* f_{n+1}}{B_n + zA_n^* f_{n+1}}\,.$$

That yields

$$|f - \widehat{f}_{n+1}| = |f - A_n/B_n| = \left| \frac{f_{n+1} z(B_n^*/B_n - (A_n^* A_n)/(B_n^2))}{1 + z f_{n+1} A_n^* B_n^{-1}} \right|.$$

We have $|A_n^*/B_n| \leqslant 1, |A_n/B_n| \leqslant 1$ in $\mathbb{D}$ (see Lemma 4.5 in [10]). Moreover, since B_n does not vanish in $\overline{\mathbb{D}}$ (by the same Lemma 4.5 in [10]), we also have $|B_n^*/B_n| \leqslant 1$ in $\mathbb{D}$ which follows from the maximum principle and identity $|B_n^*/B_n| = 1$ that holds on $\mathbb{T}$. That proves the lemma. □

Lemma 20. *If $X_n \subset \overline{\mathbb{D}}$ and $\lim_{n\to\infty} \sup_{z\in X_n} |F(z) - \widehat{F}_n(z)| = 0$, then*

$$\lim_{n\to\infty} \sup_{z\in X_n} |z(f(z) - \widehat{f}_n(z))| = 0.$$

Conversely, if $\sup_{z\in\cup_{n\geqslant 1} X_n} |f(z)| < 1$ and $\lim_{n\to\infty} \sup_{z\in X_n} |z(f(z) - \widehat{f}_n(z))| = 0$, then

$$\lim_{n\to\infty} \sup_{z\in X_n} |F(z) - \widehat{F}_n(z)| = 0\,.$$

Proof. Recall that

$$F = \frac{1+zf}{1-zf}, \qquad \widehat{F}_n = \frac{1+z\widehat{f}_n}{1-z\widehat{f}_n}.$$

Thus, we have

$$|F - \widehat{F}_n| = \left| \frac{2z(f - \widehat{f}_n)}{(1-zf)(1-z\widehat{f}_n)} \right| \leqslant \frac{2|z(f-\widehat{f}_n)|}{(1-|f|)(1-|\widehat{f}_n|)}.$$

Analogously,

$$|zf - z\widehat{f}_n| = \left| \frac{2(F - \widehat{F}_n)}{(1+F)(1+\widehat{F}_n)} \right| \leqslant 2|F - \widehat{F}_n|,$$

where we used the fact that $\operatorname{Re} F \geqslant 0$, $\operatorname{Re} \widehat{F}_n \geqslant 0$ in $\overline{\mathbb{D}}$. Now both claims are evident. □

Later, we will need the following technical result.

Lemma 21. *Suppose function G_n is analytic on $\mathfrak{D}_n = \{\eta : |\eta - in| < n\}$, continuous on $\overline{\mathfrak{D}}_n$, and $\operatorname{Re} G_n > 0$ for every $n \geqslant 1$. Assume that there are constants C_1 and C_2 such that $\operatorname{Re} C_1 > 0$,*

$$\lim_{n\to\infty} G_n(\eta) = C_1 \tag{56}$$

uniformly over compacts in $\mathbb{C}^+$,

$$\lim_{n\to\infty} \arg G_n(in - ine^{it/n}) = C_2, \quad \lim_{n\to\infty} \Big(\arg G_n(in - ine^{it/n})\Big)' = 0, \tag{57}$$

and these two limits are uniform in t over compacts in $\mathbb{R}$. Then, $C_2 = \arg C_1$ and

$$\lim_{n\to\infty} \sup_{\eta\in\overline{\mathfrak{H}}_{b,n}} |G_n(\eta) - C_1| = 0, \tag{58}$$

for every $b > 0$, where $\mathfrak{H}_{b,n} = \mathfrak{D}_n \cap \{\eta : |\eta| < b\}$.

Proof. The function $u_n = \operatorname{Im} \log G_n = \arg G_n$ is harmonic in $\mathfrak{D}_n$, continuous on $\overline{\mathfrak{D}}_n$, and $|u_n| \leqslant \pi/2$. For every point $\eta \in \mathfrak{D}_n$, we can write Poisson formula

$$u_n(\eta) = \int_{\partial \mathfrak{D}_n} u_n(\xi) d\omega_\eta(\xi),$$

where ω_η is harmonic measure at η for $\mathfrak{D}_n$ (the rescaled unit disk). The first condition in (57) and $|u_n| \leqslant \pi/2$ imply that

$$\lim_{n\to\infty} \sup_{\eta \in \overline{\mathfrak{H}}_{b,n}} |u_n(\eta) - C_2| = 0, \tag{59}$$

for every b. Thus, $C_2 = \arg C_1$. Next, we will use the fact that the function analytic on the compact simply connected domain in $\mathbb{C}$ can be recovered from the boundary value of its imaginary part (up to a real constant). Indeed, let $\lambda_n(k)$ be conformal map of $\mathbb{D}$ to $\mathfrak{H}_{b,n}$ such that the lower arc of $\partial\mathfrak{H}_{b,n}$, i.e., points $\eta : \eta \in \partial\mathfrak{H}_{b,n}$ that satisfy $|\eta - in| = n$, corresponds to lower semicircle of $\partial\mathbb{D}$, i.e., $k : k \in \partial\mathbb{D}$ for which $\operatorname{Im} k < 0$. Consider $\Gamma_n(k) = \log G_n(\lambda_n(k))$. It is analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ and, given conditions of the lemma, satisfies

$$\lim_{n\to\infty} \operatorname{Im} \Gamma_n(e^{i\theta}) = \arg C_1, \quad \lim_{n\to\infty} (\operatorname{Im} \Gamma_n(e^{i\theta}))' = 0\,, \tag{60}$$

uniformly in $\theta \in [-\pi+\delta, -\delta] \cup [\delta, \pi-\delta]$ for every $\delta > 0$. Moreover, $\lim_{n\to\infty} \Gamma_n(k) = \log C_1$ uniformly on compacts in $\mathbb{D}$ and $|\operatorname{Im} \Gamma_n| \leqslant \pi/2$ in $\overline{\mathbb{D}}$. We can recover Γ_n by the boundary values of its imaginary part as follows:

$$\Gamma_n(k) = i \int_{\mathbb{T}} \operatorname{Im} \Gamma_n(\xi) \cdot \frac{1 + \bar{\xi} k}{1 - \bar{\xi} k}\, dm(\xi) + \operatorname{Re} \Gamma_n(0)\,.$$

The conditions on $\{\Gamma_n\}$ and simple estimates on the integral above imply that $\lim_{n\to\infty} \Gamma_n(k) = \log C_1$ uniformly in $k : \{k \in \overline{\mathbb{D}}, |k-1| \geqslant \delta, |k+1| \geqslant \delta\}$, where δ is any positive number. In particular, this yields

$$\lim_{n\to\infty} \sup_{\eta \in \overline{\mathfrak{H}}_{b-1,n}} |G_n(\eta) - C_1| = 0$$

in the variable η. Since b is arbitrary positive, the lemma is proved.

Lemma 22. *In Theorem 3, if (a), (b), or (c) holds, then* $\lim_{n\to\infty} f_n(\xi) = 0$ *for almost every* $\xi \in \mathbb{T}$.

Proof. From Lemmas 15, 16, and 17, we know that conditions (a)–(c) are equivalent to each other. As before, let γ_n denote the argument of the Blaschke product $b_n = \varphi_n/\varphi_n^*$ and let $\widetilde{\gamma}_n$ be the argument of $\widetilde{b}_n = \psi_n/\psi_n^*$. The proof of Lemma 16 gives control for the derivatives of γ_n and $\widetilde{\gamma}_n$ at almost every point $\xi \in \mathbb{T}$. Without loss of generality, assume that this point ξ is equal to 1. Additionally, assume that the nontangential limit $f(1) = \lim_{z\to 1} f(z)$ exists and $|f(1)| < 1$. That last condition implies existence of nontangential limit of F at point 1 and an estimate $\operatorname{Re} F(1) > 0$. From the proof of Lemma 16, we have

$$\lim_{n\to\infty} \arg \gamma_n'(\tau/n)/n = 1, \qquad \lim_{n\to\infty} \widetilde{\gamma}_n'(\tau/n)/n = 1 \tag{61}$$

uniformly over compacts in $\mathbb{R}$.

By Lemma 15 and Lemma 11, we get $\lim_{n\to\infty} \sup_{z\in\Upsilon_{n^{-1},\rho,\alpha,\beta}} |f_n(z)| = 0$ for arbitrary $\rho \in (0,1)$ and $0 < \alpha < \beta$. From Lemma 19 and Lemma 20, one has $\lim_{n\to\infty} \sup_{z\in\Upsilon_{n^{-1},\rho,\alpha,\beta}} |F(z) - \widehat{F}_n(z)| = 0$. For $n \geqslant 1$ and η in the disk $\mathfrak{D}_n$ defined in Lemma 21, we set $G_n(\eta) = \widehat{F}_n(1 - \eta/(in))$. The existence of nontangential limit of F at point $z=1$ gives $\lim_{n\to\infty} G_n(\eta) = F(1)$ for every $\eta \in \mathbb{C}^+$. This convergence is in fact uniform over compacts in $\mathbb{C}^+$ by Lemma 11. We will apply Lemma 21 next. Notice that $\operatorname{Re} G_n > 0$ in $\mathfrak{D}_n$. If we define

$$u_n(t) = \arg G_n(in - ine^{it/n}) = \arg \widehat{F}_n(e^{it/n}) = \arg \psi_n^*(e^{it/n}) - \arg \varphi_n^*(e^{it/n}),$$

then $\lim_{n\to\infty} u_n' = 0$ uniformly over any compact in $\mathbb{R}$ as follows from (61) and a simple relation between the arguments of $b_n = \varphi_n/\varphi_n^*$ and φ_n^*. Since $|u_n| \leqslant \pi/2$, we can choose a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty} u_{n_j} = C_*, \qquad \lim_{j\to\infty} u_{n_j}' = 0 \tag{62}$$

where C is some constant and the both convergences are uniform on compact subsets of $\mathbb{R}$. Now apply Lemma 21 to G_{n_j} taking $C_1 = F(1)$ and $C_2 = C_*$ to get

$$\lim_{j\to\infty} \sup_{\eta\in\overline{\mathfrak{H}}_{b,n}} |G_{n_j}(\eta) - F(1)| = 0 \tag{63}$$

for every b, where the sets $\mathfrak{H}_{b,n}$ are defined in Lemma 21. Arguing by contradiction, we can strengthen this to

$$\lim_{n\to\infty} \sup_{\eta\in\overline{\mathfrak{H}}_{b,n}} |G_n(\eta) - F(1)| = 0 \tag{64}$$

for every b. Taking $\eta = 0$, we get $\lim_{n\to\infty} \widehat{F}_n(1) = F(1)$ and thus $\lim_{n\to\infty} \widehat{f}_n(1) = f(1)$ by Lemma 20. The last property is equivalent to $\lim_{n\to\infty} f_n(1) = 0$ by Lemma 4.8 in [10]. $\square$

In the next lemma, we will control nontangential maximal function.

Lemma 23. *Let $\rho \in (0,1)$. If g_n is analytic in $\mathbb{D}$, $|g_n| \leqslant 1$, and $\lim\limits_{n\to\infty} g_n(\xi) = 0$ for almost every $\xi \in \mathbb{T}$, then $\sup_{z\in S_\rho^*(\xi)} |g_n(z)| \to 0$ for almost every $\xi \in \mathbb{T}$.*

Proof. By Egorov's theorem, for every $j \in \mathbb{N}$, we can find $E_j \subset \mathbb{T}$, $|E_j^c| < 1/j$ and $\lim_{n\to\infty} g_n = 0$ on E_j uniformly. We can assume without loss of generality that each point of E_j is a Lebesgue point. Take $\xi \in E_j$ and let $z \in S_\rho^*(\xi)$. Write Poisson formula for harmonic function g_n: $g_n(z) = \mathcal{P}(g_n, z)$. By dominated convergence theorem, we have $\lim_{n\to\infty} g_n = 0$ uniformly on compacts in $\mathbb{D}$. Then,

$$\sup_{|z|>r, z\in S_\rho^*(\xi)} |g_n(z)| \lesssim \delta(r) + \sup_{\eta\in E_j} |g_n(\eta)|,$$

where $\lim_{r\to 1} \delta(r) = 0$ because ξ is a Lebesgue point for E_j. For the second term, we have $\lim_{n\to\infty} \sup_{\eta\in E_j} |g_n(\eta)| = 0$ due to uniform convergence to zero on E_j. Thus,

$$\sup_{z\in S_\rho^*(\xi)} |g_n(z)| = \sup_{|z|<r, z\in S_\rho^*(\xi)} |g_n(z)| + \sup_{|z|>r, z\in S_\rho^*(\xi)} |g_n(z)|$$

and $\lim_{n\to\infty} \sup_{z\in S_\rho^*(\xi)} |g_n(z)| = 0$ if we first fix r close enough to 1 and then let $n \to \infty$. Since j is arbitrary, we get statement of the lemma. $\square$

Finally, we are ready to prove Theorem 3.

Proof of Theorem 3. Lemma 15 shows that (a) implies (c). Lemma 16 shows that (c) implies (a) and (b). Lemma 17 proves that (b) implies (a). That establishes equivalence of (a), (b), and (c). Lemma 18 shows that (d) yields (a). Finally, Lemmas 22 and 23 prove that (a), (b), (c) give (d). $\square$

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